LEFT INVARIANT LORENTZIAN, HYPERBOLIC AND RIEMANNIAN METRICS ON LIE GROUPS

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Abstract. We give a classification of flat affine left invariant metric geometric structures on simply connected Lie groups of dimensions two and three. We give some examples of non flat metrics in dimensions up to four.

1. Real Affine Space

An affine space is a triple (A, V, ϕ), where V is a vector space and ϕ is a faithful and transitive action of the additive group of V on the set A. For instance, the classical real affine space is the triple $(\mathbb{R}^n, \overrightarrow{\mathbb{R}^n}, +)$, where $+ : \overrightarrow{\mathbb{R}^n} \times \mathbb{R}^n \to \mathbb{R}^n$ is given by $+(v, a) = v + a$.

Definition 1.1. *A real affine transformation is an invertible map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that $f(a + v) = T(v) + a$ for any $v \in \mathbb{R}^n$ and $a \in \mathbb{R}^n$, with $T : \overrightarrow{\mathbb{R}^n} \to \overrightarrow{\mathbb{R}^n}$ a linear transformation (by a transformation we mean an invertible map).*

The space of affine transformations, denoted by $\mathit{Aff}(\mathbb{R}^n)$, is a group isomorphic to $\mathbb{R}^n \rtimes_\theta GL_n(\mathbb{R})$, where $GL_n(\mathbb{R}^n)$ is the group of linear transformations and θ is the action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ defined by $\theta(T, v) = T(v)$. That is, the product on $\mathit{Aff}(\mathbb{R}^n)$ is given by $(v, T) \cdot (v', T') = (v + T(v'), T \circ T')$. Hence, the group $\mathit{Aff}(\mathbb{R}^n)$ is isomorphic to the matrix group $\left\{ \begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix} : A \in M_{n\times n}, \det(A) \neq 0, v \in M_{n\times 1} \right\}$.

There are special (and important) subgroups of $\mathit{Aff}(\mathbb{R}^n)$. In particular, the Euclidean group $E(n) = \mathbb{R}^n \rtimes_\theta O_n(\mathbb{R})$ where $O_n(\mathbb{R}) = \{A : A^t A = I\}$ is the group of orthogonal matrices. This group determines the Euclidean geometry.

Recall that a scalar product on $\mathbb{R}^n$ is a non-degenerate symmetric bilinear product $\langle\, , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The scalar product is positive definite if $\langle v, v \rangle > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$. Scalar products are in one to one correspondence with invertible symmetric matrices. If A is the corresponding matrix to $\langle\, , \rangle$, then $\langle v, w \rangle = v^t A w$.

Invertible symmetric $n \times n$ matrices are diagonalizable to matrices of the form $\begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}$, with $p + q = n$. The signature (p, q) is given by p number of $1's$ and q number of $-1's$, and is an invariant of the scalar product. The q number of $-1's$ is called the index of the scalar product. We say that the metric is Euclidean if its signature is $(n, 0)$ (or $(0, n)$). That is, the Euclidean space is isomorphic to $(\mathbb{R}^n, I_n)$. Hence, $\langle v, w \rangle = v^T w$.

The real hyperbolic n-dimensional space (n even) is given by $H_n = \mathbb{R}^n \rtimes_\theta HI$ where HI is the group of hyperbolic isometries, that is, $HI = \{A : A^t SA = S\}$ where $S = \begin{bmatrix} I_{n/2} & 0 \\ 0 & -I_{n/2} \end{bmatrix}$ has signature $(n/2, n/2)$.

The real Lorentz group is given by $L_n = \mathbb{R}^n \rtimes_\theta LI$ where LI is the group of Lorentzian isometries defined by $LI = \{A : A^t JA = J\}$ where $J = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$ is a matrix of signature $(n-1, 1)$. Our purpose is to determine all flat and non-flat Euclidean and Lorentzian metric geometries in dimensions two and three.

2. Affine Connections

The study of metric geometries on manifolds is done through "metric connections", i.e., Levi-Civita connections relative to a tensor metric or, in other words, connections compatible with a metric. In this section we introduce all the terminology.

Recall that a linear connection ∇ on a smooth manifold M is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(1) $\nabla_X Y$ is $C^\infty(M)$-linear in the first component,
(2) $\nabla_X Y$ is $\mathbb{R}$-linear in the second component,
(3) $\nabla_X fY = X(f)Y + f\nabla_X Y$ for $f \in C^\infty(M)$.

