

Distribution of the Product of Independent Extended Beta Variables

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Abstract

The extended beta type 1 distribution has the probability density function proportional to $x^{\alpha-1}(1-x)^{\beta-1} \exp[-\sigma/x(1-x)]$, $0 < x < 1$. In this article, we derive the probability density function of the product of two independent random variables each having an extended beta type 1 distribution. We also consider several other products involving extended beta type 1, beta type 1, beta type 2, beta type 3, Kummer-beta and inverted gamma variables.

Mathematics Subject Classification: 33E99, 60E05

Keywords: Beta distribution, extended beta function, gamma distribution, Gauss hypergeometric function, inverted gamma distribution, product

1 Introduction

The random variable X is said to have an extended beta type 1 distribution, denoted by $X \sim \text{EB1}(\alpha, \beta; \sigma)$, if its probability density function (p.d.f.) is given by (Chaudhry et al. [1]),

$$\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta; \sigma)} \exp\left[-\frac{\sigma}{x(1-x)}\right], \quad 0 < x < 1, \quad (1)$$

where $\sigma > 0$ and $B(p, q; \sigma)$ is the extended beta function defined by (Chaudhry et al. [1], Miller [7])

$$B(p, q; \sigma) = \int_0^1 t^{p-1}(1-t)^{q-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] dt, \quad (2)$$

where $-\infty < p, q < \infty$ and $\operatorname{Re}(\sigma) > 0$. For $\sigma = 0$ we must have $p > 0$, $q > 0$ and in this case the extended beta function reduces to the Euler's beta function. Further, replacing t by $1-t$ in (2), one can see that $B(a, b; \sigma) = B(b, a; \sigma)$. The rationale and justification for introducing this function are given in Chaudhry et al. [1] where several properties and a statistical application have also been studied. Miller [7] further studied this function and has given several additional results. The extended beta function has been used by Morán-Vásquez and Nagar [8] to express the density function of the product of two independent Kummer-gamma variables. Recently, Nagar, Morán-Vásquez and Gupta [15] have studied several properties of the extended beta distribution. A matrix variate generalization of the extended beta function is available in Nagar, Roldán-Correa and Gupta [14]. The extended matrix variate beta distribution has been studied by Nagar and Roldán-Correa [16].

In this article, we derive the density function of the product of two independent random variables each having an extended beta type 1 distribution. We also derive densities of several other products involving extended beta type 1, beta type 1, beta type 2, beta type 3, Kummer-beta and inverted gamma variables.

2 Some Definitions and Preliminary Results

In this section, we give some definitions and preliminary results which are used in the subsequent section. The integral representations of the confluent hypergeometric function Φ and the Gauss hypergeometric function F are given as

$$\Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \exp(zt) dt \quad (3)$$

and

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt, \quad (4)$$

respectively, where $\operatorname{Re}(a) > 0$ and $\operatorname{Re}(c-a) > 0$. Expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt| < 1$, in (3) and (4) and integrating t , series expansions for Φ and F can be obtained as

$$\Phi(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}$$

and

$$F(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1,$$

respectively, where a, b and c are complex numbers with suitable restrictions and the pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1) \cdots (a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \dots$, and $(a)_0 = 1$.

The integral representations of the Appell's first hypergeometric function F_1 and the Humbert's confluent hypergeometric function Φ_1 are given by

$$F_1(a, b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1} dv}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}},$$

$$|z_1| < 1, \quad |z_2| < 1, \tag{5}$$

and

$$\Phi_1[a, b_1; c; z_1, z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1} \exp(vz_2) dv}{(1-vz_1)^{b_1}},$$

$$|z_1| < 1, \quad |z_2| < \infty, \tag{6}$$

where $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$. Note that for $b_1 = 0$, F_1 and Φ_1 reduce to F and Φ functions, respectively. For properties and further results on these functions the reader is referred to Luke [6] and Srivastava and Karlsson [20].

