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Bivariate Generalization of the Gauss Hypergeometric Distribution

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Abstract

The bivariate generalization of the Gauss hypergeometric distribution is defined by the probability density function proportional to $x^{\alpha_1-1}y^{\alpha_2-1}(1-x-y)^{\beta-1}(1+\xi_1 x+\xi_2 y)^{-\gamma}$, $x>0$, $y>0$, $x+y<1$, where $\alpha_i>0$, $i=1,2$, $\beta>0$, $-\infty<\gamma<\infty$ and $\xi_i>-1$, $i=1,2$ are constants. In this article, we study several of its properties such as marginal and conditional distributions, joint moments and the coefficient of correlation. We compute the exact forms of Rényi and Shannon entropies for this distribution. We also derive the distributions of $X+Y$, $X/(X+Y)$, $V=X/Y$ and XY where X and Y follow a bivariate Gauss hypergeometric distribution.

Mathematics Subject Classification: 62H15, 62E15

Keywords: Beta function; beta distribution; Dirichlet distribution; Gauss hypergeometric distribution; hypergeometric function; moments; transformation

1 Introduction

The random variable X is said to have a Gauss hypergeometric distribution with parameters $\alpha > 0$, $\beta > 0$, $-\infty < \gamma < \infty$ and $\xi > -1$, denoted by $X \sim \mathrm{GH}(\alpha, \beta, \gamma, \xi)$, if its probability density function (p.d.f.) is given by

$$f_{\mathrm{GH}}(x; \alpha, \beta, \gamma, \xi) = C(\alpha, \beta, \gamma, \xi)\, \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1+\xi x)^{\gamma}}, \quad 0 < x < 1,$$

where

$$C(\alpha, \beta, \gamma, \xi) = [B(\alpha, \beta)\ {}_2F_1(\gamma, \alpha; \alpha+\beta; -\xi)]^{-1},$$
$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

and ${}_2F_1$ is the Gauss hypergeometric function (Luke [9]). This distribution was suggested by Armero and Bayarri [1] in connection with the prior distribution of the parameter ρ, $0 < \rho < 1$, which represents the traffic intensity in a $M/M/1$ queueing system. A brief introduction of this distribution is given in the encyclopedic work of Johnson, Kotz and Balakrishnan [8, p. 253]. In the context of Bayesian analysis of unreported Poisson count data, while deriving the marginal posterior distribution of the reporting probablity p, Fader and Hardie [5] have shown that $q = 1-p$ has a Gauss hypergeometric distribution. The Gauss hypergeometric distribution has also been used by Dauxois [4] to introduce conjugate priors in the Bayesian inference for linear growth birth and death processes. When either γ or ξ equals to zero, the Gauss hypergeometric density reduces to a beta type 1 density given by (Johnson, Kotz and Balakrishnan [8]),

$$f_{\mathrm{B1}}(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1.$$

Further, for $\gamma = \alpha + \beta$ and $\xi = 1$ the Gauss hypergeometric distribution simplifies to a beta type 3 distribution given by the density (Cardeño, Nagar and Sánchez [3], Sánchez and Nagar [15]),

$$f_{\mathrm{B3}}(x; \alpha, \beta) = \frac{2^{\alpha} x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)(1+x)^{\alpha+\beta}}, \quad 0 < x < 1,$$

where $\alpha > 0$ and $\beta > 0$. The matrix variate generalizations of beta type 1 and beta type 3 distributions have been defined and studied extensively. For example, see Gupta and Nagar [6,7]. For $\gamma = \alpha + \beta$ and $\xi = -(1-\lambda)$, the GH distribution slides to a a three parameter generalized beta distribution (Libby and Novic [10], Nadarajah [11], Nagar and Rada-Mora [13]) defined by the density

$$f_{\mathrm{GB1}}(x; \alpha, \beta; \lambda) = \frac{\lambda^{\alpha} x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)[1-(1-\lambda)x]^{\alpha+\beta}}, \quad 0 < x < 1,$$

where $\alpha > 0$ and $\beta > 0$.

In this article, we present a bivariate generalization of the Gauss hypergeometric distribution and study some of its properties including marginal and conditional distributions, joint moments and the coefficient of correlation. Further, for different values of parameters, we give graphically a variety of forms of the bivariate generalization of the Gauss hypergeometric density. In addition to these results, we also derive densities of several basic transformations such as $X+Y$, $X/(X+Y)$, X/Y, and XY of two variables X and Y jointly distributed as bivariate Gauss hypergeometric. These results are expressed in terms of first hypergeometric function of Appell and Gauss hypergeometric function.

It may also be noted here that bivariate generalization of the Gauss hypergeometric distribution considered here is more general than the Dirichlet distribution (a bivariate beta distribution). In Bayesian analysis, the Dirichlet distribution is used as a conjugate prior distribution for the parameters of a multinomial distribution. However, the Dirichlet family is not sufficiently rich in scope to represent many important distributional assumptions, because the Dirichlet distribution has few number of parameters. We, in this article, provide a generalization of the Dirichlet distribution with added number of parameters.

The bivariate distribution defined in this article can also be used in place of bivariate beta distributions applied in several areas; for example, in the modeling of the proportions of substances in a mixture, brand shares, i.e. the proportions of brands of some consumer product that are bought by customers, proportions of the electorate voting for the candidate in a two-candidate election and the dependence between two soil strength parameters.

2 The Density Function

A bivariate generalization of the Gauss hypergeometric distribution can be defined in the following way.

Definition 2.1. *The random variables X and Y are said to have a bivariate Gauss hypergeometric distribution with parameters $(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2)$, $\alpha_i > 0$, $i = 1, 2$, $\beta > 0$, $-\infty < \gamma < \infty$ and $\xi_i > -1$, $i = 1, 2$, denoted by $(X, Y) \sim$ $\mathrm{BGH}(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2)$, if their joint p.d.f. is given by*

$$\begin{aligned} & f_{\mathrm{BGH}}(x, y; \alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2) \\ & = C(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2)\, \frac{x^{\alpha_1 - 1} y^{\alpha_2 - 1} (1 - x - y)^{\beta - 1}}{(1 + \xi_1 x + \xi_2 y)^{\gamma}}, \\ & \quad x > 0, \quad y > 0, \quad x + y < 1, \end{aligned} \tag{1}$$

where $C(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2)$ is the normalizing constant.

Integrating the joint p.d.f. of X and Y over the space $\{(x,y): x>0, y>0, x+y<1\}$ and using the integral representation of the Appell's first hypergeometric function F_1 given in (A.4), the normalizing constant $C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$ is evaluated as

$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2) = \frac{\Gamma(\alpha_1+\alpha_2+\beta)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)} \times [F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)]^{-1}. \qquad (2)$$

Note that the Appell's first hypergeometric function F_1 in (2) can be expanded in series form if $-1<\xi_i<1$, $i=1,2$. However, if $\xi_i>1$, then we use suitably (A.6) to rewrite F_1 to have absolute value of both the arguments less than one. That is, for the purpose of series expansion, we write

$$\begin{aligned}
&F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)\\
&= (1+\xi_1)^{-\gamma} F_1\left(\gamma,\beta,\alpha_2;\alpha_1+\alpha_2+\beta;\frac{\xi_1}{1+\xi_1},\frac{\xi_1-\xi_2}{1+\xi_1}\right),\\
&\qquad \xi_1\geq 1,\quad -1<\xi_2<1,\\
&= (1+\xi_2)^{-\gamma} F_1\left(\gamma,\alpha_1,\beta;\alpha_1+\alpha_2+\beta;\frac{\xi_2-\xi_1}{1+\xi_2},\frac{\xi_2}{1+\xi_2}\right),\\
&\qquad -1<\xi_1<1,\quad \xi_2\geq 1,\\
&= (1+\xi_1)^{-\alpha_1}(1+\xi_2)^{-\alpha_2}\\
&\quad \times F_1\left(\alpha_1+\alpha_2+\beta-\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;\frac{\xi_1}{1+\xi_1},\frac{\xi_2}{1+\xi_2}\right),\\
&\qquad \xi_1\geq 1,\quad \xi_2\geq 1.
\end{aligned}$$

