An Algebraic Characterization of Affine Manifolds with G-Structure Satisfying a Homogeneity Condition

Una caracterización algebraica de las variedades afines con G-estructura que satisfacen una condición de homogeneidad

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Abstract. We give an algebraic characterization of the possible characteristic tensors of an infinitesimally homogeneous affine manifold with G-structure. Such concepts were introduced in [5].

Key words and phrases. Infinitesimally homogeneous manifold, Inner torsion, G-structures.

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Resumen. Presentamos una caracterización algebraica de los posibles tensores característicos de una variedad infinitesimalmente homogénea con G-estructura. Tales conceptos son introducidos en [5].

Palabras y frases clave. Variedad infinitesimalmente homogénea, torsión interna, G-estructuras.

1. Introduction

The concept of *infinitesimally homogeneous* affine manifold with G-structure was introduced in the recent article [5] with the aim to find a unifying language for several isometric immersion (Bonnet type) theorems that appear in the classical literature [1] (immersions into Riemannian manifolds with constant sectional curvature, immersions into Kähler manifolds of constant holomorphic curvature), and also some more recent results (see for instance [3, 2]) concerning the existence of isometric immersions in more general Riemannian manifolds. By an affine manifold with G-structure we mean a triple (M, ∇, P),

with M an n-dimensional differentiable manifold, ∇ a connection on M and P a G-structure on M, i.e., G is a Lie subgroup of $\mathrm{GL}(n)$ and P is a G-principal subbundle of the frame bundle of M. We denote by R and T, respectively, the curvature and torsion tensors of ∇. In order to handle the case in which P is not compatible with ∇, the concept of *inner torsion* was introduced in [5]: it is a tensor $\mathfrak{I}^P$ that plays the role of a covariant derivative of the G-structure P and it vanishes if and only if ∇ is compatible with P. The concept of infinitesimal homogeneity plays the same role in the theory of affine manifolds with G-structure as the concept of constant sectional curvature plays in Riemannian geometry; in fact, Riemannian manifolds with constant sectional curvature are precisely the infinitesimally homogeneous triples (M, ∇, P) in which P is the $\mathrm{O}(n)$-principal bundle of orthonormal frames and both the torsion and the inner torsion vanish. Notice that Riemannian manifolds with constant sectional curvature are those in which the (four indexed) matrix representing the curvature tensor with respect to orthonormal frames is independent of the orthonormal frame and of the point on the manifold. While it does not make sense to require that a tensor field on a manifold be constant, we can define, for manifolds endowed with a G-structure, the notion of *G-constant* tensor field: that is a tensor field whose matrix with respect to frames that belong to the G-structure is independent of the frame and of the point of the manifold. An affine manifold with G-structure (M, ∇, P) is said to be *infinitesimally homogeneous* if the tensor fields R, T and $\mathfrak{I}^P$ are all G-constant. When M is simply connected and ∇ is geodesically complete then this condition implies that the group of all affine G-structure preserving diffeomorphisms of M acts transitively on the frames that belong to P and in that case we say that the triple (M, ∇, P) is *homogeneous* [5].

The G-constant tensor fields R and T of an infinitesimally homogeneous triple (M, ∇, P) are represented, with respect to an arbitrary frame belonging to P, by multilinear maps $R_0 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $T_0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, respectively; moreover, the G-constant inner torsion $\mathfrak{I}^P$ is represented (with respect to an arbitrary frame belonging to P) by a linear map $\mathfrak{I}_0^P : \mathbb{R}^n \to \mathfrak{gl}(n)/\mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G. We call $R_0, T_0, \mathfrak{I}_0^P$ the *characteristic tensors* of (M, ∇, P). The characteristic tensors $R_0, T_0, \mathfrak{I}_0^P$ characterize locally an infinitesimally homogeneous triple (M, ∇, P), in the sense that two infinitesimally homogeneous triples having the same characteristic tensors are locally equivalent (by means of affine G-structure preserving diffeomorphisms). It is then very natural to ask what are the necessary and sufficient conditions for maps $R_0, T_0, \mathfrak{I}_0^P$ to be the characteristic tensors of an infinitesimally homogeneous triple (M, ∇, P). This paper answers such question.

The main result of this paper can be seen as part of a program of reducing a problem of classification of certain geometric objects to a problem of classification of certain algebraic objects. Other examples of such reductions are: (i) the result that two Lie groups having the same Lie algebra are locally isomorphic

and every Lie algebra is the Lie algebra of a Lie group; (ii) the result that two Riemannian symmetric spaces having the same orthogonal involutive Lie algebra (oil algebra) are locally isometric and every oil algebra is the oil algebra of a Riemannian symmetric space (see [4]).

It would be natural to ask what are the necessary and sufficient conditions for R_0, T_0, $\mathfrak{I}_0^P$ to be the characteristic tensors of a (globally) homogeneous triple (M, ∇, P). Is it true that if R_0, T_0, $\mathfrak{I}_0^P$ are the characteristic tensors of an infinitesimally homogeneous triple then they are also the characteristic tensors of some (globally) homogeneous triple? While we do not know the answer to that question, a partial answer will be given in a forthcoming paper.

2. Notation and Preliminaries

2.1. Vector spaces

Let V be a real finite-dimensional vector space. We denote by $\mathrm{GL}(V)$ the general linear group of V and by $\mathfrak{gl}(V)$ its Lie algebra. If W is another real finite-dimensional vector space, $\mathrm{Lin}_k(V; W)$ denotes the space of k-linear maps from V to W. Given multilinear maps $T \in \mathrm{Lin}_k(V; V)$, $S \in \mathrm{Lin}_k(W; W)$ and a (not necessarily invertible) linear map $\sigma : V \to W$ then T is said to be σ-*related* with S if:

$$S\big(\sigma(v_1), \ldots, \sigma(v_k)\big) = \sigma\big(T(v_1, \ldots, v_k)\big),$$

for all $v_1, \ldots, v_k \in V$. If $p : V \to W$ is a linear isomorphism we denote by $\mathcal{I}_p : \mathrm{GL}(V) \to \mathrm{GL}(W)$ the Lie group isomorphism given by conjugation with p and $\mathrm{Ad}_p = \mathrm{d}\mathcal{I}_p(\mathrm{Id}) : \mathfrak{gl}(V) \to \mathfrak{gl}(W)$ denotes the Lie algebra isomorphism given by conjugation with p.

2.2. G-structures on Manifolds

If G is a Lie subgroup of $\mathrm{GL}(n)$, by a G-*structure* on an n-dimensional real vector space V we mean a G-orbit of the action given by right composition of $\mathrm{GL}(n)$ on the set of all linear isomorphisms $p : \mathbb{R}^n \to V$. By a G-structure on an n-dimensional differentiable manifold M we mean a G-principal subbundle P of $\mathrm{FR}(TM)$, such that for each $x \in M$, P_x is a G-structure on the vector space T_xM. Let M and M' be n-dimensional differentiable manifolds endowed with G-structures P and P', respectively. A smooth map $f : M \to M'$ is said to be G-*structure preserving* if for each $x \in M$, the linear map $\mathrm{d}f_x : T_xM \to T_{f(x)}M'$ sends frames of P_x to frames that belong to $P'_{f(x)}$.