The Laguerre polynomials (Gradshteyn and Ryzhik [2, Sec. 8.97]) are given by the sum

$$L_n(x) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n}{k} x^k,$$

where $\binom{n}{k}$ is the binomial coefficient. The first few Laguerre polynomials are $L_0(x) = 1$, $L_1(x) = -x + 1$, $L_2(x) = (x^2 - 4x + 2)/2$ and $L_3(x) = (-x^3 + 9x^2 - 18x + 6)/6$. The generating function for Laguerre polynomials is given by

$$\frac{\exp[-xt/(1-t)]}{1-t} = \sum_{n=0}^{\infty} t^n L_n(x), \quad |t| < 1.$$

Replacing $\exp(-\sigma/t)$ and $\exp[-\sigma/(1-t)]$ by their respective series expansions involving Laguerre polynomials (Miller [7, Eq. 3.4a, 3.4b]), namely,

$$\exp\left(-\frac{\sigma}{t}\right) = \exp(-\sigma)t \sum_{n=0}^{\infty} L_n(\sigma)(1-t)^n, \quad |t| < 1,$$

and

$$\exp\left(-\frac{\sigma}{1-t}\right) = \exp(-\sigma)(1-t) \sum_{m=0}^{\infty} L_m(\sigma)t^m, \quad |t| < 1,$$

respectively, in (2) and integrating t by using beta integral, Miller [7, Eq. 2.3] has given an alternative representation for $B(p, q; \sigma)$ as

$$B(p, q; \sigma) = \exp(-2\sigma) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma)B(p+m+1, q+n+1),$$

where $\operatorname{Re}(p) > -1$ and $\operatorname{Re}(q) > -1$.

Finally, we define the inverted gamma, beta type 1, beta type 2, beta type 3 and Kummer-beta distributions. These definitions can be found in Johnson, Kotz and Balakrishnana [5], Nagar and Joshi [10], Nagar and Ramirez-Vanegas [11, 12], Nagar and Tabares-Herrera [13], Nagar and Zarrazola [17], Ng and Kotz [18] and Sánchez and Nagar [19].

Definition 2.1. *The random variable X is said to have an inverted gamma distribution with parameters $\theta (> 0), \kappa (> 0)$, denoted by $X \sim \operatorname{IG}(\theta, \kappa)$, if its p.d.f. is given by*

$$\{\theta^\kappa \Gamma(\kappa)\}^{-1} x^{-(\kappa+1)} \exp\left(-\frac{1}{\theta x}\right), \quad x > 0.$$

Definition 2.2. *The random variable X is said to have a beta type 1 distribution with parameters (a, b) , $a > 0, b > 0$, denoted as $X \sim \operatorname{B1}(a, b)$, if its p.d.f. is given by*

$$\{B(a, b)\}^{-1} x^{a-1} (1-x)^{b-1}, \quad 0 < x < 1,$$

where $B(a, b)$ is the beta function.

Definition 2.3. *The random variable X is said to have a beta type 2 distribution with parameters (a, b) , $a > 0, b > 0$, denoted as $X \sim \operatorname{B2}(a, b)$, if its p.d.f. is given by*

$$\{B(a, b)\}^{-1} x^{a-1} (1+x)^{-(a+b)}, \quad x > 0.$$

Definition 2.4. *The random variable X is said to have a beta type 3 distribution with parameters (a, b) , $a > 0, b > 0$, denoted as $X \sim \operatorname{B3}(a, b)$, if its p.d.f. is given by*

$$2^a \{B(a, b)\}^{-1} x^{a-1} (1-x)^{b-1} (1+x)^{-(a+b)}, \quad 0 < x < 1.$$

Definition 2.5. *The random variable X is said to have a Kummer-beta distribution, denoted by $X \sim \text{KB}(\alpha, \beta, \lambda)$, if its p.d.f. is given by*

$$\frac{x^{\alpha-1}(1-x)^{\beta-1} \exp(-\lambda x)}{B(\alpha, \beta)\Phi(\alpha, \alpha + \beta; -\lambda)}, \quad 0 < x < 1, \tag{7}$$

where $\alpha > 0, \beta > 0$ and $-\infty < \lambda < \infty$.

Note that for $\lambda = 0$ the above density simplifies to a beta type 1 density with parameters α and β . Further, using the Kummer’s relation, the Kummer-beta density (7) can also be written as

$$\frac{x^{\alpha-1}(1-x)^{\beta-1} \exp[\lambda(1-x)]}{B(\alpha, \beta)\Phi(\beta, \alpha + \beta; \lambda)}, \quad 0 < x < 1. \tag{8}$$

The matrix variate generalizations of the inverted gamma, beta type 1, beta type 2, beta type 3 and Kummer-beta distributions have been defined and studied extensively. For example, see Gupta and Nagar [3, 4], and Nagar and Gupta [9].