In particular, when $\gamma=0$ or $\xi_1=\xi_2=0$, the distribution defined by the density (1) takes the form of a Dirichlet distribution with parameters α_1, α_2 and β defined by the p.d.f.

$$\frac{\Gamma(\alpha_1+\alpha_2+\beta)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta)} x^{\alpha_1-1} y^{\alpha_2-1}(1-x-y)^{\beta-1}, \quad x>0, \quad y>0, \quad x+y<1,$$

which is a well known bivariate generalization of the beta distribution. In Bayesian probability theory, if the posterior distribution is in the same family as the prior probability distribution; the prior and posterior are then called conjugate distributions, and the prior is called a conjugate prior. In case of multinomial distribution, the usual conjugate prior is the Dirichlet distribution. If

$$P(s_1,s_2,f|x_1,x_2) = \binom{s_1+s_2+f}{s_1,s_2,f} x_1^{s_1} x_2^{s_2}(1-x_1-x_2)^f$$

and

$$p(x_1,x_2) = C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)\ \frac{x_1^{\alpha_1-1}x_2^{\alpha_2-1}(1-x_1-x_2)^{\beta-1}}{(1+\xi_1x_1+\xi_2x_2)^{\gamma}},$$

where $x_1 > 0$, $x_2 > 0$, and $x_1 + x_2 < 1$, then

$$p(x_1,x_2|s_1,s_2,f) = C(\alpha_1+s_1,\alpha_2+s_2,\beta+f,\gamma,\xi_1,\xi_2) \times \frac{x_1^{\alpha_1+s_1-1}x_2^{\alpha_2+s_2-1}(1-x_1-x_2)^{\beta+f-1}}{(1+\xi_1x_1+\xi_2x_2)^{\gamma}}.$$

Thus, the bivariate family of distributions considered in this article is the conjugate prior for the multinomial distribution.

In continuation, in Figure 1, we present a few graphs of the density function defined by the expression (1) for different values of parameters. Here one can appreciate the wide range of forms that result from the bivariate Gauss hypergeometric distribution.

Next, we derive marginal densities of X and Y.

Theorem 2.1. *Let* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$*. Then, the marginal density of* X *is given by*

$$K\frac{x^{\alpha_1-1}(1-x)^{\alpha_2+\beta-1}}{(1+\xi_1x)^{\gamma}}\,{}_2F_1\left(\gamma,\alpha_2;\alpha_2+\beta;-\frac{\xi_2(1-x)}{1+\xi_1x}\right),\quad 0<x<1,$$

where

$$K^{-1} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2+\beta)}{\Gamma(\alpha_1+\alpha_2+\beta)}F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2).$$

Similarly, the marginal density of Y *is given by*

$$M\frac{y^{\alpha_2-1}(1-y)^{\alpha_1+\beta-1}}{(1+\xi_2y)^{\gamma}}\ {}_2F_1\left(\gamma,\alpha_1;\alpha_1+\beta;-\frac{\xi_1(1-y)}{1+\xi_2y}\right),\quad 0<y<1,$$

where

$$M^{-1} = \frac{\Gamma(\alpha_2)\Gamma(\alpha_1+\beta)}{\Gamma(\alpha_1+\alpha_2+\beta)}F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2).$$

Proof. Integrating the density (1) with respect to y, the marginal density of X is derived as

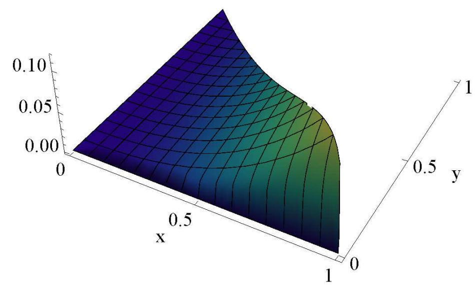
$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)\ x^{\alpha_1-1}\int_0^{1-x}\frac{y^{\alpha_2-1}(1-x-y)^{\beta-1}}{(1+\xi_1x+\xi_2y)^{\gamma}}\,\mathrm{d}y$$

$$= C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)\ \frac{x^{\alpha_1-1}(1-x)^{\alpha_2+\beta-1}}{(1+\xi_1x)^{\gamma}}\int_0^1\frac{z^{\alpha_2-1}(1-z)^{\beta-1}}{[1+(1-x)\xi_2z/(1+\xi_1x)]^{\gamma}}\,\mathrm{d}z,$$

where we have used the substitution $z = y/(1-x)$. Finally, using the integral representation of the Gauss hypergeometric function given in (A.1), the desired result is obtained. Similarly, the marginal density of Y can be derived. □

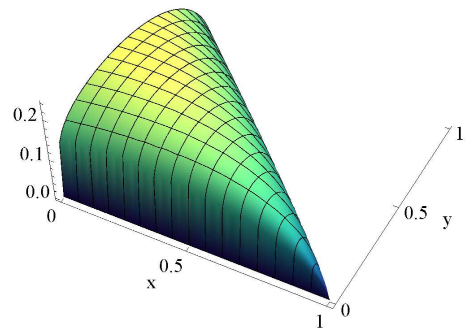
Figure 1: Graphs of the bivariate Gauss hypergeometric density

Graph 1



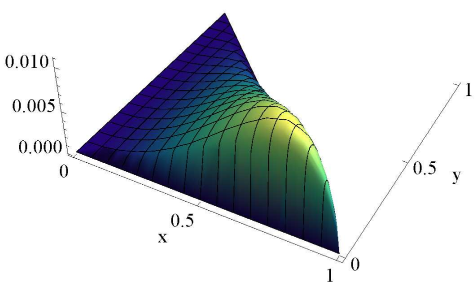
$\alpha_1 = 3, \alpha_2 = 1.2, \beta = 1, \gamma = -2, \xi_1 = 0.5, \xi_2 = 0.3$

Graph 2



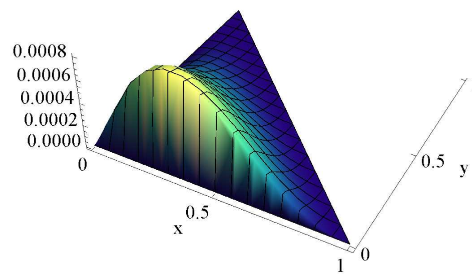
$\alpha_1 = 1, \alpha_2 = 1.2, \beta = 1.5, \gamma = -2, \xi_1 = 0.5, \xi_2 = 0.3$

Graph 3



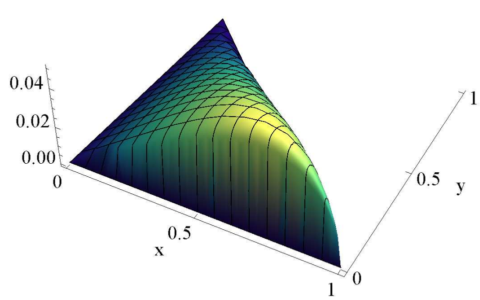
$\alpha_1 = 3, \alpha_2 = 1.2, \beta = 1.5, \gamma = -2, \xi_1 = 0.5, \xi_2 = 0.2$

Graph 4



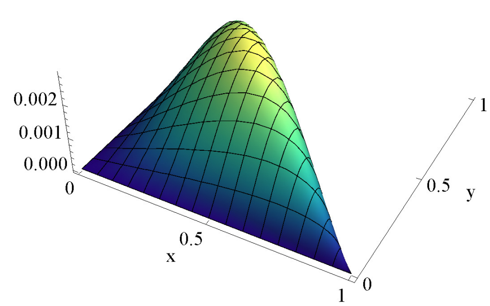
$\alpha_1 = 2, \alpha_2 = 1.2, \beta = 3.5, \gamma = -2, \xi_1 = 0.5, \xi_2 = 0.3$

Graph 5



$\alpha_1 = 2, \alpha_2 = 1.1, \beta = 1.5, \gamma = -2, \xi_1 = 0.5, \xi_2 = 0.3$

Graph 6



$\alpha_1 = 2, \alpha_2 = 2.1, \beta = 1.5, \gamma = -2, \xi_1 = 0.5, \xi_2 = 0.3$

Corollary 2.1.1. *Suppose* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$. *If* $\xi_2 = 0$, *then* $X \sim \mathrm{GH}(\alpha_1,\alpha_2+\beta,\gamma,\xi_1)$.