Over $\mathbb{R}^n$, with coordinates $(x_1, \ldots, x_n)$, the usual linear connection ∇^0 is given by

$$\nabla^0_X Y = \sum_{j=1}^n X(g_j)\frac{\partial}{\partial x_j}, \qquad for \;\; Y = \sum_{i=1}^n g_j \frac{\partial}{\partial x_j}.$$

For vector fields X, Y, Z, the torsion and curvature tensors are given, respectively, by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z.$$

Whenever a manifold M is endowed with a linear connection ∇ whose torsion tensor and curvature vanish identically, we say that ∇ is a flat affine connection and that the pair (M, ∇) is a flat affine manifold.

Let G be a Lie group. For each $a \in G$, we denote by L_a the left translation on G, given by $L_a(x) = ax$, for all $x \in G$. A vector field $X \in \mathfrak{X}(G)$ is said left-invariant if for each $a, g \in G$,

$$(L_a)_{*,g}(X(g)) = X(L_a(g)).$$

The vector space of left-invariant vector fields is isomorphic to the Lie algebra of G, i.e., it is isomorphic to $T_\epsilon G$, where ϵ is the identity element of G. For all $x \in T_\epsilon G$, the vector field x^+ defined by $x^+(g) := (L_g)_{*,\epsilon}x$ for all $g \in G$ is clearly left-invariant.

A linear connection ∇ on G is said left-invariant if, for any $X, Y \in \mathfrak{X}(G)$,

$$(L_a)_* \nabla_X Y = \nabla_{(L_a)_* X}(L_a)_* Y \tag{1}$$

Definition 2.1. *A flat affine Lie group is a Lie group endowed with a flat left-invariant affine connection.*

Let $\mathfrak{g}$ be the Lie algebra of a Lie group G. Recall that $*$ is a left-symmetric product compatible with the bracket if it satisfies

$$(x, y, z) = (y, x, z), \tag{2}$$

$$x * y - y * x = [x, y], \tag{3}$$

where (x, y, z) is the associator of x, y and z and $[x, y]$ is the bracket given in $\mathfrak{g}$.

For left invariant vector fields X, Y, defining $\nabla_X Y := (X_\epsilon * Y_\epsilon)^+$ and extending it so that $\nabla_X fY = X(f)Y + f\nabla_X Y$, for $f \in C^\infty(G)$, we have that a bilinear product on $\mathfrak{g}$ determines a left invariant linear connection on G. The connection is flat (respectively torsion free) if and only if the bilinear product verifies (2) (respectively (3)). Indeed, having a flat left invariant affine connection on G is equivalent to have a left symmetric product on $\mathfrak{g}$ compatible with the bracket. This is also equivalent to have an affine étale representation of G, i.e., a representation with an open orbit and discrete isotropy (see [2]).

3. Pseudo-Riemannian manifolds

Definition 3.1. *A metric tensor g on a smooth manifold M is a symmetric nondegenerate (0,2) tensor field of constant index. More precisely, a metric tensor g smoothly assigns to each point p of M a symmetric nondegenerate bilinear form g_p on the tangent space T_pM such that the index of g_p is the same for all p. A metric tensor is also called pseudo-metric.*

Definition 3.2. *A pseudo-Riemannian manifold (M, g) is a smooth manifold M endowed with a metric tensor g*

Remark 3.1.
1. *Two different metric tensors over the same smooth manifold determine different pseudo-Riemannian manifolds.*
2. *The common index v of g_p on a pseudo-Riemannian manifold (M, g) is called the index of (M, g). It's verified that $0 \leq v \leq dim(M)$. Whenever $v = 0$, we'll say that (M, g) is a Riemannian manifold; that is, each g_p is then a (positive definite) inner product on T_pM and the signature of g is $(n, 0)$. If $v = 1$ and $n \geq 2$, we'll say that (M, g) is a Lorentzian manifold; in this case the signature of g is $(n-1, 1)$. If $v = n/2$ with n even, (M, g) is called a Hyperbolic manifold; the signature of g in this case is $(n/2, n/2)$.*
3. *For each $X, Y \in \mathfrak{X}(M)$, $g(X, Y)$ defines a smooth function $g(X, Y) \colon M \to \mathbb{R}$ by*
$$g(X, Y)(p) = g_p(X_p, Y_p)$$
4. *Given a local coordinate system $(x^1, x^2, \ldots, x^n)$ on $U \subset M$, the coordinates of the metric tensor g on U are*
$$g_{ij} \colon = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$
for $1 \leq i, j \leq n$. For vector fields $X = \sum_{j=1}^{n} f^j \frac{\partial}{\partial x^j}$ and $Y = \sum_{j=1}^{n} h^j \frac{\partial}{\partial x^j}$, where $f^j, h^j \in C^\infty(U)$ for $1 \leq j \leq n$, we have
$$g(X, Y) = \sum_{i,j=1}^{n} g_{ij} f^i h^j.$$
5. *Since g is nondegenerate, for each $p \in U$ the matrix $(g_{ij}(p))$ is invertible, and its inverse matrix is denoted by $(g^{ij}(p))$. Since g is symmetric, $g_{ij} = g_{ji}$ and then $g^{ij} = g^{ji}$.*