Remark 1. If G is a Lie subgroup of $\mathrm{GL}(n)$ a multilinear map $\tau_0 \in \mathrm{Lin}_k(\mathbb{R}^n; \mathbb{R}^n)$ is said to be G-*invariant*, if for each $g \in G$, τ_0 is g-related with itself. Clearly, given a G-invariant tensor $\tau_0 \in \mathrm{Lin}_k(\mathbb{R}^n; \mathbb{R}^n)$ one can induce a version of τ_0 on every vector space endowed with a G-structure. More precisely, let V be a real n-dimensional vector space endowed with a G-structure P. Given any $p \in P$ let $\tau_V \in \mathrm{Lin}_k(V; V)$ be the tensor which is p-related with τ_0. The G-invariance of

τ_0 implies that τ_V does not depend on the choice of $p \in P$. In particular, when M is an n-dimensional differentiable manifold endowed with a G-structure P and $\tau_0 \in \mathrm{Lin}_k(\mathbb{R}^n; \mathbb{R}^n)$ is G-invariant, by using frames that belong to P it is possible to define a tensor field τ on M such that for each $x \in M$, the map $\tau_x \in \mathrm{Lin}_k(T_xM; T_xM)$ is the version of τ_0 in T_xM.

2.3. Connections on Vector Bundles

Let E be a vector bundle over a differentiable manifold M with typical fiber E_0. We denote by $\mathbf{\Gamma}(E)$ the set of all smooth sections of E and by $\mathrm{FR}_{E_0}(E)$ the $\mathrm{GL}(E_0)$-principal bundle over M formed by all E_0-frames of E. When $E_0 = \mathbb{R}^n$ we write $\mathrm{FR}(E)$ instead of $\mathrm{FR}_{E_0}(E)$. If $\epsilon : U \to E$ is a local section of the vector bundle E and $s : U \to \mathrm{FR}_{E_0}(E)$ is a smooth local frame for E then the *representation* of the section ϵ with respect to the smooth local frame s is a map $\widetilde{\epsilon} : U \to E_0$ defined by: $\widetilde{\epsilon}(x) = s(x)^{-1}(\epsilon(x))$, for all $x \in U$.

A smooth local frame $s : U \to \mathrm{FR}_{E_0}(E)$ defines, in a natural way, a connection $\mathbb{d}^s$ in $E|_U$, which corresponds via the trivialization of $E|_U$ defined by s to the standard derivative. More explicitly, we set:

$$\mathbb{d}^s_v \epsilon = s(x)\big(\mathrm{d}\widetilde{\epsilon}_x(v)\big),$$

for all $x \in U$, $v \in T_xM$ and all $\epsilon \in \mathbf{\Gamma}(E|_U)$, where $\widetilde{\epsilon} : U \to E_0$ denotes the representation of ϵ with respect to the local frame s.

If ∇ is a connection in E, the *Christoffel tensor* of ∇ with respect to the smooth local frame s is the smooth tensor $\Gamma = \nabla - \mathbb{d}^s \in \mathbf{\Gamma}(TM^* \otimes E^* \otimes E)$ such that:

$$\nabla_v \epsilon = \mathbb{d}^s_v \epsilon + \Gamma_x\big(v, \epsilon(x)\big),$$

for all $x \in U$, $v \in T_xM$ and all $\epsilon \in \mathbf{\Gamma}(E|_U)$. Denoting by ω the smooth $\mathfrak{gl}(E_0)$-valued connection form on $\mathrm{FR}_{E_0}(E)$ associated to ∇, we have the following:

$$\Gamma_x(v) = s(x) \circ \overline{\omega}_x(v) \circ s(x)^{-1} \in \mathfrak{gl}(E_x), \qquad (1)$$

for all $x \in U$, $v \in T_xM$, where $\overline{\omega} = s^*\omega$ denotes the pullback by s of the connection form ω.

Remark 2. If ∇ is a (symmetric) connection on TM and $\mathfrak{t} : \mathbf{\Gamma}(TM) \times \mathbf{\Gamma}(TM) \to \mathbf{\Gamma}(TM)$ is an arbitrary $C^\infty(M)$-bilinear (symmetric) map, $\nabla' = \nabla + \mathfrak{t}$ is also a (symmetric) connection on TM and a simple calculation shows that, (see [6]):

$$R'(X,Y)Z = R(X,Y)Z + (\nabla_X \mathfrak{t})(Y,Z) - (\nabla_Y \mathfrak{t})(X,Z) + [\mathfrak{t}(X), \mathfrak{t}(Y)]Z, \quad (2)$$

$$T'(X,Y) = T(X,Y) + \mathfrak{t}(X)Y - \mathfrak{t}(Y)X, \qquad (3)$$

for each $X, Y, Z \in \mathbf{\Gamma}(TM)$. Where R' and T' denote the curvature and torsion tensors of ∇', respectively; R and T denote the curvature and torsion tensors of ∇, respectively.

3. Infinitesimally Homogeneous Manifolds

Let (M, ∇, P) be an n-dimensional affine manifold with G-structure P, the inner torsion of P with respect to the connection ∇ was introduced in [6], this notion gives rise to a tensor field $\mathfrak{I}^P$ on M that measures the lack of compatibility of the connection ∇ with P, since this notion plays an important role in this work, we present below its definition in a brief way.

For each $x \in M$, we denote by G_x the Lie subgroup of $\mathrm{GL}(T_x M)$ consisting of G-structure preserving endomorphisms of $T_x M$. Clearly $G_x = \mathcal{I}_p(G)$, for all $p \in P_x$, so that G_x is a Lie subgroup of $\mathrm{GL}(T_x M)$. We denote by $\mathfrak{g}_x \subset \mathfrak{gl}(T_x M)$ the Lie algebra of G_x. It is clear that $\mathrm{Ad}_p(\mathfrak{g}) = \mathfrak{g}_x$, for all $p \in P_x$, where $\mathfrak{g} \subset \mathfrak{gl}(n)$ denotes the Lie algebra of G. Since $\mathrm{Ad}_p : \mathfrak{gl}(n) \to \mathfrak{gl}(T_x M)$ carries $\mathfrak{g}$ onto $\mathfrak{g}_x$; therefore, it induces an isomorphism:

$$\overline{\mathrm{Ad}}_p : \mathfrak{gl}(n)/\mathfrak{g} \longrightarrow \mathfrak{gl}(T_x M)/\mathfrak{g}_x.$$

Let $s : U \subset M \to P$ be a smooth local section of P, with $x \in U$ and set $s(x) = p$. If ω denotes the $\mathfrak{gl}(n)$-valued connection form on $\mathrm{FR}(TM)$ associated with ∇ and $\overline{\omega} = s^*\omega$. The map

$$T_x M \xrightarrow{\overline{\omega}_x} \mathfrak{gl}(\mathfrak{n}) \xrightarrow{\mathfrak{q}} \mathfrak{gl}(\mathfrak{n})/\mathfrak{g} \xrightarrow{\overline{\mathrm{Ad}}_p} \mathfrak{gl}(T_x M)/\mathfrak{g}_x \qquad (4)$$

$$\mathfrak{I}_x^P$$

does not depend on the choice of the local section s. The linear map $\mathfrak{I}_x^P$ defined by (4) is called the *inner torsion* of the G-structure P at the point x with respect to the connection ∇. It follows from (1), that if $s : U \to P$ is a smooth local section with $x \in U$ and Γ denotes the Christoffel tensor of ∇ with respect to s then the inner torsion $\mathfrak{I}_x^P$ is precisely the composition of the $\Gamma_x : T_x M \to \mathfrak{gl}(T_x M)$ with the quotient map $\mathfrak{gl}(T_x M) \to \mathfrak{gl}(T_x M)/\mathfrak{g}_x$. This observation gives a simple way of computing inner torsions, (see [5]).