3 Products of Two Independent Random Variables

In this section, we derive distributions of products of two independent random variables when at least one of them has extended beta type 1 distribution. First, we re-write the extended beta type 1 density in series involving Laguerre polynomials.

Replacing $\exp(-\sigma/x)$ and $\exp[-\sigma/(1-x)]$ by their respective series expansions involving Laguerre polynomials (Miller [7, Eq. 3.4a, 3.4b]), namely,

$$\exp\left(-\frac{\sigma}{x}\right) = \exp(-\sigma)x \sum_{n=0}^{\infty} L_n(\sigma)(1-x)^n, \quad |x| < 1,$$

and

$$\exp\left(-\frac{\sigma}{1-x}\right) = \exp(-\sigma)(1-x) \sum_{m=0}^{\infty} L_m(\sigma)x^m, \quad |x| < 1,$$

respectively, in (1), the extended beta type 1 density can also be written as

$$\begin{aligned} & \{B(\alpha, \beta; \sigma) \exp(2\sigma)\}^{-1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma) \\ & \times x^{\alpha+m+1-1}(1-x)^{\beta+n+1-1}, \quad 0 < x < 1, \end{aligned} \tag{9}$$

where $\alpha > -1$ and $\beta > -1$.

We will use the above representation of the extended beta type 1 density in deriving a number of results.

Theorem 3.1. *Let X_1 and X_2 be independent, $X_1 \sim \text{EB1}(\alpha_1, \beta_1; \sigma_1)$, $\alpha_1 > -1$, $\beta_1 > -1$ and $X_2 \sim \text{EB1}(\alpha_2, \beta_2; \sigma_2)$, $\alpha_2 > -1$, $\beta_2 > -1$. Then, the p.d.f. of $Z = X_1 X_2$ is*

$$\begin{aligned} & K_1 z^{\alpha_1} (1-z)^{\beta_1 + \beta_2 + 1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_m(\sigma_1) L_n(\sigma_1) L_r(\sigma_2) L_s(\sigma_2) \\ & \times z^m (1-z)^{n+s} B(\beta_1 + n + 1, \beta_2 + s + 1) \\ & \times F(\beta_2 + s + 1, \alpha_1 + \beta_1 + m + n + 1 - \alpha_2 - r; \beta_1 + \beta_2 + n + s + 2; 1-z), \end{aligned}$$

where $0 < z < 1$ and

$$K_1 = \{B(\alpha_1, \beta_1; \sigma_1) B(\alpha_2, \beta_2; \sigma_2) \exp[2(\sigma_1 + \sigma_2)]\}^{-1}.$$

Proof. Using (9), the joint p.d.f. of X_1 and X_2 is given by

$$\begin{aligned} & K_1 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_m(\sigma_1) L_n(\sigma_1) L_r(\sigma_2) L_s(\sigma_2) \\ & \times x_1^{\alpha_1 + m + 1 - 1} x_2^{\alpha_2 + r + 1 - 1} (1-x_1)^{\beta_1 + n + 1 - 1} (1-x_2)^{\beta_2 + s + 1 - 1}, \end{aligned} \quad (10)$$

where $0 < x_1 < 1$ and $0 < x_2 < 1$. Making the transformation $Z = X_1 X_2$, $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \rightarrow z, x_2) = 1/x_2$ in (10), the joint p.d.f. of Z and X_2 is obtained as

$$\begin{aligned} & K_1 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_m(\sigma_1) L_n(\sigma_1) L_r(\sigma_2) L_s(\sigma_2) \\ & \times \frac{z^{\alpha_1 + m + 1 - 1} (1-x_2)^{\beta_2 + s + 1 - 1} (x_2 - z)^{\beta_1 + n + 1 - 1}}{x_2^{\alpha_1 + \beta_1 + m + n + 1 - \alpha_2 - r}}, \quad 0 < z < x_2 < 1. \end{aligned} \quad (11)$$

To find the marginal p.d.f. of Z , we integrate (11) with respect to x_2 to get

$$\begin{aligned} & K_1 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_m(\sigma_1) L_n(\sigma_1) L_r(\sigma_2) L_s(\sigma_2) \\ & \times z^{\alpha_1 + m + 1 - 1} \int_z^1 \frac{(1-x_2)^{\beta_2 + s + 1 - 1} (x_2 - z)^{\beta_1 + n + 1 - 1}}{x_2^{\alpha_1 + \beta_1 + m + n + 1 - \alpha_2 - r}} dx_2. \end{aligned} \quad (12)$$