(6) The pseudo-metric g can be represented on U by

$$g = \sum_{i,j=1}^{n} g_{ij} dx^i \otimes dx^j.$$

On a pseudo-Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ there is a unique linear connection ∇ such that

$$[X,Y] = \nabla_X Y - \nabla_Y X \tag{4}$$

$$X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \tag{5}$$

for all $X,Y,Z \in \mathfrak{X}(M)$ (see page 61 in [4]). Property (4) states that ∇ has vanishing torsion and property (5) means that ∇ is compatible with the metric $\langle\,,\,\rangle$. Such ∇ is called the Levi-Civita connection of $(M, \langle\,,\,\rangle)$.

A pseudo-Riemannian metric $\langle\,,\,\rangle$ on a Lie group G is called left-invariant if

$$\langle u, v\rangle_a = \langle (L_x)_{*,a} u, (L_x)_{*,a} v\rangle_x$$

for all $x, a \in G$ and for all $u, v \in T_a G$. The compatibility condition (5) becomes, in the case of a left invariant pseudo-Riemannian metric, into

$$\langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle = 0 \tag{6}$$

Remark 3.2. *If g is a left invariant pseudo-Riemannian metric on a Lie group G, then the matrix representation of the bilinear form g_x with respect to the basis $\{e_i^+(x)\}$ is the same for every $x \in G$.*

A flat affine Lie group (G, ∇^+) is Riemannian (respectively Lorentzian, Hyperbolic) if there exists a left invariant Riemannian (respectively Lorentzian, Hyperbolic) metric $\langle\,,\,\rangle$ such that ∇^+ is the Levi-Civita connection of $(G, \langle\,,\,\rangle)$.

4. Two dimensional metric geometries

We want to classify flat and non-flat affine left invariant metric connections on two dimensional simply connected Lie groups. It is known that, up to isomorphism, there are just two simply connected Lie groups, namely $(\mathbb{R}^2, +)$ and $(\mathit{Aff}(\mathbb{R})_0, \cdot)$. Here, the subindex below $\mathit{Aff}(\mathbb{R})$ indicates that we are taking only the connected component which contains the identity. The product on $\mathit{Aff}(\mathbb{R})_0$ is given by $(a,b)\cdot(c,d) = (ac, ad+b)$.

4.1. Flat affine left invariant metric connections.

Up to isomorphism, $\mathbb{R}^2$ admits six flat left invariant affine connections given by the following left symmetric products on the Lie algebra $\mathbb{R}^2$ (see [1]):

$$\begin{aligned}
&R_1 : \begin{cases} e_1\cdot e_1 = 0, & e_1\cdot e_2 = 0,\\ e_2\cdot e_1 = 0, & e_2\cdot e_2 = 0,\end{cases} \qquad
&&R_2 : \begin{cases} e_1\cdot e_1 = 0, & e_1\cdot e_2 = 0,\\ e_2\cdot e_1 = 0, & e_2\cdot e_2 = e_1,\end{cases}\\
&R_3 : \begin{cases} e_1\cdot e_1 = 0, & e_1\cdot e_2 = 0,\\ e_2\cdot e_1 = 0, & e_2\cdot e_2 = e_2,\end{cases} \qquad
&&R_4 : \begin{cases} e_1\cdot e_1 = e_1, & e_1\cdot e_2 = e_2,\\ e_2\cdot e_1 = e_2, & e_2\cdot e_2 = 0,\end{cases}\\
&R_5 : \begin{cases} e_1\cdot e_1 = e_1, & e_1\cdot e_2 = 0,\\ e_2\cdot e_1 = 0, & e_2\cdot e_2 = e_2,\end{cases} \qquad
&&R_6 : \begin{cases} e_1\cdot e_1 = e_1, & e_1\cdot e_2 = e_2,\\ e_2\cdot e_1 = e_2, & e_2\cdot e_2 = -e_1.\end{cases}
\end{aligned} \tag{7}$$