Using the marginal density of Y given in the Theorem 2.1, we obtain the conditional density function of the variable X given $Y = y$, $0 < y < 1$, as

$$\frac{\Gamma(\alpha_1+\beta)}{\Gamma(\alpha_1)\Gamma(\beta)}\frac{(1+\xi_2 y)^{\gamma}x^{\alpha_1-1}(1-x-y)^{\beta-1}(1+\xi_1 x+\xi_2 y)^{-\gamma}}{(1-y)^{\alpha_1+\beta-1}\ {}_2F_1\left(\gamma,\alpha_1,\alpha_1+\beta,-\xi_1(1-y)/(1+\xi_2 y)\right)},$$

where $0 < x < 1-y$.

By using the integral representation of the first hypergeometric function of Appell given in (A.4), the (r,s)-the joint moment of the variables X and Y having bivariate Gauss hypergeometric distribution, can be expressed as

$$\begin{aligned}\mathrm{E}(X^rY^s) =& \frac{\Gamma(\alpha_1+r)\Gamma(\alpha_2+s)\Gamma(\alpha_1+\alpha_2+\beta)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_1+\alpha_2+\beta+r+s)}\\ &\times\frac{F_1(\gamma,\alpha_1+r,\alpha_2+s;\alpha_1+\alpha_2+\beta+r+s;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)},\end{aligned} \tag{3}$$

where r and s are for non-negative integers.

Substituting appropriately for r and s in (3), we obtain

$$\mathrm{E}(X) = \frac{\alpha_1}{\alpha_1+\alpha_2+\beta}\frac{F_1(\gamma,\alpha_1+1,\alpha_2;\alpha_1+\alpha_2+\beta+1;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)},$$

$$\mathrm{E}(Y) = \frac{\alpha_2}{\alpha_1+\alpha_2+\beta}\frac{F_1(\gamma,\alpha_1,\alpha_2+1;\alpha_1+\alpha_2+\beta+1;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)},$$

$$\begin{aligned}\mathrm{E}(X^2) =& \frac{\alpha_1(\alpha_1+1)}{(\alpha_1+\alpha_2+\beta)(\alpha_1+\alpha_2+\beta+1)}\\ &\times\frac{F_1(\gamma,\alpha_1+2,\alpha_2;\alpha_1+\alpha_2+\beta+2;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)},\end{aligned}$$

$$\begin{aligned}\mathrm{E}(Y^2) =& \frac{\alpha_2(\alpha_2+1)}{(\alpha_1+\alpha_2+\beta)(\alpha_1+\alpha_2+\beta+1)}\\ &\times\frac{F_1(\gamma,\alpha_1,\alpha_2+2;\alpha_1+\alpha_2+\beta+2;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)},\end{aligned}$$

$$\begin{aligned}\mathrm{E}(XY) =& \frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2+\beta)(\alpha_1+\alpha_2+\beta+1)}\\ &\times\frac{F_1(\gamma,\alpha_1+1,\alpha_2+1;\alpha_1+\alpha_2+\beta+2;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)}.\end{aligned}$$

Now, using the respective definitions, we compute Var(X), Var(Y) and Cov(X, Y) as

$$\mathrm{Var}(X) = \frac{\alpha_1}{\alpha_1+\alpha_2+\beta}\left\{\frac{\alpha_1+1}{\alpha_1+\alpha_2+\beta+1}\right.$$
$$\times\frac{F_1(\gamma,\alpha_1+2,\alpha_2;\alpha_1+\alpha_2+\beta+2;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)}$$
$$\left.-\frac{\alpha_1}{\alpha_1+\alpha_2+\beta}\left[\frac{F_1(\gamma,\alpha_1+1,\alpha_2;\alpha_1+\alpha_2+\beta+1;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)}\right]^2\right\},$$

and

$$\mathrm{Var}(Y) = \frac{\alpha_2}{\alpha_1+\alpha_2+\beta}\left\{\frac{\alpha_2+1}{\alpha_1+\alpha_2+\beta+1}\right.$$
$$\times\frac{F_1(\gamma,\alpha_1,\alpha_2+2;\alpha_1+\alpha_2+\beta+2;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)}$$
$$\left.-\frac{\alpha_2}{\alpha_1+\alpha_2+\beta}\left[\frac{F_1(\gamma,\alpha_1,\alpha_2+1;\alpha_1+\alpha_2+\beta+1;-\xi_1,-\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)}\right]^2\right\},$$

$$\mathrm{Cov}(X,Y) = \frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2+\beta}\left\{\frac{F_1(\gamma,\alpha_1+1,\alpha_2+1;\alpha_1+\alpha_2+\beta+2;-\xi_1,-\xi_2)}{(\alpha_1+\alpha_2+\beta+1)F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)}\right.$$
$$-\frac{F_1(\gamma,\alpha_1+1,\alpha_2;\alpha_1+\alpha_2+\beta+1;-\xi_1,-\xi_2)}{\alpha_1+\alpha_2+\beta}$$
$$\left.\times\frac{F_1(\gamma,\alpha_1,\alpha_2+1;\alpha_1+\alpha_2+\beta+1;-\xi_1,-\xi_2)}{[F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)]^2}\right\}.$$

Utilizing the above expressions for variances and covariance, the correlation coefficient between X and Y can be calculated using the formula

$$\rho_{XY} = \frac{\mathrm{Cov}(X,Y)}{[\mathrm{Var(X)Var(Y)}]^{1/2}}.$$

Table 1 contains numerical values of the correlation between X and Y for different values of parameters. As expected, all the values in the table are negative because of the condition $X+Y<1$. Further, by selecting properly values of parameters, it is possible to find values of the correlation close to 0 or -1. Note that for different values of parameters we obtain diverse correlations between X and Y. For example, when α_1 or α_2 is big or β is small, the value of ρ_{XY} is close to -1 which suggests a considerable linear dependence between these two variables. In Figure 2 several graphs illustrate the behavior of the correlation as a function of one of the parameters. The tendency of a fixed value when values of parameters are considerably big can be observed.

					ξ_2			
α_1	α_2	β	γ	ξ_1	-0.7	-0.3	0.5	0.9
0.1	0.3	0.1	1	-0.7	-0.80	-0.77	-0.71	-0.68
0.1	1	0.5	0.5	-0.3	-0.37	-0.35	-0.34	-0.33
0.5	5	1	1	0	-0.54	-0.53	-0.52	-0.52
1.5	10	3	-1	0.5	-0.51	-0.51	-0.52	-0.52
3	15	5	-0.5	1.5	-0.53	-0.54	-0.54	-0.54
5	15	10	1	5	-0.43	-0.43	-0.43	-0.42

Table 1: Correlation coefficient between X and Y with joint density (1) for different values of parameters

3 Entropies

In this section, exact forms of Rényi and Shannon entropies are obtained for the bivariate generalization of the Gauss hypergeometric distribution defined in Section 2.