The geometry of an affine manifold with G-structure (M, ∇, P) is described by three tensors of M: the torsion T of ∇, the curvature R of ∇ and the inner torsion $\mathfrak{I}^P$. An important class of examples of affine manifolds with G-structure is defined by the property that these three tensors T, R and $\mathfrak{I}^P$ be *constant* when written in frames of the G-structure P. When this is the case, (M, ∇, P) is said to be *infinitesimally homogeneous*. This statement is made more precise in the following definition.

Definition 1. An n-dimensional affine manifold with G-structure, (M, ∇, P) is said to be *infinitesimally homogeneous* if there exist maps $R_0 \in \mathrm{Lin}_3(\mathbb{R}^n, \mathbb{R}^n)$, $T_0 \in \mathrm{Lin}_2(\mathbb{R}^n, \mathbb{R}^n)$ and a linear map $\mathfrak{I}_0 : \mathbb{R}^n \to \mathfrak{gl}(n)/\mathfrak{g}$ such that: for every $x \in M$, every $p \in P_x$ relates T_0 with T_x, R_0 with R_x and $\overline{\mathrm{Ad}}_p \circ \mathfrak{I}_0 = \mathfrak{I}_x^P \circ p$.

The maps T_0, R_0, $\mathfrak{I}_0$ as referred above are called the *characteristic tensors* of the infinitesimally homogeneous manifold (M, ∇, P).

Clearly, the characteristic tensors T_0, R_0, $\mathfrak{I}_0$ of an infinitesimally homogeneous manifold (M, ∇, P) are invariant by the action of the *structural group* G. Therefore, it follows from the G-invariance condition that the following relations hold:

$$R_0(u, v) = \mathrm{Ad}_g \cdot R_0\big(g^{-1} \cdot u, g^{-1} \cdot v\big); \tag{5}$$

$$T_0(u, v) = g \cdot T_0\big(g^{-1} \cdot u, g^{-1} \cdot v\big); \tag{6}$$

$$\mathrm{Ad}_g\big(\lambda(g^{-1} \cdot u)\big) - \lambda(u) \in \mathfrak{g}, \tag{7}$$

for all $g \in G$, all $u, v \in \mathbb{R}^n$. Where $\lambda : \mathbb{R}^n \to \mathfrak{gl}(n)$ is an arbitrary lifting of $\mathfrak{I}_0$. Notice that Relation (7) does not depend on λ. In fact, let λ, δ be liftings of $\mathfrak{I}_0$. Write $\lambda = \delta + L$, where L is a $\mathfrak{g}$-valued linear map defined in $\mathbb{R}^n$. An easy computation shows that:

$$\mathfrak{g} \ni \mathrm{Ad}_g\big(\lambda(g^{-1} \cdot u)\big) - \lambda(u) =$$
$$\mathrm{Ad}_g\big(\delta(g^{-1} \cdot u)\big) - \delta(u) + \underbrace{\mathrm{Ad}_g\big(L(g^{-1} \cdot u)\big) - L(u)}_{\in\ \mathfrak{g}},$$

for all $g \in G$, $u \in \mathbb{R}^n$. Which shows the independence on λ.

By differentiating (5), (6), and (7) we obtain the following:

Lemma 1. *Let $\lambda : \mathbb{R}^n \to \mathfrak{gl}(n)$ be an arbitrary lifting of $\mathfrak{I}_0$. Then for all $L \in \mathfrak{g}$ and all $u, v \in \mathbb{R}^n$, the following conditions hold:*

(1) $\big[L, R_0(u, v)\big] - R_0(L \cdot u, v) - R_0(u, L \cdot v) = 0$;

(2) $L \circ T_0(u, v) - T_0(L \cdot u, v) - T_0(u, L \cdot v) = 0$;

(3) $\big[L, \lambda(u)\big] - \lambda(L \cdot u) \in \mathfrak{g}$.

4. Algebraic Relation Between the Characteristic Tensors

It is a natural question to ask whether one can give a (local) *classification* of infinitesimally homogeneous manifolds with prescribed group G and prescribed characteristic tensors T_0, R_0, $\mathfrak{I}_0$. We solve this question in this paper by giving necessary and sufficient conditions for maps T_0, R_0, $\mathfrak{I}_0$ to be the characteristic tensors of an infinitesimally homogeneous manifold. Our plan for developing the necessary condition is the following: we show that to give a classification of infinitesimally homogeneous manifolds with prescribed group G is equivalent to finding an infinitesimally homogeneous manifold without torsion whose structural group is G, and to give a classification of the G-invariant maps $\mathfrak{t}_0 \in \mathrm{Lin}_2(\mathbb{R}^n, \mathbb{R}^n)$. Once, this is done, in order to obtain the aimed condition, it will be sufficient to consider the case of symmetric connections (equivalently $T_0 = 0$). This is the purpose of this section, and the sufficient conditions will be developed in the following section.

4.1. Covariant Derivative for G-Constant Tensors

Let (M, ∇, P) be an homogeneous affine manifold with G-structure P. If $\mathfrak{I}^P = 0$, i.e., the covariant derivative of P is zero, it follows that every G-constant tensor is parallel with respect to ∇. On the other hand, if ∇ is not compatible with P, i.e., the covariant derivative of P is not zero, it is not true that every G-constant tensor is parallel with respect to ∇. Hence, we need to find a simply way to calculate the covariant derivative for G-constant tensors on this case, i.e., when $\mathfrak{I}^P \neq 0$. Which we do next.

Denoting by $\underline{\mathfrak{Vec}}$ the category whose objects are real finite-dimensional vector spaces and whose morphisms are linear isomorphisms. Given a smooth functor $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ and any object V of $\underline{\mathfrak{Vec}}$, $\mathfrak{F}$ induces a Lie group homomorphism $\mathfrak{F} : \mathrm{GL}(V) \longrightarrow \mathrm{GL}\big(\mathfrak{F}(V)\big)$, whose differential at the identity is a Lie algebra homomorphism that will be denoted by $\underline{\mathfrak{f}} : \mathfrak{gl}(V) \longrightarrow \mathfrak{gl}\big(\mathfrak{F}(V)\big)$.