The Lie group $\mathit{Aff}(\mathbb{R})$ admits, up to isomorphism, two 1-parameter families of flat left invariant affine connections and four other such connections (see [1]). These connections are determined by the following two families and the four extra products on the Lie algebra $\mathit{aff}(\mathbb{R})$:

$$F_1(\alpha):\begin{cases} e_1\cdot e_1=\alpha e_1, & e_1\cdot e_2=e_2,\\ e_2\cdot e_1=0, & e_2\cdot e_2=0,\end{cases}\ \alpha\in\mathbb{R},$$
$$F_2(\alpha):\begin{cases} e_1\cdot e_1=\alpha e_1, & e_1\cdot e_2=(\alpha+1)e_2,\\ e_2\cdot e_1=\alpha e_2, & e_2\cdot e_2=0,\end{cases}\ \alpha\in\mathbb{R}\setminus\{0\} \tag{8}$$

$$P_1:\begin{cases} e_1\cdot e_1=e_1+e_2, & e_1\cdot e_2=e_2,\\ e_2\cdot e_1=0, & e_2\cdot e_2=0,\end{cases}\qquad P_2:\begin{cases} e_1\cdot e_1=-e_1+e_2, & e_1\cdot e_2=0,\\ e_2\cdot e_1=-e_2, & e_2\cdot e_2=0,\end{cases}$$
$$P_3:\begin{cases} e_1\cdot e_1=e_1, & e_1\cdot e_2=e_2,\\ e_2\cdot e_1=0, & e_2\cdot e_2=e_1,\end{cases}\qquad P_4:\begin{cases} e_1\cdot e_1=e_1, & e_1\cdot e_2=e_2,\\ e_2\cdot e_1=0, & e_2\cdot e_2=-e_1.\end{cases} \tag{8}$$

4.1.1. *Riemannian metric connections.* The connection defined on $\mathbb{R}^2$ by product R_1 on (7) is trivially compatible with any metric and, in particular, with any Riemannian metric.

4.1.2. *Lorentzian metric connections.* There is a unique flat affine left invariant metric connection on $\mathit{Aff}(\mathbb{R})$ determined by product $F_1(-1)$ on (8). This connection is compatible with the Lorentzian metric determined by the matrix $C=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

4.2. **Non-flat affine left invariant metric connections.** Up to isomorphism, there is only one non-flat affine left invariant metric connection in dimension two. This connection is given by the product $e_2\cdot e_1=-e_2, e_2\cdot e_2=e_1$ on $\mathit{Aff}(\mathbb{R})_0$, and it's the Levi-Civita connection of the left invariant metric given by matrix $P=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

5. Three dimensional metric geometries

If G is a three dimensional simply connected Lie group, then it is isomorphic to one of the groups listed below:

(1) $(\mathbb{R}^3,+)$.
(2) $\mathit{Aff}(\mathbb{R})_0\times\mathbb{R}$.
(3) Heisenberg group $H_3(\mathbb{R})$, given by product $(x,y,z)\cdot(x',y',z')=(x+x',y+y',z+z'+xy')$.
(4) $G_4=(\mathbb{R}^3,\cdot)$ with product given by $(x,y,z)\cdot(x',y',z')=(x+x'e^z+zy'e^z,y+y'e^z,z+z')$.
(5) $G_5(\lambda)=(\mathbb{R}^3,\cdot)$ with product given by $(x,y,z)\cdot(x',y',z')=(x+x'e^z,y+y'e^{\lambda z},z+z')$.
(6) $G_6(\beta)=(\mathbb{R}^3,\cdot)$ with product given by
$(x,y,z)\cdot(x',y',z')=(x+x'e^{\beta z}\cos z+y'e^{\beta z}\sin z,y-x'e^{\beta z}\sin z+y'e^{\beta z}\cos z,z+z')$.
(7) The special linear group $SL(2,\mathbb{R})$.
(8) The rotation group $SO(3)$.

We give a complete classification of flat left invariant affine metric connection and we exhibit some left invariant affine metric connections on each of the Lie groups above. For each of the Lie groups, we present every left invariant connection in the form of the associated left symmetric product on its Lie Algebra.

5.1. **Flat affine left invariant metric connections.**

5.1.1. $(\mathbb{R}^3,+)$ *with Lie Algebra* $\mathbb{R}^3$. The trivial left symmetric product on $\mathbb{R}^3$ defines a flat affine connection compatible with every possible left invariant metric.