Let $(\mathcal{X}, \mathcal{B}, \mathcal{P})$ be a probability space. Consider a p.d.f. f associated with $\mathcal{P}$, dominated by $\sigma-$finite measure μ on $\mathcal{X}$. Denote by $H_{SH}(f)$ the well-known Shannon entropy introduced in Shannon [16]. It is define by

$$H_{SH}(f) = -\int_{\mathcal{X}} f(x) \ln f(x)\, d\mu. \tag{4}$$

One of the main extensions of the Shannon entropy was defined by Rényi [14]. This generalized entropy measure is given by

$$H_R(\eta, f) = \frac{\ln G(\eta)}{1-\eta} \qquad (\text{for } \eta > 0 \text{ and } \eta \neq 1), \tag{5}$$

where

$$G(\eta) = \int_{\mathcal{X}} f^{\eta} d\mu.$$

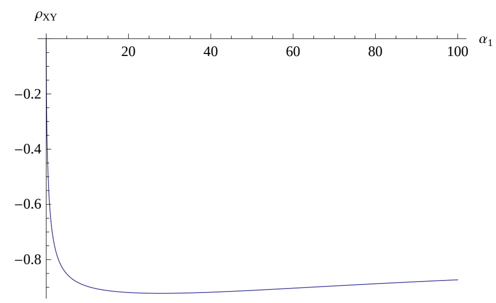
The additional parameter η is used to describe complex behavior in probability models and the associated process under study. Rényi entropy is monotonically decreasing in η, while Shannon entropy (4) is obtained from (5) for $\eta \uparrow 1$. For details see Nadarajah and Zografos [12], Zografos [18], and Zografos and Nadarajah [19].

First, we give the following lemma useful in deriving these entropies.

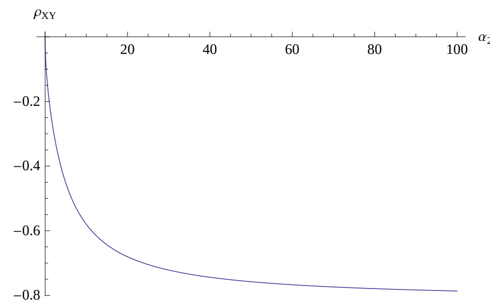
Lemma 3.1. *Let* $g(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2) = \lim_{\eta \to 1} h(\eta)$, *where*

$$h(\eta) = \frac{\mathrm{d}}{\mathrm{d}\eta} F_1(\eta\gamma, \eta(\alpha_1 - 1) + 1, \eta(\alpha_2 - 1) + 1; \eta(\alpha_1 + \alpha_2 + \beta - 3) + 3; -\xi_1, -\xi_2).$$

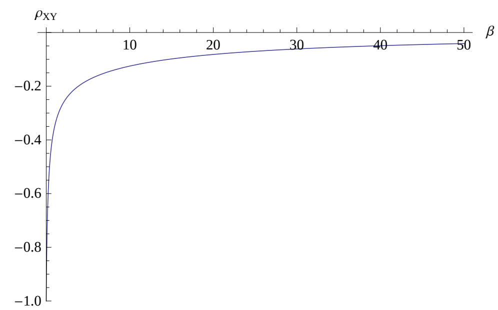
Figure 2: Graphs of the correlation coefficient $\rho_{XY}(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2) = \rho_{XY}$ for different values of $\alpha_1, \alpha_2, \beta, \gamma, \xi_1$ and ξ_2



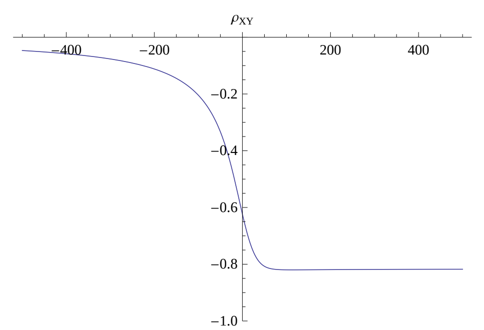
$\rho_{XY}(\alpha_1, 2.1, 1.5, -10, 1, 10)$



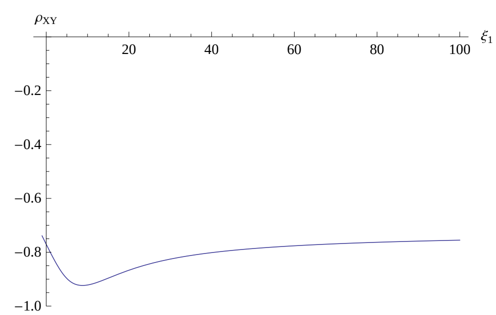
$\rho_{XY}(3, \alpha_2, 1.5, -10, -0.9, 0)$



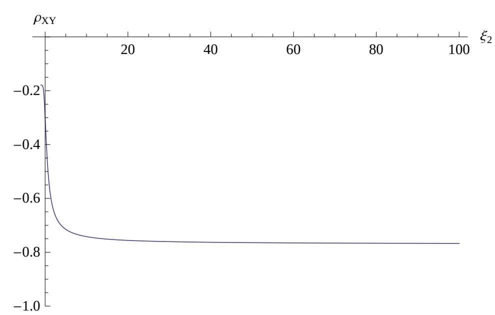
$\rho_{XY}(3, 2.1, \beta, -10, -0.9, 0)$



$\rho_{XY}(3, 2.1, 1.5, \gamma, 0, -0.2)$



$\rho_{XY}(3, 2.1, 1.5, -10, \xi_1, 10)$



$\rho_{XY}(3, 2.1, 1.5, -10, -0.9, \xi_2)$

Then, for $-1 < \xi_1 < 1$, $-1 < \xi_2 < 1$,

$$
\begin{aligned}
&g(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2)\\
&= \sum_{j+k=1}^{\infty} \frac{(\alpha_1)_j (\alpha_2)_k (\gamma)_{j+k}}{(\alpha_1+\alpha_2+\beta)_{j+k} j!\, k!} (-\xi_1)^j (-\xi_2)^k [(\alpha_1 - 1)\psi(\alpha_1 + j)\\
&\quad + (\alpha_2 - 1)\psi(\alpha_2 + k) + \gamma\psi(\gamma + j + k) + (\alpha_1 + \alpha_2 + \beta - 3)\psi(\alpha_1 + \alpha_2 + \beta)\\
&\quad - (\alpha_1 - 1)\psi(\alpha_1) - (\alpha_2 - 1)\psi(\alpha_2) - \gamma\psi(\gamma)\\
&\quad - (\alpha_1 + \alpha_2 + \beta - 3)\psi(\alpha_1 + \alpha_2 + \beta + j + k)], \qquad (6)
\end{aligned}
$$

for $\xi_1 > 1,\ -1 < \xi_2 < 1,$

$$\begin{aligned} & g(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2) \\ & = (1+\xi_1)^{-\gamma} \sum_{j+k=1}^{\infty} \frac{(\beta)_j(\alpha_2)_k(\gamma)_{j+k}}{(\alpha_1+\alpha_2+\beta)_{j+k} j!\,k!} \left(\frac{\xi_1}{1+\xi_1}\right)^j \left(\frac{\xi_1-\xi_2}{1+\xi_1}\right)^k \\ & \quad \times [-\gamma \ln(1+\xi_1) + (\beta-1)\psi(\beta+j) + (\alpha_2-1)\psi(\alpha_2+k) \\ & \quad + \gamma\psi(\gamma+j+k) + (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta) \\ & \quad - (\beta-1)\psi(\beta) - (\alpha_2-1)\psi(\alpha_2) - \gamma\psi(\gamma) \\ & \quad - (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta+j+k)], \end{aligned} \tag{7}$$