Let E be a vector bundle with typical fiber E_0 on M. Given a smooth functor $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ we denote by $\mathfrak{F}(E) = \bigcup_{x \in M} \mathfrak{F}(E_x)$, the vector bundle with typical fiber $\mathfrak{F}(E_0)$ obtained from E by using $\mathfrak{F}$.

Given a smooth funtor $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ we have the following:

Lemma 2. *Let $\mathfrak{t}$ be a smooth G-constant section of $\mathfrak{F}(TM)$. Then*

$$\nabla_v \mathfrak{t} = \underline{\mathfrak{f}}(L) \cdot \mathfrak{t}_x, \tag{8}$$

for all $x \in M$, $v \in T_x M$, where $L \in \mathfrak{gl}(T_x M)$ is such that $\mathfrak{I}^P_x(v) = L + \mathfrak{g}_x$.

Proof. Clearly $\mathfrak{t}$ can be thought as an $\mathrm{FR}(TM)$-valued 0-form on M, which is associated to a 0-form $\phi : \mathrm{FR}(TM) \to \mathfrak{F}(\mathbb{R}^n)$ such that: $\phi(p) = \mathfrak{F}(p)^{-1}(\mathfrak{t}_x)$ for all $x \in M$, $p \in \mathrm{FR}(TM)$. Moreover, the covariant exterior differential $\mathrm{D}\phi$ is associated to the covariant exterior differential $\mathrm{D}\mathfrak{t}$ of $\mathfrak{t}$. More explicitly, we have:

$$\mathrm{d}\phi_p(\zeta) = \mathrm{D}\phi_p(\zeta) = \mathfrak{F}(p)^{-1}(\mathrm{D}\mathfrak{t})_x \cdot v = \mathfrak{F}(p)^{-1} \nabla_v \mathfrak{t}, \tag{9}$$

for all $x \in M$, $p \in P_x$, $v \in T_x M$ and ζ an horizontal vector such that $\mathrm{d}\Pi_p(\zeta) = v$, where $\Pi : \mathrm{FR}(TM) \to M$ denotes the canonical projection. To obtain the desired result, we must calculate $\mathrm{d}\phi_p(\zeta)$. If $X \in \mathfrak{gl}(n)$ is such that $\overline{\mathrm{Ad}}_p(X + \mathfrak{g}) = \mathfrak{I}^P_x(v)$ then:

$$\zeta = (\mathrm{d}\Pi_p, \omega_p)^{-1}(v, X) - (\mathrm{d}\Pi_p, \omega_p)^{-1}(0, X) = \underbrace{(\mathrm{d}\Pi_p, \omega_p)^{-1}(v, X)}_{\in\, T_p P} - \mathrm{d}\beta_p(1) \cdot X,$$

where β_p denotes the map given by the action of $\mathrm{GL}(n)$ on p. Since $\phi|_P$ is constant, we have:

$$\mathrm{d}\phi_p(\zeta) = -\mathrm{d}\phi_p\big(\mathrm{d}\beta_p(1) \cdot X\big) = \underline{\mathfrak{f}}(X) \cdot \mathfrak{t}_0. \tag{10}$$

But (8) follows directly from equalities (9), (10).   ☑

Example 1. Let $\mathfrak{F} : \underline{\mathfrak{Vec}} \to \underline{\mathfrak{Vec}}$ be the functor defined by:

$$\underline{\mathfrak{F}}(V) = \mathrm{Lin}_k(V; \mathrm{Lin}(V))$$

for each object V of $\underline{\mathfrak{Vec}}$. Let (M, ∇, P) be an n-dimensional affine manifold with G-structure. If $\mathfrak{t}_0 \in \mathrm{Lin}_k(\mathbb{R}^n; \mathfrak{gl}(n))$ is a G-constant tensor, denoting by $\mathfrak{t}_x$ the induced version of $\mathfrak{t}_0$ on T_xM, by using (2) we have:

$$\nabla_v \mathfrak{t} = \big[L, \mathfrak{t}_x(\cdot, \ldots, \cdot)\big] - \mathfrak{t}_x(L\cdot, \cdot, \ldots, \cdot) - \cdots - \mathfrak{t}_x(\cdot, \cdot, \ldots, L\cdot),$$

where $L \in \mathfrak{gl}(T_xM)$ is such that $\mathfrak{I}_x^P(v) = L + \mathfrak{g}_x$. On the other hand, it is clear that an arbitrary lifting $\lambda : \mathbb{R}^n \to \mathfrak{gl}(n)$ of $\mathfrak{I}_0$, induces a derivation $\mathcal{D}_{\lambda(X)}$ on the tensor algebra over the vector space $\mathbb{R}^n$, for all $X \in \mathbb{R}^n$; an easy computation shows that:

$$\big(\mathcal{D}_{\lambda(X)}\mathfrak{t}_0\big) = \underline{\mathfrak{f}}\big(\lambda(X)\big) \cdot \mathfrak{t}_0.$$

Therefore, if λ is an arbitrary lifting of $\mathfrak{I}_0$, given $x \in M$, $p \in P_x$ and $X \in \mathbb{R}^n$ such that $v = p(X)$ and $\mathrm{Ad}_p\big(\lambda(X)\big) = L$ we have:

$$\mathrm{Ad}_p\big(\mathcal{D}_{\lambda(X)}\mathfrak{t}_0\big) = (\nabla_v \mathfrak{t}) \circ (p, \ldots, p).$$

4.2. Infinitesimally Homogeneous Manifolds without Torsion

Let (M, ∇, P) be an n-dimensional affine manifold with G-structure and assume that ∇ is a symmetric connection. Let $\mathfrak{t}_0 \in \mathrm{Lin}_2(\mathbb{R}^n, \mathbb{R}^n)$ be a G-invariant skew-symmetric tensor. For each $x \in M$, we denote by $\mathfrak{t}_x$ the induced version of $\mathfrak{t}_0$ on T_xM. In view of Remark 2, it is clear that $\nabla' = \nabla + \frac{1}{2}\mathfrak{t}$ defines a connection on M whose torsion is $\mathfrak{t}$. We devote this section to prove the following.

Lemma 3. *With the same notation as above, if (M, ∇, P) is an infinitesimally homogeneous manifold then the triple (M, ∇', P) is also infinitesimally homogeneous.*

Proof. It is enough to prove that there exist tensors T_0', R_0', $\mathfrak{I}_0'$ as in (1). We take $T_0' = \mathfrak{t}_0$. On the other hand, $\mathfrak{t}$ can be identified with a smooth $\mathrm{Lin}(TM)$-valued covariant 1-tensor field on M. Let $s : U \to P$ be a smooth local section of P. We denote by Γ' and Γ, respectively, the Christoffel tensors of ∇' and ∇ with respect to s. Given $x \in U$, it is clear that $\Gamma_x' = \Gamma_x + \mathfrak{t}_x$, by composing this with the canonical projection $\mathfrak{q} : \mathfrak{gl}(T_xM) \to \mathfrak{gl}(T_xM)/\mathfrak{g}_x$ we obtain:

$$\mathfrak{I}_x'^P = \mathfrak{I}_x^P + \mathfrak{q} \circ \mathfrak{t}_x.$$

Therefore, we can take $\mathfrak{I}_0' = \mathfrak{I}_0 + \mathfrak{q} \circ \mathfrak{t}_0$. On the other hand, we denote by R' and R, respectively, the curvature tensors of ∇' and ∇. Let λ be an arbitrary

lifting of $\mathfrak{I}_0$, $x \in U$ and set $s(x) = p$. From (2) and by using Lemma 2 we have that the following holds:

$$
\begin{aligned}
R'_x(p\cdot, p\cdot) &= R_x(p\cdot, p\cdot) + (\mathrm{D}\mathfrak{t})_x(p\cdot, p\cdot) + [\mathfrak{t}_x(p\cdot), \mathfrak{t}_x(p\cdot)] \\
&= \mathrm{Ad}_p \circ \big(R_0(\cdot, \cdot) + \mathrm{Alt}\big(\mathcal{D}_{\lambda(\cdot)}\mathfrak{t}_0\big) \cdot + [\mathfrak{t}_0(\cdot), \mathfrak{t}_0(\cdot)]\big).
\end{aligned}
$$