5.1.2. *$Aff(\mathbb{R})\times\mathbb{R}$ with Lie Algebra $aff(\mathbb{R})\times\mathbb{R}$.* If $*$ is a flat symmetric product on $aff(\mathbb{R})\times\mathbb{R}$ associated to a flat left invariant affine connection on $Aff(\mathbb{R})\times\mathbb{R}$, then it is isomorphic to the product

$$e_1\cdot e_1=-e_1,\; e_1\cdot e_2=e_2.$$

This product determines a left-invariant affine connection which is the Levi-Civita connection of the Lorentzian metric given by matrix $M=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$.

5.1.3. *$H_3(\mathbb{R})$ with Lie Algebra $\mathfrak{h}_3$ given by $[e_1,e_2]=e_3$.* Up to isomorphism, the only flat left invariant affine connection on $H_3(\mathbb{R})$ is the one determined by the left symmetric product

$$e_1\cdot e_1=-e_2,\; e_1\cdot e_2=e_3.$$

The associated connection is the Levi-Civita connection of the Lorentzian left invariant metric given by $M=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$.

5.1.4. *G_4 with Lie Algebra $\mathfrak{g}_4$ given by $[e_3,e_1]=e_1,[e_3,e_2]=e_1+e_2$.* The product

$$e_2\cdot e_2=e_1, e_2\cdot e_3=-e_2, e_3\cdot e_1=e_1, e_3\cdot e_2=e_1, e_3\cdot e_3=-e_2-e_3,$$

determines the only flat affine left invariant connection on G_4. This connection is the Levi-Civita connection of the left invariant Lorentzian metric given by matrix $M=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$.

5.1.5. *$G_5(\lambda=1)$ with Lie Algebra $\mathfrak{g}_5(1)$ given by $[e_3,e_1]=e_1,[e_3,e_2]=e_2$.* Every flat affine left invariant connection on $G_5(1)$ is isomorphic to the connection given by product

$$e_2\cdot e_2=e_1, e_2\cdot e_3=-e_2, e_3\cdot e_1=e_1, e_3\cdot e_3=-e_3.$$

This connection is the Levi-Civita connection of the left invariant Lorentzian metric given by matrix $M=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$.

5.1.6. *$G_5(\lambda=-1)$ with Lie Algebra $\mathfrak{g}_5(-1)$ given by $[e_3,e_1]=e_1,[e_3,e_2]=-e_2$.* Up to isomorphism, there is only one flat affine left invariant affine connection on $G_5(-1)$. This connection is given by the product

$$e_3\cdot e_1=e_1, e_3\cdot e_2=-e_2,$$

which is the Levi-Civita connection of the left invariant Lorentzian metric given by matrix $M=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$

5.1.7. $G_6(\beta = 0)$ *with Lie Algebra* $\mathfrak{g}_6(0)$ *given by* $[e_3, e_1] = -e_2, [e_3, e_2] = e_1$. This group is isomorphic to the Euclidean group $E(2)$. There is only one flat affine left invariant connection on $G_6(0)$ given by product

$$e_3 \cdot e_1 = -e_2, e_3 \cdot e_2 = e_1.$$

This connection is the Levi-Civita connection of both of the Riemannian and Lorentzian metrics given, respectively, by matrices $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

To summarize this subsection, we classify the groups that admits a flat affine left invariant metric connection in terms of the index of the metric.

The trivial left symmetric product on $\mathbb{R}^3$ defines a connection compatible with every possible left invariant metric. The group $G_6(\beta = 0)$ admits a flat left-invariant connection compatible with the Riemannian metric $g := dx^2 + dy^2 + dz^2$

The simply connected Lie groups that admit a flat left-invariant affine connection compatible with a Lorentzian metric are listed below, together with the respective metric tensor:

Group	Metric
$(\mathbb{R}^3, +)$	$g = dx^2 + dy^2 - dz^2$
$Aff(\mathbb{R}) \times \mathbb{R}$	$g = \frac{1}{x^2}(dx \otimes dy + dy \otimes dx) + dz^2$
$H_3(\mathbb{R})$	$g = -x(dx \otimes dy + dy \otimes dx) + dx \otimes dz + dz \otimes dx + dy^2$
G_4	$g = e^{-z}(dx \otimes dz + dz \otimes dx) + e^{-2z}dy^2 - ze^{-z}(dy \otimes dz + dz \otimes dy)$
$G_5(\lambda = 1)$	$g = e^{-z}(dx \otimes dz + dz \otimes dx) + e^{-2z}dy^2$
$G_5(\lambda = -1)$	$g = dx \otimes dy + dy \otimes dx + dz^2$
$G_6(\beta = 0)$	$g = dx^2 + dy^2 - dz^2$

5.2. **Some non-flat affine left invariant metric connections.**

5.2.1. $(\mathbb{R}^3, +)$. The trivial left symmetric product determines the only left invariant metric connection on $\mathbb{R}^2$. This connection is Levi-Civita for every left invariant metric.