for $-1 < \xi_1 < 1,\ \xi_2 > 1,$

$$\begin{aligned} & g(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2) \\ & = (1+\xi_2)^{-\gamma} \sum_{j+k=1}^{\infty} \frac{(\alpha_1)_j(\beta)_k(\gamma)_{j+k}}{(\alpha_1+\alpha_2+\beta)_{j+k} j!\,k!} \left(\frac{\xi_2-\xi_1}{1+\xi_2}\right)^j \left(\frac{\xi_2}{1+\xi_2}\right)^k \\ & \quad \times [-\gamma \ln(1+\xi_2) + (\alpha_1-1)\psi(\alpha_1+j) + (\beta-1)\psi(\beta+k) \\ & \quad + \gamma\psi(\gamma+j+k) + (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta) \\ & \quad - (\alpha_1-1)\psi(\alpha_1) - (\beta-1)\psi(\beta) - \gamma\psi(\gamma) \\ & \quad - (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta+j+k)], \end{aligned} \tag{8}$$

and for $\xi_1 > 1,\ \xi_2 > 1,$

$$\begin{aligned} & g(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2) \\ & = (1+\xi_1)^{-\alpha_1}(1+\xi_2)^{-\alpha_2} \sum_{j+k=1}^{\infty} \frac{(\alpha_1)_j(\alpha_2)_k(\alpha_1+\alpha_2+\beta-\gamma)_{j+k}}{(\alpha_1+\alpha_2+\beta)_{j+k} j!\,k!} \\ & \quad \times \left(\frac{\xi_1}{1+\xi_1}\right)^j \left(\frac{\xi_2}{1+\xi_2}\right)^k [-(\alpha_1-1)\ln(1+\xi_1) - (\alpha_2-1)\ln(1+\xi_2) \\ & \quad + (\alpha_1-1)\psi(\alpha_1+j) + (\alpha_2-1)\psi(\alpha_2+k) \\ & \quad + (\alpha_1+\alpha_2+\beta-\gamma-3)\psi(\alpha_1+\alpha_2+\beta-\gamma+j+k) \\ & \quad + (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta) - (\alpha_1-1)\psi(\alpha_1) - (\alpha_2-1)\psi(\alpha_2) \\ & \quad - (\alpha_1+\alpha_2+\beta-\gamma)\psi(\alpha_1+\alpha_2+\beta-\gamma) \\ & \quad - (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta+j+k)], \end{aligned} \tag{9}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ *is the digamma function.*

Proof. Using the series expansion of F_1, we write

$$\begin{aligned} h(\eta) &= \frac{\mathrm{d}}{\mathrm{d}\eta} \sum_{j,k=0}^{\infty} \Delta_{j,k}(\eta) \frac{(-\xi_1)^j (-\xi_2)^k}{j!\, k!} \\ &= \sum_{j,k=0}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}\eta} \Delta_{j,k}(\eta) \right] \frac{(-\xi_1)^j (-\xi_2)^k}{j!\, k!}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \Delta_{j,k}(\eta) = & \frac{\Gamma(\eta\gamma + j + k)\Gamma[\eta(\alpha_1 - 1) + 1 + j]\Gamma[\eta(\alpha_2 - 1) + 1 + k]}{\Gamma[\eta(\alpha_1 + \alpha_2 + \beta - 3) + 3 + j + k]} \\ & \times \frac{\Gamma[\eta(\alpha_1 + \alpha_2 + \beta - 3) + 3]}{\Gamma(\eta\gamma)\Gamma[\eta(\alpha_1 - 1) + 1]\Gamma[\eta(\alpha_2 - 1) + 1]}. \end{aligned}$$

Now, differentiating the logarithm of $\Delta_j(\eta)$ w.r.t. η, we arrive at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\eta} \Delta_{j,k}(\eta) = \Delta_{j,k}(\eta) [& (\alpha_1 - 1)\psi(\eta(\alpha_1 - 1) + 1 + j) \\ & + (\alpha_2 - 1)\psi(\eta(\alpha_2 - 1) + 1 + k) + \gamma\psi(\eta\gamma + j + k) \\ & + (\alpha_1 + \alpha_2 + \beta - 3)\psi(\eta(\alpha_1 + \alpha_2 + \beta - 3) + 3) - \gamma\psi(\eta\gamma) \\ & - (\alpha_1 - 1)\psi(\eta(\alpha_1 - 1) + 1) - (\alpha_2 - 1)\psi(\eta(\alpha_2 - 1) + 1) \\ & - (\alpha_1 + \alpha_2 + \beta - 3)\psi(\eta(\alpha_1 + \alpha_2 + \beta - 3) + 3 + j + k)]. \end{aligned} \tag{11}$$

Finally, substituting (11) in (10) and taking $\eta \to 1$, we obtain (6). To obtain (7), (8) and (9), we use (A.6) to write

$$\begin{aligned} & F_1(\eta\gamma, \eta(\alpha_1 - 1) + 1, \eta(\alpha_2 - 1) + 1; \eta(\alpha_1 + \alpha_2 + \beta - 3) + 3; -\xi_1, -\xi_2) \\ & = (1 + \xi_1)^{-\eta\gamma} \\ & \quad \times F_1 \left(\eta\gamma, \eta(\beta - 1) + 1, \eta(\alpha_2 - 1) + 1; \eta(\alpha_1 + \alpha_2 + \beta - 3) + 3; \frac{\xi_1}{1 + \xi_1}, \frac{\xi_1 - \xi_2}{1 + \xi_1} \right) \\ & = (1 + \xi_2)^{-\eta\gamma} \\ & \quad \times F_1 \left(\eta\gamma, \eta(\alpha_1 - 1) + 1, \eta(\beta - 1) + 1; \eta(\alpha_1 + \alpha_2 + \beta - 3) + 3; \frac{\xi_2 - \xi_1}{1 + \xi_2}, \frac{\xi_2}{1 + \xi_2} \right) \\ & = (1 + \xi_1)^{-\eta(\alpha_1 - 1) - 1} (1 + \xi_2)^{-\eta(\alpha_2 - 1) - 1} \\ & \quad \times F_1 \Big(\eta(\alpha_1 + \alpha_2 + \beta - \gamma - 3) + 3, \eta(\alpha_1 - 1) + 1, \eta(\alpha_2 - 1) + 1; \\ & \qquad \eta(\alpha_1 + \alpha_2 + \beta - 3) + 3; \frac{\xi_1}{1 + \xi_1}, \frac{\xi_2}{1 + \xi_2} \Big) \end{aligned}$$

and proceed similarly. $\square$

Theorem 3.1. *For the bivariate Gauss hypergeometric distribution defined by the p.d.f. (1), the Rényi and the Shannon entropies are given by*

$$\begin{aligned}
&H_R(\eta, f_{\mathrm{BGH}})\\
&= \frac{1}{1-\eta}[\eta \ln C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2) + \ln\Gamma[\eta(\alpha_1-1)+1]\\
&\quad + \ln\Gamma[\eta(\alpha_2-1)+1] + \ln\Gamma[\eta(\beta-1)+1] - \ln\Gamma[\eta(\alpha_1+\alpha_2+\beta-3)+3]\\
&\quad + \ln F_1(\eta\gamma, \eta(\alpha_1-1)+1, \eta(\alpha_2-1)+1; \eta(\alpha_1+\alpha_2+\beta-3)+3; -\xi_1,-\xi_2)]
\end{aligned}$$

and

$$\begin{aligned}
&H_{SH}(f_{\mathrm{BGH}})\\
&= -\ln C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2) - [(\alpha_1-1)\psi(\alpha_1) + (\alpha_2-1)\psi(\alpha_2) + (\beta-1)\psi(\beta)\\
&\quad - (\alpha_1+\alpha_2+\beta-3)\psi(\alpha_1+\alpha_2+\beta)] - \frac{g(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)}{F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi_1,-\xi_2)},
\end{aligned}$$

respectively, where $g(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$ is given by (6) if $-1<\xi_1,\xi_2<1$, by (7) if $\xi_1>1$, $-1<\xi_2<1$, by (8) if $-1<\xi_1<1$, $\xi_2>1$, by (9) if $\xi_1>1$, $\xi_2>1$, and $\psi(z)=\Gamma'(z)/\Gamma(z)$ is the digamma function.