In (12) change of variable $u = (1-x_2)/(1-z)$ yields

$$\begin{aligned} & K_1 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_m(\sigma_1) L_n(\sigma_1) L_r(\sigma_2) L_s(\sigma_2) \\ & \times z^{\alpha_1 + m + 1 - 1} (1-z)^{\beta_1 + \beta_2 + n + s + 2 - 1} \int_0^1 \frac{u^{\beta_2 + s + 1 - 1} (1-u)^{\beta_1 + n + 1 - 1}}{[1 - (1-z)u]^{\alpha_1 + \beta_1 + m + n + 1 - \alpha_2 - r}} du. \end{aligned}$$

Finally, applying (4), we obtain the desired result. \square

Theorem 3.2. Let X_1 and X_2 be independent, $X_1 \sim \text{EB1}(\alpha_1, \beta_1; \sigma)$, $\alpha_1 > -1$, $\beta_1 > -1$ and $X_2 \sim \text{KB}(\alpha_2, \beta_2, \lambda)$. Then, the p.d.f. of $Z = X_1X_2$ is

$$K_2 z^{\alpha_1} (1-z)^{\beta_1+\beta_2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma) L_n(\sigma) z^m (1-z)^n B(\beta_1+n+1, \beta_2) \\ \times \Phi_1[\beta_2, \alpha_1 + \beta_1 - \alpha_2 + m + n + 2; \beta_1 + \beta_2 + n + 1; 1-z, \lambda(1-z)],$$

where $0 < z < 1$ and

$$K_2 = \{\exp(2\sigma) B(\alpha_1, \beta_1; \sigma) B(\alpha_2, \beta_2) \Phi(\beta_2, \alpha_2 + \beta_2; \lambda)\}^{-1}.$$

Proof. Using (9) and (8), the joint p.d.f. of X_1 and X_2 is given by

$$K_2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma) L_n(\sigma) x_1^{\alpha_1+m+1-1} x_2^{\alpha_2-1} \\ \times (1-x_1)^{\beta_1+n+1-1} (1-x_2)^{\beta_2-1} \exp[\lambda(1-x_2)], \tag{13}$$

where $0 < x_1 < 1$ and $0 < x_2 < 1$. By transforming $Z = X_1X_2$ and $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \rightarrow z, x_2) = 1/x_2$ in (13), the joint p.d.f. of Z and X_2 is obtained as

$$K_2 z^{\alpha_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma) L_n(\sigma) z^m \frac{(1-x_2)^{\beta_2-1} (x_2-z)^{\beta_1+n+1-1} \exp[\lambda(1-x_2)]}{x_2^{\alpha_1+\beta_1-\alpha_2+m+n+2}}, \tag{14}$$

where $0 < z < x_2 < 1$. Now, integrating x_2 in (14), the marginal p.d.f. of Z is derived as

$$K_2 z^{\alpha_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma) L_n(\sigma) z^m \\ \times \int_z^1 \frac{(1-x_2)^{\beta_2-1} (x_2-z)^{\beta_1+n+1-1} \exp[\lambda(1-x_2)]}{x_2^{\alpha_1+\beta_1-\alpha_2+m+n+2}} dx_2. \tag{15}$$

Now, substituting $u = (1-x_2)/(1-z)$ in (15), the marginal p.d.f. of Z is re-written as

$$K_2 z^{\alpha_1} (1-z)^{\beta_1+\beta_2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma) L_n(\sigma) z^m (1-z)^n \\ \times \int_0^1 \frac{u^{\beta_2-1} (1-u)^{\beta_1+n+1-1} \exp[\lambda(1-z)u]}{[1-(1-z)u]^{\alpha_1+\beta_1+m+n+2-\alpha_2}} du. \tag{16}$$

Finally, application of (6) yields the desired result. □

Corollary 3.2.1. Let X_1 and X_2 be independent, $X_1 \sim \text{EB1}(\alpha_1, \beta_1; \sigma)$, $\alpha_1 > -1$, $\beta_1 > -1$ and $X_2 \sim \text{B1}(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1 X_2$ is

$$\frac{z^{\alpha_1}(1-z)^{\beta_1+\beta_2}}{\exp(2\sigma)B(\alpha_1, \beta_1; \sigma)B(\alpha_2, \beta_2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma)z^m(1-z)^n B(\beta_1+n+1, \beta_2) \\ \times F(\beta_2, \alpha_1 + \beta_1 - \alpha_2 + m + n + 2; \beta_1 + \beta_2 + n + 1; 1-z), \quad 0 < z < 1.$$