5.2.2. $Aff(\mathbb{R})_0 \times \mathbb{R}$. The following left symmetric products determine non-flat affine left invariant connections on $Aff(\mathbb{R})_0 \times \mathbb{R}$. Each of these connections is Levi-Civita of the left invariant metric given by the respective matrix:

$$e_2 \cdot e_1 = -e_2, e_2 \cdot e_2 = e_1, \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e_2 \cdot e_1 = -e_2, e_2 \cdot e_2 = -e_1, \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e_2 \cdot e_1 = -e_2, e_2 \cdot e_2 = e_3, \qquad \text{with } g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$e_1 \cdot e_2 = -e_2 \cdot e_1 = \frac{1}{2}e_2, e_1 \cdot e_3 = e_3 \cdot e_1 = -\frac{1}{2}e_3, e_2 \cdot e_3 = e_3 \cdot e_2 = \frac{1}{2}e_1 \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

5.2.3. $H_3(\mathbb{R})$. The group $H_3(\mathbb{R})$ admits the following non-flat affine left invariant metric connections:

$$e_1\cdot e_2=-e_2\cdot e_1=\frac{1}{2}e_3, e_1\cdot e_3=e_3\cdot e_1=\frac{-1}{2}e_2, e_2\cdot e_3=e_3\cdot e_2=\frac{1}{2}e_1, \quad \text{with } g=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix};$$

$$e_1\cdot e_2=-e_2\cdot e_1=\frac{1}{2}e_3, e_1\cdot e_3=e_3\cdot e_1=\frac{1}{2}e_2, e_2\cdot e_3=e_3\cdot e_2=\frac{-1}{2}e_1, \quad \text{with } g=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix};$$

$$e_1\cdot e_2=-e_2\cdot e_1=\frac{1}{2}e_3, e_1\cdot e_3=e_3\cdot e_1=\frac{1}{2}e_2, e_2\cdot e_3=e_3\cdot e_2=\frac{1}{2}e_1, \quad \text{with } g=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}.$$

5.2.4. G_4. This group admits the following non-flat affine left invariant metric connections:

$\cdot$	e_1	e_2	e_3
e_1	e_3	$\frac{1}{2}e_3$	$-e_1-\frac{1}{2}e_2$
e_2	$\frac{1}{2}e_3$	e_3	$-\frac{1}{2}e_1-e_2$
e_3	$-\frac{1}{2}e_2$	$\frac{1}{2}e_1$	0

with $g=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	$-e_3$	$-\frac{1}{2}e_3$	$-e_1-\frac{1}{2}e_2$
e_2	$-\frac{1}{2}e_3$	$-e_3$	$-\frac{1}{2}e_1-e_2$
e_3	$-\frac{1}{2}e_2$	$\frac{1}{2}e_1$	0

with $g=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	e_3	$\frac{1}{2}e_3$	$-e_1+\frac{1}{2}e_2$
e_2	$\frac{1}{2}e_3$	$-e_3$	$-\frac{1}{2}e_1-e_2$
e_3	$\frac{1}{2}e_2$	$\frac{1}{2}e_1$	0

with $g=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	$-e_3$	$-\frac{1}{2}e_3$	$-e_1+\frac{1}{2}e_2$
e_2	$-\frac{1}{2}e_3$	e_3	$-\frac{1}{2}e_1-e_2$
e_3	$\frac{1}{2}e_2$	$\frac{1}{2}e_1$	0

with $g=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	0	e_3	$-e_1$
e_2	e_3	e_3	$-e_1-e_2$
e_3	0	0	0

with $g=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	e_2	$\frac{1}{2}e_2$	$-e_1-\frac{1}{2}e_3$
e_2	$\frac{1}{2}e_2$	0	$-\frac{1}{2}e_1$
e_3	$-\frac{1}{2}e_3$	$\frac{1}{2}e_1+e_2$	$-e_3$

with $g=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$.