Therefore, in order to obtain the desired result we can take

$$
R'_0 = R_0 + \mathcal{D}\mathfrak{t}_0 + [\mathfrak{t}_0, \mathfrak{t}_0]. \qquad \qquad \text{☑}
$$

4.3. The Necessary Conditions

We are now ready to give necessary conditions which must be satisfied by the characteristic tensors of an infinitesimally homogeneous manifold. To do this, throughout the subsection we consider a fixed n-dimensional infinitesimally homogeneous manifold (M, ∇, P) with structural group G. From Lemma 3 it follows that we may assume without loss of generality that ∇ is a symmetric connection with curvature R. We denote by $R_0, \mathfrak{I}_0$ the characteristic tensors of (M, ∇, P). Clearly, a necessary condition is that $R_0, \mathfrak{I}_0$ are G-invariant.

Let ω be the $\mathfrak{gl}(n)$-valued connection form on $\mathrm{FR}(TM)$ associated with ∇, let Ω be its curvature form and let θ be the canonical form of $\mathrm{FR}(TM)$. Given a smooth local frame $s : U \to P$ then, setting $\overline{\omega} = s^*(\omega)$, $\overline{\Omega} = s^*\Omega$, $\overline{\theta} = s^*\theta$, we have:

$$
\overline{\Omega} = \mathrm{d}\overline{\omega} + \overline{\omega} \wedge \overline{\omega}, \qquad \mathrm{d}\overline{\theta} = -\overline{\omega} \wedge \overline{\theta}.
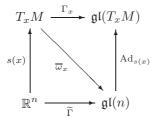$$

Moreover, the infinitesimal homogeneity implies that:

$$
\overline{\Omega}_x(X, Y) = s(x) \circ R_x(X, Y) \circ s(x)^{-1} = R_0\big(s(x)^{-1}X, s(x)^{-1}Y\big),
$$

$$
\mathfrak{q} \circ \overline{\omega}_x = \overline{\mathrm{Ad}}_{s(x)^{-1}} \circ \mathfrak{I}_x^P = \mathfrak{I}_0 \circ \overline{\theta},
$$

for all $x \in U$, $X, Y \in T_xM$, where $\mathfrak{q} : \mathfrak{gl}(n) \to \mathfrak{gl}(n)/\mathfrak{g}$ denotes the canonical projection and $\mathfrak{g}$ denotes the Lie algebra of G. Clearly when the linear map $\mathfrak{I}^P$ vanishes, $\overline{\Omega}$ is a $\mathfrak{g}$-valued 2-form on M. Under the previous conditions, in order to handle the general case in which P is not compatible with ∇ we get:

$$
\begin{aligned}
\mathfrak{q} \circ \overline{\Omega} &= \mathrm{d}(\mathfrak{q} \circ \overline{\omega}) + \mathfrak{q} \circ \overline{\omega} \wedge \overline{\omega} \\
&= \mathrm{d}(\mathfrak{I}_0 \circ \overline{\theta}) + \mathfrak{q} \circ \overline{\omega} \wedge \overline{\omega} \\
&= \mathfrak{I}_0 \circ \mathrm{d}\overline{\theta} + \mathfrak{q} \circ \overline{\omega} \wedge \overline{\omega} \\
&= -\mathfrak{I}_0 \circ (\overline{\omega} \wedge \overline{\theta}) + \mathfrak{q} \circ \overline{\omega} \wedge \overline{\omega}.
\end{aligned} \qquad (11)
$$

Given $x \in U$, let $\widetilde{\Gamma} : \mathbb{R}^n \to \mathfrak{gl}(n)$ be the map defined by requiring the diagram

$$
\begin{array}{ccc}
T_xM & \xrightarrow{\ \Gamma_x\ } & \mathfrak{gl}(T_xM) \\
\Big\uparrow{\scriptstyle s(x)} & \searrow{\scriptstyle\overline{\omega}_x} & \Big\uparrow{\scriptstyle \mathrm{Ad}_{s(x)}} \\
\mathbb{R}^n & \xrightarrow[\ \widetilde{\Gamma}\ ]{} & \mathfrak{gl}(n)
\end{array}
$$

to be commutative. Therefore, $\mathfrak{I}_0 = \mathfrak{q} \circ \widetilde{\Gamma}$ and substituting in (11) we obtain the following relation:

$$\overline{\Omega}_x + \widetilde{\Gamma} \circ \big(\overline{\omega}_x \wedge \overline{\theta}_x\big) - \overline{\omega}_x \wedge \overline{\omega}_x \in \mathfrak{g}.$$

Thus, given vectors $u, v \in \mathbb{R}^n$ the relation above can be written as:

$$R_0(u,v) - \big[\widetilde{\Gamma}(u), \widetilde{\Gamma}(v)\big] + \widetilde{\Gamma}\big(\widetilde{\Gamma}(u)v - \widetilde{\Gamma}(v)u\big) \in \mathfrak{g}. \tag{12}$$

This relation does not depend on the choice of $\widetilde{\Gamma}$. Namely, let λ be an arbitrary lifting of $\mathfrak{I}_0$ and δ be a $\mathfrak{g}$-valued linear map in $\mathbb{R}^n$ such that $\widetilde{\Gamma} = \lambda + \delta$. By replacing this into (12), we obtain

$$\mathfrak{g} \ni R_0(u,v) - \big[\lambda(u), \lambda(v)\big] + \lambda\big(\lambda(u)v - \lambda(v)u\big) + \mathcal{A}(\delta) + \mathcal{B}(\delta), \tag{13}$$

where:

$$\mathcal{A}(\delta) = \big(\big[\delta(v), \lambda(u)\big] - \lambda\big(\delta(v) \cdot u\big)\big) - \big(\big[\delta(u), \lambda(v)\big] - \lambda\big(\delta(u) \cdot v\big)\big),$$
$$\mathcal{B}(\delta) = \delta\big(\widetilde{\Gamma}(u)v - \widetilde{\Gamma}(v)u\big) - \big[\delta(u), \delta(v)\big].$$