Corollary 3.2.2. Let the random variables X_1 and X_2 be independent, $X_1 \sim \text{B1}(\alpha_1, \beta_1)$ and $X_2 \sim \text{KB}(\alpha_2, \beta_2, \lambda)$. Then, the p.d.f. of $Z = X_1 X_2$ is

$$\frac{z^{\alpha_1}(1-z)^{\beta_1+\beta_2}}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)\Phi(\beta_2, \alpha_2 + \beta_2; \lambda)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z^m(1-z)^n B(\beta_1+n+1, \beta_2) \\ \times \Phi_1[\beta_2, \alpha_1 + \beta_1 - \alpha_2 + m + n + 2; \beta_1 + \beta_2 + n + 1; 1-z, \lambda(1-z)],$$

where $0 < z < 1$.

Nagar and Zarrazola [17] have also derived the density of $Z = X_1 X_2$, where X_1 and X_2 are independent, $X_1 \sim \text{B1}(\alpha_1, \beta_1)$ and $X_2 \sim \text{KB}(\alpha_2, \beta_2, \lambda)$. The form of the density derived by them is given by

$$\frac{\Gamma(\beta_1 + \alpha_1)\Gamma(\beta_2 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1 + \beta_2)} \{\Phi(\beta_2, \alpha_2 + \beta_2; \lambda)\}^{-1} z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1} \\ \times \Phi_1[\beta_2, \alpha_1 + \beta_1 - \alpha_2; \beta_1 + \beta_2; 1-z, \lambda(1-z)], \quad 0 < z < 1.$$

This expression can be obtained by substituting $\sigma = 0$ in (16), summing infinite series as

$$\sum_{m=0}^{\infty} \left[\frac{z}{1-(1-z)u} \right]^m = \frac{1-(1-z)u}{(1-z)(1-u)} \quad (17)$$

and

$$\sum_{n=0}^{\infty} \left[\frac{(1-z)(1-u)}{1-(1-z)u} \right]^m = \frac{1-(1-z)u}{z}, \quad (18)$$

and integrating the resulting expression by using (6),

Theorem 3.3. Let X_1 and X_2 be independent, $X_1 \sim \text{EB1}(\alpha_1, \beta_1; \sigma)$, $\alpha_1 > -1$, $\beta_1 > -1$ and $X_2 \sim \text{B2}(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1 X_2$ is given by

$$\frac{z^{\alpha_2-1}(1+z)^{-(\alpha_2+\beta_2)}}{B(\alpha_1, \beta_1; \sigma)B(\alpha_2, \beta_2)} \sum_{r=0}^{\infty} \frac{(\alpha_2 + \beta_2)_r}{(1+z)^r r!} B(\beta_1 + r, \alpha_1 + \beta_2; \sigma), \quad z > 0.$$

Proof. Since X_1 and X_2 are independent, their joint p.d.f. is given by

$$K_3 \frac{x_1^{\alpha_1-1}(1-x_1)^{\beta_1-1}x_2^{\alpha_2-1}}{(1+x_2)^{\alpha_2+\beta_2}} \exp\left[-\frac{\sigma}{x_1(1-x_1)}\right], \quad 0 < x_1 < 1, \quad x_2 > 0,$$

where

$$K_3 = \{B(\alpha_1, \beta_1; \sigma)B(\alpha_2, \beta_2)\}^{-1}.$$

Now, transforming $Z = X_1X_2$ and $W = 1 - X_1$ with the Jacobian $J(x_1, x_2 \rightarrow w, z) = 1/(1-w)$, we obtain the joint p.d.f. of W and Z as

$$K_3 \frac{z^{\alpha_2-1}}{(1+z)^{\alpha_2+\beta_2}} \frac{w^{\beta_1-1}(1-w)^{\alpha_1+\beta_2-1}}{[1-w/(1+z)]^{\alpha_2+\beta_2}} \exp\left[-\frac{\sigma}{w(1-w)}\right], \quad 0 < w < 1. \quad (19)$$

Now, expanding $[1-w/(1+z)]^{-(\alpha_2+\beta_2)}$ in series form and integrating w using (1) and substituting for K_3 in (19), we obtain the desired result. \square

Corollary 3.3.1. *Let X_1 and X_2 be independent random variables, $X_1 \sim B1(\alpha_1, \beta_1)$ and $X_2 \sim B2(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1X_2$ is given by*

$$\frac{B(\beta_1, \alpha_1 + \beta_2)}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \frac{z^{\alpha_2-1}}{(1+z)^{\alpha_2+\beta_2}} F\left(\beta_1, \alpha_2 + \beta_2; \alpha_1 + \beta_1 + \beta_2; \frac{1}{1+z}\right),$$

where $z > 0$.