Proof. For $\eta>0$ and $\eta\neq 1$, using the density of (X,Y) given by (1), we have

$$\begin{aligned}
G(\eta) &= \int_0^1\int_0^{1-x}[f_{\mathrm{BGH}}(x,y;\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)]^\eta \,\mathrm{d}y\,\mathrm{d}x\\
&= [C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)]^\eta \int_0^1\int_0^{1-x}\frac{x^{\eta(\alpha_1-1)}y^{\eta(\alpha_2-1)}(1-x-y)^{\eta(\beta-1)}}{(1+\xi_1 x+\xi_2 y)^{\eta\gamma}}\,\mathrm{d}y\,\mathrm{d}x\\
&= \frac{[C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)]^\eta}{C(\eta(\alpha_1-1)+1,\eta(\alpha_2-1)+1,\eta(\beta-1)+1;\eta\gamma,\xi_1,\xi_2)}\\
&= [C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)]^\eta \frac{\Gamma[\eta(\alpha_1-1)+1]\Gamma[\eta(\alpha_2-1)+1]\Gamma[\eta(\beta-1)+1]}{\Gamma[\eta(\alpha_1+\alpha_2+\beta-3)+3]}\\
&\quad\times F_1(\eta\gamma,\eta(\alpha_1-1)+1,\eta(\alpha_2-1)+1;\eta(\alpha_1+\alpha_2+\beta-3)+3;-\xi_1,-\xi_2),
\end{aligned}$$

where the last line has been obtained by using (2). Now, taking logarithm of $G(\eta)$ and using (5) we get expression for $H_R(\eta, f_{\mathrm{BGH}})$. The Shannon entropy is obtained from the Rényi entropy by taking $\eta\uparrow 1$ and using L'Hopital's rule. □

4 Some Transformations

In this section we obtain expressions for densities of sum, quotient and product of two random variables whose joint p.d.f. is given by (1). First, we give definitions of beta type 1 and beta type 2 distributions.

Definition 4.1. *A random variable X is said to have a beta type 1 distribution with parameters $\alpha > 0$ and $\beta > 0$, denoted by $X \sim \mathrm{B1}(\alpha, \beta)$, if its p.d.f. is given by*

$$f_{\mathrm{B1}}(x; \alpha, \beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 < x < 1.$$

Definition 4.2. *A random variable Y is said to have a beta type 2 distribution with parameters $\alpha > 0$ and $\beta > 0$, denoted by $Y \sim \mathrm{B2}(\alpha, \beta)$, if its p.d.f. is given by*

$$f_{\mathrm{B2}}(y; \alpha, \beta) = \frac{y^{\alpha-1}(1+y)^{-(\alpha+\beta)}}{B(\alpha, \beta)}, \quad 0 < x < 1.$$

Now, we turn our attention to the problem of obtaining distributions of sum, quotient and product.

Theorem 4.1. *Let $(X, Y) \sim \mathrm{BGH}(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2)$ and define $S = X+Y$, and $U = X/(X+Y)$. Then, the p.d.f. of S is given by*

$$K_1 \frac{s^{\alpha_1+\alpha_2-1}(1-s)^{\beta-1}}{(1+\xi_2 s)^{\gamma}} \, {}_2F_1\left(\gamma, \alpha_1; \alpha_1+\alpha_2; \frac{\xi_2-\xi_1}{1+\xi_2 s} s\right), \quad 0 < s < 1, \tag{12}$$

where

$$K_1^{-1} = B(\alpha_1+\alpha_2, \beta) F_1(\gamma, \alpha_1, \alpha_2; \alpha_1+\alpha_2+\beta; -\xi_1, -\xi_2).$$

Further, the density of U is given by

$$K_2 \, u^{\alpha_1-1}(1-u)^{\alpha_2-1} \, {}_2F_1(\gamma, \alpha_1+\alpha_2; \alpha_1+\alpha_2+\beta; (\xi_2-\xi_1)u - \xi_2), \tag{13}$$

where $0 < u < 1$ and

$$K_2^{-1} = B(\alpha_1, \alpha_2) \; F_1(\gamma, \alpha_1, \alpha_2; \alpha_1+\alpha_2+\beta; -\xi_1, -\xi_2).$$

Proof. Making the transformation $S = X + Y$ and $U = X/(X + Y)$, whose Jacobian is $J(x, y \to s, u) = s$ in (1), the joint density of S and U is given by

$$C(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2) \; \frac{s^{\alpha_1+\alpha_2-1}(1-s)^{\beta-1} u^{\alpha_1-1}(1-u)^{\alpha_2-1}}{[1+(\xi_1-\xi_2)su+\xi_2 s]^{\gamma}}, \tag{14}$$

where $0 < s < 1$ and $0 < u < 1$. To compute the marginal density of S we need to integrate the above expression with respect to u to obtain

$$C(\alpha_1, \alpha_2, \beta, \gamma, \xi_1, \xi_2) \; \frac{s^{\alpha_1+\alpha_2-1}(1-s)^{\beta-1}}{(1+\xi_2 s)^{\gamma}} \int_0^1 \frac{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}{[1+(\xi_1-\xi_2)su/(1+\xi_2 s)]^{\gamma}} \, \mathrm{d}u,$$

where $0 < s < 1$. Now, evaluating the above integral using (A.1) and simplifying, we get the desired result. On the other hand, to obtain the marginal density of U, we integrate the joint density (14) with respect to s, to obtain

$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)\ u^{\alpha_1-1}(1-u)^{\alpha_2-1}\int_0^1 \frac{s^{\alpha_1+\alpha_2-1}(1-s)^{\beta-1}}{[1+\{(\xi_1-\xi_2)u+\xi_2\}s]^{\gamma}}\,\mathrm{d}s,$$

where $0 < u < 1$. Now, evaluation of the above integral by using (A.1) yields the desired result. □

Corollary 4.1.1. *If* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$, *then the p.d.f. of* $V = X/Y$ *is given by*

$$K_2\,\frac{v^{\alpha_1-1}}{(1+v)^{\alpha_1+\alpha_2}}\,{}_2F_1\left(\gamma,\alpha_1+\alpha_2;\alpha_1+\alpha_2+\beta;-\frac{\xi_1 v+\xi_2}{1+v}\right),\quad v>0. \tag{15}$$

Proof. We can write

$$\frac{X}{X+Y}=\frac{X/Y}{X/Y+1}$$

which, in terms of $U = X/(X+Y)$ and $V = X/Y$ can be written as $U = V/(1+V)$. Therefore, making the transformation $U = V/(1+V)$ with the Jacobian $J(u \to v) = (1+v)^{-2}$ in the density of U given in (13), we get the density of V. □

Corollary 4.1.2. *If* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi,\xi)$, *then* $X+Y \sim \mathrm{GH}(\alpha_1+\alpha_2,\beta,\gamma,\xi)$ *and* $X/(X+Y) \sim \mathrm{B1}(\alpha_1,\alpha_2)$.

Proof. Substituting $\xi_1 = \xi_2 = \xi$ in (12) and (13) and simplifying the resulting expressions using ${}_2F_1(\gamma,\alpha_1;\alpha_1+\alpha_2;0) = 1$ and $F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi,-\xi) = {}_2F_1(\gamma,\alpha_1+\alpha_2;\alpha_1+\alpha_2+\beta;-\xi)$, the desired result is obtained. □

Corollary 4.1.3. *If* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi,\xi)$, *then* $V = X/Y \sim \mathrm{B2}(\alpha_1,\alpha_2)$.