So that Lemma 1 guarantees that $\mathcal{A}(\delta) \in \mathfrak{g}$; moreover, $\mathcal{B}(\delta) \in \mathfrak{g}$ because δ is a $\mathfrak{g}$-valued linear map. Therefore for an arbitrary lifting λ of $\mathfrak{I}_0$ the following relation holds:

$$R_0(u,v) - \big[\lambda(u), \lambda(v)\big] + \lambda\big(\lambda(u)v - \lambda(v)u\big) \in \mathfrak{g},$$

this shows the independence on the lifting; hence we have proved the following:

Theorem 1. *Let M be an n-dimensional differentiable manifold, G a Lie subgroup of $\mathrm{GL}(n)$ with Lie algebra $\mathfrak{g}$ and assume that M is endowed with a symmetric connection ∇ and a G-structure $P \subset \mathrm{FR}(TM)$. Assume that (M, ∇, P) is an infinitesimally homogeneous manifold with characteristic tensors $R_0, \mathfrak{I}_0$. Then given an arbitrary lifting λ of $\mathfrak{I}_0$, the following relation holds:*

$$R_0(u,v) - \big[\lambda(u), \lambda(v)\big] + \lambda\big(\lambda(u)v - \lambda(v)u\big) \in \mathfrak{g},$$

for all $u, v \in \mathbb{R}^n$.

5. Infinitesimally Homogeneous Manifolds with Prescribed Group and Prescribed Characteristic Tensors

We devote this section to obtain sufficient conditions for maps T_0, R_0, $\mathfrak{I}_0$ to be the characteristic tensors of an infinitesimally homogeneous manifold. Therefore, throughout section we will consider a fixed real finite-dimensional vector space $\mathfrak{m}$, a Lie subgroup $H \subset \mathrm{GL}(\mathfrak{m})$ with Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$ and H-invariant maps $R_0 \in \mathrm{Lin}_2\big(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m})\big)$, $\mathfrak{I}_0 : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})/\mathfrak{h}$. As we said above, our goal is to obtain conditions for the maps R_0, $\mathfrak{I}_0$ to be the characteristic tensors of an infinitesimally homogeneous manifold (M, ∇, P).

Let $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ be an arbitrary lifting of $\mathfrak{I}_0$. As in Section 3, by using the H-invariance of $\mathfrak{I}_0$ we conclude that the following relation holds:

$$[L, \lambda(X)] - \lambda(L \cdot X) \in \mathfrak{h}, \tag{14}$$

for all $L \in \mathfrak{h}$, all $X, Y \in \mathfrak{m}$. An analogous relation to (12) is:

$$R_0(X, Y) - \big[\lambda(X), \lambda(Y)\big] + \lambda\big(\lambda(X)Y - \lambda(Y)X\big) \in \mathfrak{h} \tag{15}$$

for all $X, Y \in \mathfrak{m}$. Neither relation (14) nor relation (15) depend on the choice of λ.

Assuming that (15) holds, we have the following:

Definition 2. Setting $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m}$. We endow $\mathfrak{a}$ with a bracket operation which is defined below. For each $X, Y \in \mathfrak{m}$, each $L, T \in \mathfrak{h}$ we set:

(1) $[X, Y]^{\mathfrak{m}} = \lambda(X) \cdot Y - \lambda(Y) \cdot X$;

(2) $[X, Y]^{\mathfrak{h}} = R_0(X, Y) + \lambda\big(\lambda(X) \cdot Y - \lambda(Y) \cdot X\big) - \big[\lambda(X), \lambda(Y)\big]$;

(3) $[L, X]^{\mathfrak{m}} = L \cdot X$;

(4) $[L, X]^{\mathfrak{h}} = \big[L, \lambda(X)\big] - \lambda(L \cdot X)$;

(5) $[L, T]$ is the Lie bracket of $\mathfrak{h}$;

(6) $[L, X] = -[X, L]$.

We will prove that the vector space $\mathfrak{a}$ endowed with the bracket operation as above is a Lie algebra. Before we proceed, we will present some algebraic preliminaries.

Definition 3. We say that the map R_0 satisfies the *Bianchi identities* if the following equalities hold:

(B_1) $\mathfrak{S} R_0(X, Y) \cdot Z = 0$;

(B_2) $\mathfrak{S}\big(\mathcal{D}_{\lambda(X)} R_0\big)(Y, Z) = 0$.

Where for $X \in \mathfrak{m}$, $\mathcal{D}_{\lambda(X)}$ denotes the derivation on the tensor algebra over the vector space $\mathfrak{m}$ induced by $\lambda(X)$ and $\mathfrak{S}$ denotes the sum over all cyclic permutations of X, Y, Z.

Remark 3. For $X, Y, Z \in \mathfrak{m}$ and $L \in \mathfrak{h}$ we set:

$$\mathcal{S}_{[L,X,Y]} = \big[L, \lambda(X)\big] \cdot Y - \lambda(Y) \cdot (L \cdot X).$$
$$\mathcal{T}_{[X,Y,Z]} = \big[\lambda(X), \lambda(Y)\big] \cdot Z - \lambda(Z) \cdot [X,Y]^{\mathfrak{m}}.$$

Thus, it is not difficult to see that:

$$\mathcal{S}_{[L,X,Y]} - \mathcal{S}_{[L,Y,X]} = L\big([X,Y]^{\mathfrak{m}}\big). \tag{16}$$

We can also easily see that:

$$\mathfrak{S}\mathcal{T}_{[X,Y,Z]} = 0. \tag{17}$$

Remark 4. For $X, Y, Z \in \mathfrak{m}$ by using the Bianchi identities we obtain:

$$\mathfrak{S}\Big(\big[\lambda(Z), R_0(X,Y)\big] - R_0\big([X,Y]^{\mathfrak{m}}, Z\big)\Big) = 0. \tag{18}$$

Lemma 4. *Using the same notation and terminology as above, suppose that the H-invariant maps $R_0, \mathfrak{J}_0$ satisfy the following conditions:*

(1) R_0 is skew-symmetric;

(2) given an arbitrary lifting $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ of $\mathfrak{J}_0$, the map R_0 satisfies the Bianchi identities and relation (15) holds.

Then the vector space $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m}$ endowed with the bracket operation $[\cdot, \cdot]$, defined as in Definition (2), is a Lie algebra.

Proof. Since $[\cdot, \cdot]$ is skew-symmetric, it is enough to show that it satisfies the Jacobi identity. To do that, we divide the proof in three cases. First we consider the case that $L, T \in \mathfrak{h}$, $X \in \mathfrak{m}$. From Definition 2:

$$\big[[X,L],T\big] = -\big[[L,\lambda(X)],T\big] - \lambda\big(T(L \cdot X)\big) + T(L \cdot X) \tag{19}$$
$$\big[[L,T],X\big] = \big[[L,T],\lambda(X)\big] - \lambda\big([L,T] \cdot X\big) + [L,T] \cdot X \tag{20}$$

interchanging T and L in (19) we get:

$$\big[[T,X],L\big] = \big[[T,\lambda(X)],L\big] + \lambda\big(L(T \cdot X)\big) - L(T \cdot X). \tag{21}$$

The conclusion follows from (19), (20) and (21) by applying the Jacobi identity in $\mathfrak{gl}(\mathfrak{m})$.