5.2.5. $G_5(\lambda)$. For each λ, the group $G_5(\lambda)$ admits the following affine left invariant metric connections:

$$e_1 \cdot e_1 = e_3, e_1 \cdot e_3 = -e_1, e_2 \cdot e_2 = \lambda e_3, e_2 \cdot e_3 = -\lambda e_2 \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e_1 \cdot e_1 = -e_3, e_1 \cdot e_3 = -e_1, e_2 \cdot e_2 = -\lambda e_3, e_2 \cdot e_3 = -\lambda e_2 \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix};$$

$$e_1 \cdot e_1 = e_3, e_1 \cdot e_3 = -e_1, e_2 \cdot e_2 = -\lambda e_3, e_2 \cdot e_3 = -\lambda e_2 \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e_1 \cdot e_1 = -e_3, e_1 \cdot e_3 = -e_1, e_2 \cdot e_2 = \lambda e_3, e_2 \cdot e_3 = -\lambda e_2 \qquad \text{with } g = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e_2 \cdot e_2 = \lambda e_1, e_2 \cdot e_3 = -\lambda e_2, e_3 \cdot e_1 = e_1, e_3 \cdot e_3 = -e_3 \qquad \text{with } g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix};$$

$$e_1 \cdot e_1 = e_2, e_1 \cdot e_3 = -e_1, e_3 \cdot e_2 = \lambda e_2, e_3 \cdot e_3 = -\lambda e_3 \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix};$$

$\cdot$	e_1	e_2	e_3
e_1	0	$\frac{1+\lambda}{2}e_3$	$-\frac{1+\lambda}{2}e_1$
e_2	$\frac{1+\lambda}{2}e_3$	0	$-\frac{1+\lambda}{2}e_2$
e_3	$\frac{1-\lambda}{2}e_1$	$-\frac{1-\lambda}{2}e_2$	0

$$\text{with } g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5.2.6. $G_6(\beta)$. For each β, the group $G_6(\beta)$ admits the following affine left invariant metric connections:

$\cdot$	e_1	e_2	e_3
e_1	$\pm\beta e_3$	0	$-\beta e_1$
e_2	0	$\pm\beta e_3$	$-\beta e_2$
e_3	$-e_2$	e_1	0

$$\text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix};$$

$\cdot$	e_1	e_2	e_3
e_1	βe_3	e_3	$-\beta e_1 + e_2$
e_2	e_3	$-\beta e_3$	$-e_1 - \beta e_2$
e_3	0	0	0

$$\text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$\cdot$	e_1	e_2	e_3
e_1	0	$-\frac{1}{2}e_1$	$\frac{1}{2}e_2$
e_2	$-\frac{1}{2}e_1$	βe_1	$-\beta e_2 + \frac{1}{2}e_3$
e_3	$\beta e_1 - \frac{1}{2}e_2$	$e_1 + \frac{1}{2}e_3$	$-e_2 - \beta e_3$

$$\text{with } g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

5.2.7. $SL(2,\mathbb{R})$. This group does not admit flat affine left invariant metric connections. However, it admits the following affine left invariant metric connections:

$\cdot$	e_1	e_2	e_3
e_1	$\pm 2e_3$	$\frac{1}{2}e_3$	$-2e_1 \mp \frac{1}{2}e_2$
e_2	$-\frac{1}{2}e_3$	$\mp 2e_3$	$\pm\frac{1}{2}e_1 + 2e_2$
e_3	$\mp\frac{1}{2}e_2$	$\pm\frac{1}{2}e_1$	0

with $g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	$2e_3$	$\frac{1}{2}e_3$	$-2e_1 + \frac{1}{2}e_2$
e_2	$-\frac{1}{2}e_3$	$2e_3$	$\frac{1}{2}e_1 + 2e_2$
e_3	$\frac{1}{2}e_2$	$\frac{1}{2}e_1$	0

with $g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	0	$\frac{1}{2}e_3$	$-\frac{1}{2}e_1$
e_2	$-\frac{1}{2}e_3$	0	$\frac{1}{2}e_2$
e_3	$\frac{3}{2}e_1$	$-\frac{3}{2}e_1$	0

with $g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

$\cdot$	e_1	e_2	e_3
e_1	$-e_2$	e_3	0
e_2	0	$-2e_1$	$2e_2$
e_3	$2e_1$	0	$-2e_3$

with $g = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

5.2.8. $SO(3)$. The group $SO(3)$ does not admit flat affine left invariant metric connections. It admits the following non flat affine left invariant metric connections:

$$e_1 \cdot e_2 = -e_2 \cdot e_1 = \frac{1}{2}e_3, e_1 \cdot e_3 = -e_3 \cdot e_1 = \frac{-1}{2}e_2, e_2 \cdot e_3 = -e_3 \cdot e_2 = \frac{1}{2}e_1, \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$e_1 \cdot e_2 = -e_2 \cdot e_1 = \frac{1}{2}e_3, 3e_1 \cdot e_3 = e_3 \cdot e_1 = \frac{3}{2}e_2, 3e_2 \cdot e_3 = e_3 \cdot e_2 = -\frac{3}{2}e_1, \qquad \text{with } g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Appendix A. Four dimensional simply connected solvable Lie groups