The above corollary is also available in Nagar and Zarrazola [17].

Theorem 3.4. *Let X_1 and X_2 be independent, $X_1 \sim EB1(\alpha_1, \beta_1; \sigma)$, $\alpha_1 > -1$, $\beta_1 > -1$ and $X_2 \sim B3(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1X_2$ is*

$$K_4 \frac{z^{\alpha_1}(1-z)^{\beta_1+\beta_2}}{2^{\alpha_2+\beta_2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma)z^m(1-z)^n B(\beta_1 + n + 1, \beta_2) \\ \times F_1\left(\beta_2, \alpha_1 + \beta_1 - \alpha_2 + m + n + 2, \alpha_2 + \beta_2; \beta_1 + \beta_2 + n + 1; 1-z, \frac{1-z}{2}\right),$$

where $0 < z < 1$ and

$$K_4 = 2^{\alpha_2} \{\exp(2\sigma)B(\alpha_1, \beta_1; \sigma)B(\alpha_2, \beta_2)\}^{-1}.$$

Proof. Using the independence, the joint p.d.f. of X_1 and X_2 is given by

$$K_4 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma) \frac{x_1^{\alpha_1+m+1-1}x_2^{\alpha_2-1}(1-x_1)^{\beta_1+n+1-1}(1-x_2)^{\beta_2-1}}{(1+x_2)^{\alpha_2+\beta_2}}, \quad (20)$$

where $0 < x_1 < 1$ and $0 < x_2 < 1$. Now, transforming $Z = X_1X_2$, $X_2 = X_2$ with the Jacobian $J(x_1, x_2 \rightarrow z, x_2) = 1/x_2$ in (20) and integrating the resulting expression with respect to x_2 , the density of Z is obtained as

$$K_4 z^{\alpha_1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma)z^m \int_z^1 \frac{(1-x_2)^{\beta_2-1}(x_2-z)^{\beta_1+n+1-1}}{x_2^{\alpha_1+\beta_1+m+n+2-\alpha_2}(1+x_2)^{\alpha_2+\beta_2}} dx_2,$$

where $0 < z < x_2 < 1$. Now, substituting $u = (1-x_2)/(1-z)$ in the above expression, we obtain

$$K_4 \frac{z^{\alpha_1}(1-z)^{\beta_1+\beta_2}}{2^{\alpha_2+\beta_2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} L_m(\sigma)L_n(\sigma)z^m(1-z)^n \times \int_0^1 \frac{u^{\beta_2-1}(1-u)^{\beta_1+n+1-1}}{[1-(1-z)u]^{\alpha_1+\beta_1+m+n+2-\alpha_2} [1-u(1-z)/2]^{\alpha_2+\beta_2}} du. \tag{21}$$

Finally, applying (5), we get the desired result. □

Corollary 3.4.1. *Let the random variables X_1 and X_2 be independent, $X_1 \sim B1(\alpha_1, \beta_1)$ and $X_2 \sim B3(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1X_2$ is*

$$\frac{z^{\alpha_1}(1-z)^{\beta_1+\beta_2}}{2^{\beta_2}B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z^m(1-z)^n B(\beta_1+n+1, \beta_2) \times F_1 \left(\beta_2, \alpha_1 + \beta_1 - \alpha_2 + m + n + 2, \alpha_2 + \beta_2; \beta_1 + \beta_2 + n + 1; 1 - z, \frac{1-z}{2} \right),$$

where $0 < z < 1$.