Proof. Substituting $\xi_1 = \xi_2 = \xi$ in (15) and simplifying the resulting expressions by using $F_1(\gamma,\alpha_1,\alpha_2;\alpha_1+\alpha_2+\beta;-\xi,-\xi) = {}_2F_1(\gamma,\alpha_1+\alpha_2;\alpha_1+\alpha_2+\beta;-\xi)$, we obtain the desired result. □

The distribution of XY has been studied by several authors especially when X and Y are independent random variables and come from the same family. However, there is relatively little work of this kind when X and Y are correlated random variables. For a bivariate random vector (X,Y), the distribution of the product XY is of interest in problems in biological and physical sciences, econometrics, and classification. The following theorem gives partial result on the distribution of the product of two random variables distributed jointly as bivariate Gauss hypergeometric.

Theorem 4.2. *If* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$, $\xi_1 \neq 0$ *and* $\xi_1\xi_2 \leq 1$, *then the p.d.f. of the random variable* $Z = XY$*, is given by*

$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)\frac{B(\beta,\beta)b^{\alpha_1+\gamma-\alpha_2-\beta}(b-a)^{2\beta-1}z^{\alpha_2-1}}{(b-p)^\gamma(b-q)^\gamma}\sum_{r=0}^\infty\left(\frac{b-a}{b}\right)^r$$
$$\times\frac{(\beta)_r(\alpha_2+\beta-\alpha_1-\gamma)_r}{(2\beta)_r r!}\ F_1\left(\beta+r,\gamma,\gamma;2\beta+r;\frac{b-a}{b-p},\frac{b-a}{b-q}\right),\quad z<1/4,$$

where

$$a=\frac{1-\sqrt{1-4z}}{2},\qquad b=\frac{1+\sqrt{1-4z}}{2},$$
$$p=\frac{-1-\sqrt{1-4\xi_1\xi_2 z}}{2\xi_1},\qquad q=\frac{-1+\sqrt{1-4\xi_1\xi_2 z}}{2\xi_1}.$$

Proof. Consider the transformation $X = X$ and $Z = XY$. Given that $x+y<1$ and $y = z/x$, we have $x^2 - x + z < 0$, and by taking $a = (1-\sqrt{1-4z})/2$ and $b = (1+\sqrt{1-4z})/2$, these conditions can be expressed as $a < x < b$ and $z < 1/4$. Further, the Jacobian of this transformation is $J(x, y \to x, z) = 1/x$. Thus, substituting appropriately in (1), the joint p.d.f. of X and Z is given by

$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)\frac{x^{\alpha_1+\gamma-\alpha_2-\beta}z^{\alpha_2-1}(-x^2+x-z)^{\beta-1}}{(\xi_1x^2+x+\xi_2z)^\gamma},\quad a<x<b,\quad z<1/4,$$

Now, integrating with respect to x, we get the marginal density of Z as

$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)z^{\alpha_2-1}\int_a^b\frac{x^{\alpha_1+\gamma-\alpha_2-\beta}(-x^2+x-z)^{\beta-1}}{(\xi_1x^2+x+\xi_2z)^\gamma}\mathrm{d}x,\quad z<1/4$$

which can be rewritten as

$$C(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)z^{\alpha_2-1}\int_a^b\frac{x^{\alpha_1+\gamma-\alpha_2-\beta}[-(x-a)(x-b)]^{\beta-1}}{[(x-p)(x-q)]^\gamma}\mathrm{d}x,$$

where we have assumed that $\xi_1\xi_2 \leq 1$ and $\xi_1 \neq 0$, with

$$p=\frac{-1-\sqrt{1-4\xi_1\xi_2 z}}{2\xi_1},\quad q=\frac{-1+\sqrt{1-4\xi_1\xi_2 z}}{2\xi_1}.$$

Since $a < x < b$, we have $0 < (x-b)/(a-b) < 1$. Further, if $u = (x-b)/(a-b)$, then $x = (a-b)u + b$ and $\mathrm{d}x = (a-b)\mathrm{d}u$. Furthermore, $(x-b) = (a-b)u$ and $x-a = (b-a)(1-u)$, and the above integral is expressed as

$$\frac{b^{\alpha_1+\gamma-\alpha_2-\beta}(b-a)^{2\beta-1}}{(b-p)^\gamma(b-q)^\gamma}\int_0^1\frac{u^{\beta-1}(1-u)^{\beta-1}[1-(b-a)u/b]^{\alpha_1+\gamma-\alpha_2-\beta}}{\{[1-(b-a)u/(b-p)][1-(b-a)u/(b-q)]\}^\gamma}\mathrm{d}u$$
$$=\frac{B(\beta,\beta)b^{\alpha_1+\gamma-\alpha_2-\beta}(b-a)^{2\beta-1}}{(b-p)^\gamma(b-q)^\gamma}\sum_{r=0}^\infty\frac{(\beta)_r(\alpha_2+\beta-\alpha_1-\gamma)_r}{(2\beta)_r r!}\left(\frac{b-a}{b}\right)^r$$
$$\times\ F_1\left(\beta+r,\gamma,\gamma;2\beta+r;\frac{b-a}{b-p},\frac{b-a}{b-q}\right),$$

where the last line has been obtained by expanding $[1-(b-a)u/b]^{-(\alpha_2+\beta-\alpha_1-\gamma)}$ in power series and applying (A.5). Finally, substituting appropriately, we get the desired result. □

Corollary 4.2.1. *If* $(X,Y) \sim \mathrm{BGH}(\alpha_1,\alpha_2,\beta,\gamma,\xi_1,\xi_2)$ *with* $\xi_1 = \xi_2 = 1$, *then the p.d.f. of the random variable* $Z = XY$ *is given by*

$$\begin{aligned} &C(\alpha_1,\alpha_2,\beta,\gamma,1,1)\frac{B(\beta,\beta)(1-4z)^{\beta-1/2}z^{\alpha_2-1}}{2^{\alpha_1+\gamma-\alpha_2-\beta}(1+\sqrt{1-4z})^{\alpha_2+\beta-\alpha_1}} \\ &\times \sum_{r=0}^{\infty} \frac{2^r(\beta)_r(\alpha_2+\beta-\alpha_1-\gamma)_r}{(2\beta)_r r!}\left(\frac{\sqrt{1-4z}}{1+\sqrt{1-4z}}\right)^r \\ &\times F_1\left(\beta+r,\gamma,\gamma;2\beta+r;\frac{\sqrt{1-4z}}{1+\sqrt{1-4z}},\sqrt{1-4z}\right), \quad z<1/4. \end{aligned}$$

Appendix

The integral representation of the Gauss hypergeometric function is given as

$$\begin{aligned} {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,\mathrm{d}t,& \\ \mathrm{Re}(c) > \mathrm{Re}(a) > 0,\ |\arg(1-z)| < \pi.& \quad \text{(A.1)} \end{aligned}$$

Note that, by expanding $(1-zt)^{-b}$, $|zt| < 1$, in (A.1) and integrating t the series expansion for ${}_2F_1$ can be obtained as

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \quad |z|<1, \qquad \text{(A.2)}$$

where the Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1,2,\ldots,$ and $(a)_0 = 1$. From (A.2), it can easily be observed that

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z).$$

For properties and further results the reader is referred to Luke [9].

The first hypergeometric function of Appell is defined in a series form as

$$F_1(a,b_1,b_2;c;z_1,z_2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{z_1^m z_2^n}{m!n!}, \qquad \text{(A.3)}$$

where $|z_1| < 1$ and $|z_2| < 1$.