Now in the case that $X, Y \in \mathfrak{m}$, $L \in \mathfrak{h}$. From Definition 2:

$$\big[[X,Y],L\big]^{\mathfrak{m}} = -L\big([X,Y]^{\mathfrak{m}}\big) \tag{22}$$

$$\big[[X,Y],L\big]^{\mathfrak{h}} = \big[[\lambda(X),\lambda(Y)],L\big] + \lambda\big(L \cdot [X,Y]^{\mathfrak{m}}\big) - [R_0(X,Y),L] \tag{23}$$

by using Remark 3 we obtain:

$$\big[[Y,L],X\big]^{\mathfrak{m}} = -\mathcal{S}_{[L,Y,X]} \tag{24}$$

$$\big[[Y,L],X\big]^{\mathfrak{h}} = \big[[\lambda(Y),L],\lambda(X)\big] + \lambda\big(\mathcal{S}_{[L,Y,X]}\big) - R_0(X,L \cdot Y) \tag{25}$$

interchanging X and Y in (24), (25) we get:

$$\big[[L,X],Y\big]^{\mathfrak{m}} = \mathcal{S}_{[L,X,Y]} \tag{26}$$

$$\big[[L,X],Y\big]^{\mathfrak{h}} = \big[[L,\lambda(X)],\lambda(Y)\big] - \lambda\big(\mathcal{S}_{[L,X,Y]}\big) + R_0(Y,L \cdot X). \tag{27}$$

It follows from (22), (24) and (26) by using (16) that:

$$\mathfrak{S}\big[[X,Y],L\big]^{\mathfrak{m}} = 0.$$

On the other hand, it follows from (23), (25) and (27) by using (16), (18) and the Jacobi identity in $\mathfrak{gl}(\mathfrak{m})$ that:

$$\mathfrak{S}\big[[X,Y],L\big]^{\mathfrak{h}} = 0.$$

Finally, we consider the case $X, Y, Z \in \mathfrak{m}$. It follows directly from 2 that:

$$\mathfrak{S}\big[[X,Y],Z\big]^{\mathfrak{m}} = 0.$$

For the $\mathfrak{h}$ component we have:

$$\big[[X,Y],Z\big]^{\mathfrak{h}} = \big[[\lambda(X),\lambda(Y)],\lambda(Z)\big] - R_0\big([X,Y]^{\mathfrak{m}},Z\big)$$
$$- \big[R_0(X,Y),\lambda(Z)\big] - \lambda\big(\mathcal{T}_{[X,Y,Z]} - R_0(X,Y)Z\big).$$

Hence from (17) and (18) by using the Jacobi identity in $\mathfrak{gl}(\mathfrak{m})$ we can conclude:

$$\mathfrak{S}\big[[X,Y],Z\big]^{\mathfrak{h}} = 0. \qquad \qquad ☑$$

Remark 5. The Lie bracket defined in Definition 2 does not depend on the choice of λ. In fact, if $[\cdot,\cdot]_\lambda$ denotes the Lie Bracket in $\mathfrak{a}$ obtained by using the arbitrary lifting λ of $\mathfrak{I}_0$, given another lifting $\widetilde{\lambda}$ there exists a linear map $\delta : \mathfrak{m} \to \mathfrak{h}$ such that $\lambda = \widetilde{\lambda} + \delta$. The map $\varphi : \mathfrak{a} \to \big(\mathfrak{a},[\cdot,\cdot]_{\widetilde{\lambda}}\big)$ defined by the matrix:

$$\begin{bmatrix} \mathrm{Id}_{\mathfrak{h}} & \delta \\ 0 & \mathrm{Id}_{\mathfrak{m}} \end{bmatrix},$$

is an isomorphism of vector spaces, moreover, a direct computation shows that $[\cdot,\cdot]_\lambda = \varphi^*[\cdot,\cdot]_{\widetilde{\lambda}}$ so that φ is an isomorphism of Lie algebras. Which shows the assertion.

5.1. Existence of an Infinitesimally Homogeneous Manifold

The main goal of this subsection is to show the existence of an infinitesimally homogeneous manifold with prescribed structural group and prescribed characteristic tensors. To do this, let $\mathfrak{m}$ be a real finite-dimensional vector space, let $H \subset \mathrm{GL}(\mathfrak{m})$ be a Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$. Let $R_0 \in \mathrm{Lin}_2\big(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m})\big)$, $\mathfrak{I}_0 : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})/\mathfrak{h}$, be maps satisfying the following conditions:

(1) $R_0, \mathfrak{I}_0$ are $H-$ invariants;

(2) R_0 is skew–symmetric;

(3) given an arbitrary lifting $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ of $\mathfrak{I}_0$, R_0 satisfies the Bianchi identities and the relation (15) holds.

Now we are going to obtain an infinitesimally homogeneous manifold with structural group H whose characteristic tensor are $R_0, \mathfrak{I}_0$. It follows from Lemma 4 that the vector space $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m}$ endowed with the bracket defined on Definition 2 is a Lie algebra.

Let $\overline{\lambda} : \mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ be a map defined by:

$$\overline{\lambda}(X) = \begin{cases} \lambda(X), & \text{se } X \in \mathfrak{m}; \\ \overline{\mathrm{ad}}_X, & \text{se } X \in \mathfrak{h}. \end{cases} \tag{28}$$

where $\overline{\mathrm{ad}}$ denotes the isotropic representation of $\mathfrak{h}$ on $\mathfrak{m}$, more precisely $\overline{\mathrm{ad}}_X(Y) = \mathfrak{p}_{\mathfrak{m}}\big([X,Y]\big) = X(Y)$ for all $X \in \mathfrak{h}$, $Y \in \mathfrak{m}$.

Lemma 5. *If $L \in \mathfrak{h}$ and $\mathfrak{X} \in \mathfrak{a}$. Then*

$$\big[\overline{\lambda}(L), \overline{\lambda}(\mathfrak{X})\big] = \overline{\lambda}\big([L, \mathfrak{X}]\big).$$

Proof. We set $\mathfrak{X} = T + X$, for $T \in \mathfrak{h}$, $X \in \mathfrak{m}$.

$$\begin{aligned} \overline{\lambda}\big([L, \mathfrak{X}]\big) &= \overline{\mathrm{ad}}_{[L,T]} + \overline{\mathrm{ad}}_{\mathfrak{p}_{\mathfrak{h}}([L,X])} + \lambda(L \cdot X) \\ &= \big[\overline{\mathrm{ad}}_L, \overline{\mathrm{ad}}_T\big] + \big[\overline{\mathrm{ad}}_L, \lambda(X)\big] \\ &= \big[\overline{\lambda}(L), \overline{\lambda}(\mathfrak{X})\big]. \qquad\qquad\qquad \text{☑} \end{aligned}$$