If G is a non-abelian solvable Lie group, the Lie algebra $\mathfrak{g}$ of G is not abelian and solvable. Then, if G is a non-abelian solvable connected Lie group , it is isomorphic to the Lie group $(\mathbb{R}^4, \cdot)$, where $\cdot$ is one of the following products:

R_1: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y', z+z'+xy', w+w')$,
R_2: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y'e^x+xz'e^x, z+z'e^x, w+w')$,
R_3: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y'e^x, z+z'e^{\lambda x}, w+w')$, with $\lambda \in [-1,1]$,
R_4: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y'e^{\gamma x}\cos x+z'e^{\gamma x}\sin x, z-y'e^{\gamma x}\sin x+z'e^{\gamma x}\cos x, w+w')$, with $\gamma \geq 0$,
R_5: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y'e^x, z+z', w+w'e^z)$,
R_6: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y', z+z'e^x\cos y - w'e^x\sin y, w+z'e^x\sin y + w'e^x\cos y)$,
R_7: $(x,y,z,w)\cdot(x',y',z',w') = (x+x', y+y'+wx', z+z'+\frac{w^2}{2}x'+wy', w+w')$,
R_8: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^x+xy'e^x+\frac{1}{2}x^2z'e^x, y+y'e^x+xz'e^x, z+z'e^x, w+w')$,

R_9: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^w, y+y'e^{\mu w}+wz'e^{\mu w}, z+z'e^{\mu w}, w+w')$, with $\mu\in\mathbb{R}$,
R_{10}: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^w, y+y'e^{\alpha w}, z+z'e^{\beta w}, w+w')$, with $-1<\alpha\le\beta\le 1, \alpha\beta\neq 0$ or $-1=\alpha\le\beta\le 0$,
R_{11}: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^w, y+y'e^{\gamma w}\cos(\delta w)+z'e^{\gamma d}\sin(\delta w), z-y'e^{\gamma w}\sin(\delta w)+z'e^{\gamma w}\cos(\delta w), w+w')$, with $\gamma\in\mathbb{R}$, $\delta>0$,
R_{12}: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^w, y+y'e^{-w}, z+z'+xy'e^{-w}, w+w')$,
R_{13}: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^{\lambda w}, y+y'e^{(1-\lambda)w}, z+z'e^w+xy'e^{(1-\lambda)w}, w+w')$, with $\lambda\ge\frac{1}{2}$,
R_{14}: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^{\frac{\delta w}{2}}\cos w+y'e^{\frac{\delta w}{2}}\sin w, y-x'e^{\frac{\delta w}{2}}\sin w+y'e^{\frac{\delta w}{2}}\cos w, z+z'e^{\delta w}-xx'e^{\frac{\delta w}{2}}\sin w+xy'e^{\frac{\delta w}{2}}\cos w, w+w')$, with $\delta\ge 0$,
R_{15}: $(x,y,z,w)\cdot(x',y',z',w') = (x+x'e^{\frac{w}{2}}+wy'e^{\frac{w}{2}}, y+y'e^{\frac{w}{2}}, z+z'e^w+xy'e^{\frac{w}{2}}, w+w')$.

For a complete classification of four dimensional solvable Lie algebras see [5]. From now on, G_i will denote the Lie group $\mathbb{R}^4$ with the product R_i from above.

A.1. **Pseudo-Riemannian Structures.** The Lie groups G_1 and G_3 with $\lambda=-1$, admit flat affine left invariant connections compatible with left invariant pseudo-Riemannian metrics. The left symmetric products

$\cdot$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	e_4	0	0
e_3	0	0	0	0
e_4	0	0	0	0

$\cdot$	e_1	e_2	e_3	e_4
e_1	0	0	0	0
e_2	0	e_1	0	0
e_3	0	0	0	0
e_4	0	0	0	0

defined on the Lie algebra $\mathfrak{g}_1$ of G_1, determine flat affine left invariant connections compatible with the left invariant metrics given respectively by the matrices $\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$ and $\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$.

In the case of G_3, for $\lambda=-1$, the left symmetric product

$\cdot$	e_1	e_2	e_3	e_4
e_1	0	e_2	$-e_3$	0
e_2	0	0	0	0
e_3	0	0	0	0
e_4	0	0	0	0

defined on its Lie algebra $\mathfrak{g}_3$, determines flat affine left invariant structures compatible with both of the left invariant metrics given by the matrices above.

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