Substituting $\sigma = 0$ in (21), summing infinite series by using (17) and (18), and integrating the resulting expression by applying (5), the density of $Z = X_1X_2$, where X_1 and X_2 are independent, $X_1 \sim B1(\alpha_1, \beta_1)$ and $X_2 \sim B3(\alpha_2, \beta_2)$, can also be derived as

$$\frac{\Gamma(\alpha_1 + \beta_1)\Gamma(\alpha_2 + \beta_2)}{2^{\beta_2}\Gamma(\alpha_1)\Gamma(\beta_2)\Gamma(\beta_1 + \beta_2)} z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1} \times F_1 \left(\beta_2, \alpha_1 + \beta_1 - \alpha_2, \alpha_2 + \beta_2; \beta_1 + \beta_2; 1 - z, \frac{1-z}{2} \right), \quad 0 < z < 1.$$

The above result has also been obtained by Sánchez and Nagar [19].

Theorem 3.5. *Let the random variables X_1 and X_2 be independent. Further, $X_1 \sim EB(\alpha, \beta; \sigma)$ and $X_2 \sim IG(\theta, \kappa)$. Then, the p.d.f. of $Z = X_1X_2$ is*

$$\frac{\exp(-1/\theta z) z^{-\kappa-1}}{\theta^\kappa \Gamma(\kappa) B(\alpha, \beta; \sigma)} \sum_{r=0}^{\infty} \frac{1}{(\theta z)^r r!} B(\beta + r, \alpha + \kappa; \sigma), \quad z > 0.$$

Proof. The joint p.d.f. of X_1 and X_2 is given by

$$K_5 x_1^{\alpha-1} (1-x_1)^{\beta-1} x_2^{-(\kappa+1)} \exp \left[-\frac{\sigma}{x_1(1-x_1)} - \frac{1}{\theta x_2} \right], \tag{22}$$

where $0 < x_1 < 1$, $x_2 > 0$ and

$$K_5 = \{ \theta^\kappa \Gamma(\kappa) B(\alpha, \beta; \sigma) \}^{-1}.$$

Now, transforming $Z = X_1 X_2$, $X_1 = X_1$ with the Jacobian $J(x_1, x_2 \rightarrow x_1, z) = 1/x_1$ in (22), we obtain the joint p.d.f. of Z and X_1 as

$$K_5 z^{-(\kappa+1)} x_1^{\alpha+\kappa-1} (1-x_1)^{\beta-1} \exp \left[-\frac{\sigma}{x_1(1-x_1)} - \frac{x_1}{\theta z} \right],$$

where $z > 0$ and $0 < x_1 < 1$. Now, integrating x_1 , we get the marginal density of Z as

$$\begin{aligned} & K_5 z^{-(\kappa+1)} \int_0^1 v^{\alpha+\kappa-1} (1-v)^{\beta-1} \exp \left[-\frac{\sigma}{v(1-v)} - \frac{v}{\theta z} \right] dv \\ &= K_5 z^{-\kappa-1} \exp \left(-\frac{1}{\theta z} \right) \int_0^1 w^{\beta-1} (1-w)^{\alpha+\kappa-1} \exp \left[-\frac{\sigma}{w(1-w)} + \frac{w}{\theta z} \right] dw \end{aligned}$$

where the last line has been obtained by substituting $w = 1 - v$. Finally, expanding $\exp(w/\theta z)$ in series form, integrating the resulting expression using (2), substituting for K_5 and simplifying, we obtain the desired result. \square

Corollary 3.5.1. *Let the random variables X_1 and X_2 be independent. Further, $X_1 \sim B1(\alpha, \beta)$ and $X_2 \sim IG(\theta, \kappa)$. Then, the p.d.f. of $Z = X_1 X_2$ is given by*

$$\frac{\Gamma(\alpha + \beta)\Gamma(\alpha + \kappa)}{\theta^\kappa \Gamma(\kappa)\Gamma(\alpha)\Gamma(\alpha + \beta + \kappa)} z^{-\kappa-1} \exp \left(-\frac{1}{\theta z} \right) \Phi \left(\beta, \alpha + \kappa + \beta; \frac{1}{\theta z} \right),$$

where $z > 0$.

4 Conclusion

In this article, we have derived the density function of the product of two independent random variables each having extended beta type 1 distribution. We have also derived densities of several other products involving extended beta type 1, beta type 1, beta type 2, beta type 3, Kummer-beta and inverted gamma variables. We applied the traditional method of transformation of random variables to derive these results and used several special functions to express density functions.

Acknowledgements. The research work of DKN and LES was supported by the Sistema Universitario de Investigación, Universidad de Antioquia under the project no. IN10231CE.

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Received: October 15, 2014; Published: November 18, 2014