The first hypergeometric function of Appell can be expressed as a double integral,

$$\iint\limits_{\substack{u,v>0\\u+v<1}} \frac{u^{b_1-1}v^{b_2-1}(1-u-v)^{c-b_1-b_2-1}}{(1-uz_1-vz_2)^a}\,\mathrm{d}u\,\mathrm{d}v = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(c-b_1-b_2)}{\Gamma(c)}F_1(a,b_1,b_2;c;z_1,z_2) \tag{A.4}$$

for $\mathrm{Re}(b_1)>0$, $\mathrm{Re}(b_2)>0$, and $\mathrm{Re}(c-b_1-b_2)>0$.

It can also be shown that

$$\begin{aligned} F_1(a,b_1,b_2;c;z_1,z_2) &= \sum_{r_1=0}^{\infty}\frac{(a)_{r_1}(b_1)_{r_1}}{(c)_{r_1}}\frac{z_1^{r_1}}{r_1!}{}_2F_1(a+r_1,b_2;c+r_1;z_2)\\ &= \sum_{r_2=0}^{\infty}\frac{(a)_{r_2}(b_2)_{r_2}}{(c)_{r_2}}\frac{z_2^{r_2}}{r_2!}{}_2F_1(a+r_2,b_1;c+r_2;z_1), \end{aligned}$$

and further, using the results

$$\frac{(a)_{r_1+r_2}}{(c)_{r_1+r_2}} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 v^{a+r_1+r_2-1}(1-v)^{c-a-1}\,\mathrm{d}v,$$

where $\mathrm{Re}(c)>\mathrm{Re}(a)>0$, and

$$\sum_{r_i=0}^{\infty}\frac{(b_i)_{r_i}(vz_i)^{r_i}}{r_i!} = (1-vz_i)^{-b_i},\ |vz_i|<1, i=1,2,$$

in (A.3), it is possible to obtain

$$F_1(a,b_1,b_2;c;z_1,z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1\frac{v^{a-1}(1-v)^{c-a-1}\,\mathrm{d}v}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}},\quad |z_1|<1, |z_2|<1, \tag{A.5}$$

where $\mathrm{Re}(c)>\mathrm{Re}(a)>0$.

The function F_1 reduces to a ${}_2F_1$ in the following cases:

$$\begin{aligned} F_1(a,0,b_2;c;z_1,z_2) = F_1(a,b_1,b_2;c;0,z_2) &= {}_2F_1(a,b_2;c;z_2),\\ F_1(a,b_1,0;c;z_1,z_2) = F_1(a,b_1,b_2;c;z_1,0) &= {}_2F_1(a,b_1;c;z_1). \end{aligned}$$

Further,

$$F_1(a,b_1,b_2;c;z,z) = {}_2F_1(a,b_1+b_2;c;z).$$

Some transformations of the function F_1 that eventually facilitate calculations for different values of the parameters and variables are

$$\begin{aligned}
&F_1(a, b_1, b_2; c; z_1, z_2)\\
&= (1-z_1)^{-b_1}(1-z_2)^{-b_2} F_1\left(c-a, b_1, b_2; c; -\frac{z_1}{1-z_1}, -\frac{z_2}{1-z_2}\right)\\
&= (1-z_1)^{-a} F_1\left(a, c-b_1-b_2, b_2; c; -\frac{z_1}{1-z_1}, -\frac{z_1-z_2}{1-z_1}\right)\\
&= (1-z_2)^{-a} F_1\left(a, b_1, c-b_1-b_2; c; -\frac{z_2-z_1}{1-z_2}, -\frac{z_2}{1-z_2}\right)\\
&= (1-z_1)^{c-a-b_1}(1-z_2)^{-b_2} F_1\left(c-a, c-b_1-b_2, b_2; c; z_1, \frac{z_1-z_2}{1-z_2}\right)\\
&= (1-z_1)^{-b_1}(1-z_2)^{c-a-b_2} F_1\left(c-a, b_1, c-b_1-b_2; c; \frac{z_2-z_1}{1-z_1}, z_2\right). \qquad \text{(A.6)}
\end{aligned}$$

For several other properties and results on F_1 the reader is referred to Srivastava and Karlsson [17] and Bailey [2].

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References

[1] C. Armero and M. Bayarri, Prior assessments for predictions in queues, *The Statistician*, **43** (1994), no. 1, 139–153. http://dx.doi.org/10.2307/2348939

[2] W. N. Bailey, *Generalized Hypergeometric Series*, Stechert-Hafner Service Agency, New York, 1964.

[3] Liliam Cardeño, Daya K. Nagar and Luz Estela Sánchez, Beta type 3 distribution and its multivariate generalization, *Tamsui Oxford Journal of Mathematical Sciences*, **21**(2005), no. 2, 225–241.

[4] J.-Y. Dauxois, Bayesian inference for linear growth birth and death processes, *Journal of Statistical Planning and Inference*, **121**(2004), no. 1, 1–19. http://dx.doi.org/10.1016/s0378-3758(02)00520-7

[5] Peter S. Fader and Bruce G. S. Hardie, A note on modelling underreported Poisson counts, *Journal of Applied Statistics*, **27**(2000), no. 8, 953–964. http://dx.doi.org/10.1080/02664760050173283

[6] A. K. Gupta and D. K. Nagar, *Matrix Variate Distributions*, Chapman & Hall/CRC, Boca Raton, 2000.

[7] A. K. Gupta and D. K. Nagar, Properties of matrix variate beta type 3 distribution, *International Journal of Mathematics and Mathematical Sciences*, **2009**(2009), Art. ID 308518, 18 pp. http://dx.doi.org/10.1155/2009/308518

[8] N. L. Johnson, S. Kotz and N. Balakrishnan, *Continuous Univariate Distributions-2*, Second Edition, John Wiley & Sons, New York, 1994.

[9] Y. L. Luke, *The Special Functions and their Approximations, Vol. 1*, Academic Press, New York, 1969.

[10] D. L. Libby and M. R. Novic, Multivariate generalized beta distributions with applications to utility assessment, *Journal of Educational Statistics*, **7** (1982), no. 4, 271–294. http://dx.doi.org/10.2307/1164635

[11] S. Nadarajah, Sums, products and ratios of generalized beta variables, *Statistical Papers*, **47**(2006), no. 1, 69–90. http://dx.doi.org/10.1007/s00362-005-0273-8

[12] S. Nadarajah and K. Zografos, Expressions for Rényi and Shannon entropies for bivariate distributions, *Information Sciences*, **170**(2005), no. 2-4, 173–189. http://dx.doi.org/10.1016/j.ins.2004.02.020

[13] Daya K. Nagar and Erika Alejandra Rada-Mora, Properties of multivariate beta distributions, *Far East Journal of Theoretical Statistics*, **24**(2008), no. 1, 73–94.

[14] A. Rényi, On measures of entropy and information, *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, University of California Press, Berkeley, California, pp. 547–561 (1961).

[15] Luz Estela Sánchez and Daya K. Nagar, Distribution of the product and the quotient of independent beta type 3 variables, *Far East Journal of Theoretical Statistics*, **17**(2005), no. 2, 239–251.

[16] C. E. Shannon, A mathematical theory of communication, *Bell System Technical Journal*, **27**(1948), 379–423, 623–656. http://dx.doi.org/10.1002/j.1538-7305.1948.tb01338.x, http://dx.doi.org/10.1002/j.1538-7305.1948.tb00917.x

[17] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, John Wiley & Sons, New York, 1985.

[18] K. Zografos, On maximum entropy characterization of Pearson's type II and VII multivariate distributions, *Journal of Multivariate Analysis*, **71**(1999), no. 1, 67–75. http://dx.doi.org/10.1006/jmva.1999.1824

[19] K. Zografos and S. Nadarajah, Expressions for Rényi and Shannon entropies for multivariate distributions, *Statistics & Probability Letters*, **71**(2005), no. 1, 71–84. http://dx.doi.org/10.1016/j.spl.2004.10.023

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