Let A be a Lie group such that $T_1 A = \mathfrak{a}$. Let $M' \subset A$ be a submanifold of A through 1 such that $T_1 M' = \mathfrak{m}$. Let $\mathfrak{p}_{\mathfrak{m}}{}^L$ be the left invariant 1-form on A induced by the linear projection $\mathfrak{p}_{\mathfrak{m}} : \mathfrak{a} = \mathfrak{h} \oplus \mathfrak{m} \to \mathfrak{m}$. Setting $\overline{\kappa} = \mathfrak{p}_{\mathfrak{m}}{}^L|_{M'}$ then:

$$\overline{\kappa}_1(X) = \mathfrak{p}_{\mathfrak{m}}{}^L(X) = \mathfrak{p}_{\mathfrak{m}}(X) = X$$

for all $X \in \mathfrak{m}$. Let M be a neighborhood of 1 in M' such that for all $x \in M$ the map $\overline{\kappa}_x : T_x M \to \mathfrak{m}$ is a linear isomorphism. Then, the map $s : M \to \mathrm{FR}_{\mathfrak{m}}(TM)$

defined by $s(x) = \overline{\kappa}_x^{-1} : \mathfrak{m} \to T_x M$, for all $x \in M$ gives us a global section of the GL($\mathfrak{m}$)-principal bundle $\mathrm{FR}_\mathfrak{m}(TM)$ over M. Given $x \in M$, the set

$$P_x = s(x) \cdot H = \{s(x) \circ h \mid h \in H\},$$

is an H-structure on $T_x M$ and $P = \bigcup_{x \in M} P_x$ defines an H-structure on M.

In order to construct ∇, let $\overline{\lambda}^L$ the left invariant 1-form on A induced by the linear map $\overline{\lambda}$ defined in (28). Setting $\overline{\omega} = \overline{\lambda}^L|_M$, it is clear that $\overline{\omega}$ is a $\mathfrak{gl}(\mathfrak{m})$-valued smooth 1-form on M. Let ω be the unique $\mathfrak{gl}(\mathfrak{m})$-valued 1-form on $\mathrm{FR}_\mathfrak{m}(TM)$ such that $s^*\omega = \overline{\omega}$. Then ω is a connection form on $\mathrm{FR}_\mathfrak{m}(TM)$.

So far, we have obtained an affine manifold with H-structure (M, ∇, P), where ∇ denotes the linear connection associated with the connection form ω. We claim that (M, ∇, P) is an infinitesimally homogeneous manifold whose characteristic tensors are $R_0, \mathfrak{I}_0$. In fact, given $x \in M$ and $X \in T_x M$, we have:

$$\overline{\omega}_x(X) = \overline{\lambda}_x^L(X) = \overline{\lambda}(x^{-1} \cdot X) = \underbrace{\overline{\mathrm{ad}}_{\mathfrak{p}_\mathfrak{h}}(x^{-1} \cdot X)}_{\in\, \mathfrak{h}} + \lambda\big(\mathfrak{p}_\mathfrak{m}(x^{-1} \cdot X)\big),$$

therefore, in the quotient $\mathfrak{gl}(\mathfrak{m})/\mathfrak{h}$ the following equality holds:

$$\overline{\omega}_x(X) = \lambda\big(\mathfrak{p}_\mathfrak{m}(x^{-1} \cdot X)\big);$$

clearly $\mathfrak{p}_\mathfrak{m}(x^{-1} \cdot X) = \overline{\kappa}_x(X) = s(x)^{-1} \cdot X$. Thus we have:

$$\mathfrak{I}_x^P(X) = \overline{\mathrm{Ad}}_{s(x)}\big(\mathfrak{q} \circ \lambda \circ s(x)^{-1} \cdot X\big) = \overline{\mathrm{Ad}}_{s(x)}\big(\mathfrak{I}_0 \circ s(x)^{-1} \cdot X\big).$$

On the other hand, we set $\overline{\Omega} = s^*\Omega$, where Ω denotes the curvature form of ω. For each $x \in M$, $X, Y \in T_x M$. Setting $x^{-1} \cdot X = L + \overline{\kappa}_x \cdot X$, $x^{-1} \cdot Y = T + \overline{\kappa}_x \cdot Y$, for $L, T \in \mathfrak{h}$. It follows from Lemma 5 that:

$$-\overline{\omega}_x([X,Y]) = -\overline{\lambda}\big([L, T + \overline{\kappa}_x \cdot Y] + [\overline{\kappa}_x \cdot X, T] + [\overline{\kappa}_x \cdot X, \overline{\kappa}_x \cdot Y]\big)$$
$$= -\big[\overline{\lambda}(L), \overline{\lambda}(T + \overline{\kappa}_x \cdot Y)\big] - \big[\overline{\lambda}(\overline{\kappa}_x \cdot X), \overline{\lambda}(T)\big] - \overline{\lambda}\big[\overline{\kappa}_x \cdot X, \overline{\kappa}_x \cdot Y\big].$$

moreover:

$$\big[\overline{\omega}_x(X), \overline{\omega}_x(Y)\big] = \big[\overline{\lambda}(L), \overline{\lambda}(T + \overline{\kappa}_x \cdot Y)\big] + \big[\overline{\lambda}(\overline{\kappa}_x \cdot X), \overline{\lambda}(T)\big]$$
$$+ \big[\overline{\lambda}(\overline{\kappa}_x \cdot X), \overline{\lambda}(\overline{\kappa}_x \cdot Y)\big].$$

since

$$\overline{\Omega}_x(X, Y) = \mathrm{d}\overline{\omega}_x(X, Y) + \big[\overline{\omega}_x(X), \overline{\omega}_x(Y)\big] = -\overline{\omega}_x([X, Y]) + \big[\overline{\omega}_x(X), \overline{\omega}_x(Y)\big],$$

it follows from the two previous equalities that:

$$\overline{\Omega}_x(X, Y) = -\overline{\lambda}\big[\overline{\kappa}_x \cdot X, \overline{\kappa}_x \cdot Y\big] + \big[\overline{\lambda}(\overline{\kappa}_x \cdot X), \overline{\lambda}(\overline{\kappa}_x \cdot Y)\big] = R_0(\overline{\kappa}_x \cdot X, \overline{\kappa}_x \cdot Y)$$

which shows the claim. The following theorem summarizes all subsections:

Theorem 2. *Let $\mathfrak{m}$ be a real finite-dimensional vector space, let $H \subset \mathrm{GL}(\mathfrak{m})$ be a Lie subgroup with Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m})$. Let $R_0 \in \mathrm{Lin}_2\big(\mathfrak{m}, \mathfrak{gl}(\mathfrak{m})\big)$, $\mathfrak{I}_0 : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})/\mathfrak{h}$, be maps satisfying the following conditions:*

(1) *$R_0, \mathfrak{I}_0$ are $H-$ invariants;*

(2) *R_0 is skew–symmetric;*

(3) *given an arbitrary lifting $\lambda : \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m})$ of $\mathfrak{I}_0$, the map R_0 satisfies the Bianchi identities and the relation*

$$R_0(X, Y) - [\lambda(X), \lambda(Y)] + \lambda\big(\lambda(X)Y - \lambda(Y)X\big) \in \mathfrak{h}$$

holds.

Then there exists an infinitesimally homogeneous manifold (M, ∇, P) with structural group H, whose characteristic tensors are R_0, $\mathfrak{I}_0$.

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