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Extended Higgs sector on noncommutative geometry and nonassociative lepton chiral symmetry

PHD THESIS

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Contents

Abstract

Connes' noncommutative geometry (NCG) provides a generalization of Riemannian geometry by turning our attention away from manifolds to instead focus on the algebra of functions defined on them. In this setting, the main object is the so-called 'spectral triple' $\{A, H, D\}$, which consists of an algebra A and a Dirac operator D represented on a Hilbert space H. Furthermore, this framework enjoys a great physical interest since it offers a geometric reinterpretation for the standard model (SM) of particle physics coupled with gravity. In particular, in this picture, there is an underlying finite space attached to each space-time point where the Higgs boson appears as the 'connection' associated with this new 'dimensionless' space. However, after the discovery of the Higgs boson in 2012, the spectral approach to the SM revealed an inconsistent value for the Higgs boson mass.

In this thesis, we build particle physics models that might meaningfully contribute to the Higgs sector in the noncommutative geometry standard model (NCG SM). In chapter 3, we show that there are as many Higgs doublets as Yukawa couplings are in the fluctuated Dirac operator. Then, we construct the two Higgs doublet model (2HDM) in the NCG context. We deduce the boundary conditions for the renormalization group equations (RGEs) of the scalar couplings to calculate the mass spectrum for each one of these models. In particular, we study the parameter space of the required couplings to have a phenomenologically viable noncommutative geometry two Higgs doublet model (NCG 2HDM) type II.

In chapter 4, we focus on the nonassociative 'Bison algebra' $\mathbb{B}_2$, which has the automorphisms group $SU(2)\times U(1)$. We identify a natural representation for the SM leptons together with an exotic fermion degree of freedom. Then, we define a Hermitian Dirac operator containing the minimal Yukawa interaction. We end up with a 'twisted' spectral triple describing the electroweak theory for the SM lepton sector.

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Dedication

To the memory of *Anatilde Marín*[1] and *Jose Omar Giraldo*.[2]

[1]My grandmother (1937-2016).
[2]My grandfather (1944-2020).

Introduction

The standard model of particle physics (SM) and Einstein's general theory of relativity are the cornerstones of our understanding of the physical world. They are strongly supported by the empirical evidence found so far. In particular, the SM tells us what are quantum fields composing the (ordinary) matter in the universe and how they interact. This theory is a chiral one which means that the left and right-handed particles transform differently under the SM symmetries. Through the Brout-Englert-Higgs mechanism [1], the SM tell us how the bosons $W^{\pm}$ and Z, the charged leptons (like the electron), and the quarks acquire masses. Despite this achievement, there are some unanswered questions like, how neutrinos get mass? [2], or which are the fundamental dark matter particles? [3]. From a more theoretical point of view, one may also be wonder about why is the SM a chiral theory or why is there three fermion families?

Therefore, it is necessary to go beyond the SM by proposing new fields and interactions or by exploring new theoretical settings to find any track for the explanation to these phenomenons. One of the most popular extensions is the 2-Higgs doublets model (2HDM), which includes a new Higgs doublet with wide phenomenological implications. The interest in models with two Higgs has increased in the last years because it could be a source to explaining (at the same time) both dark matter and neutrinos mass problems [4, 5, 6]. If well there are many constraints coming from the experimental data collected so far, there are not many theoretical restrictions when constructing such new physics models. For that reason, it would be desirable to have a mathematical framework that can be used as a guideline.

Conne's noncommutative geometry (NCG) [7] offers us an algebraic reinterpretation of the Riemannian geometry which allows extrapolating the main geometrical concepts to discrete spaces (with a finite number of elements). When the product between these continuous (curved) and discrete (finite) spaces are treated in the 'spectral' formulation, it is possible to get the Einstein Hilbert action coupled with the SM [8, 9]. Beyond this possible framework for unifying the four fundamental forces, it is interesting to ask about the reliability of the noncommutative geometry standard model (NCG SM) itself. In this setting, the main object of study is the so-called spectral triple, which consists mainly of an algebra A, a Hilbert space H and a Dirac operator D. The group of automorphisms of A contains the gauge group symmetry of the theory, while H provides the fermionic degrees of freedom. On the other hand, D contains the Yukawa couplings as well as the correct gauge and scalar content. In NCG the dynamics of the bosonic and fermionic degrees of freedom are described by the 'spectral action' and the fermion action functional respectively [10], which are derived so that they should only depend on the spectrum of the Dirac operator. Similar to what happens in ordinary gauge theory, in NCG the automorphism covariance in guaranteed by replacing the Dirac operator by the 'fluctuated' Dirac operator. In particular, the action obtained in NCG unifies the gauge couplings as it is done in grand unified theories like $SU(5)$ or $SO(10)$ [11, 12]. In addition to this, new conditions on the Yukawa and scalar couplings turn up and are interpreted as high energy boundary conditions for the renormalization group equations (RGEs). When these RGEs are running from high to low energies, it is possible to make predictions on the mass spectrum for both fermions and scalar particles. As an important fact, the mass of the top quark obtained is consistent with its experimental value of 173 GeV. Even before the discovery at CERN [13], the Higgs boson particle was predicted to exist by NCG but with a wrong mass of approximately 170 GeV [14]. This shortcoming has been solved by turning one of the constant entries of the Dirac operator into a singlet scalar field, although at the cost of the reliability of the 1-loop top quark mass [15, **?**]. Further studies have remedied the theoretical foundations of this singlet field, such as it is done in [**?**, **?**, **?**] by considering a larger algebra, or as in [**?**, **?**, **?**] by relaxing the first-order axiom. One of the most reliable ways is the developed in [16, 17], where the concept of differential graded $*$-algebra, besides to capture the NCG axioms more simply and elegantly (along with a 'second-order' axiom which avoids the non-geometric massless photon condition), induces a complex scalar field, which is singlet under the SM symmetry, but charged under an extra Abelian gauge symmetry.

Despite all its achievements, the usual NCG does not provides any explanation for the number of families neither for the chiral nature of the SM. However, the recent efforts to generalize NCG to nonassociative geometry [18, 19, 20, 21], has shown to be a very promising route to finding answers to these kinds of theoretical questions. In this way, the Exceptional Jordan algebra can explain the number of SM fermion generations [22, 23, 24].

With this in mind, our goal in this research is to build two different models that substantially contribute to the Higgs sector in the NCG SM. In our first approach we build a phenomenologically viable two doublet model in noncommutative geometry (NCG 2HDM). First, we show that if we use the second-order axiom instead of the non-geometric massless photon condition, it is possible to obtain a Higgs doublet per Yukawa coupling present in the fluctuated Dirac operator. Once presented the form of the most general scalar potential, we proceed to build the 2HDMs that do not change the flavor in neutral currents: Lepton-Specific, Flipped, and Type-II. Unlike the SM, in the 2HDM we count with eight scalar degrees of freedom. Three of them are absorbed to give the masses to the gauge bososns $W^{\pm}$ and Z. The extra scalar fields define the 2HDM mass spectrum which is given by the charged $H^{\pm}$, CP-even (H, h) and CP-odd A (or pseudoscalar) fields.[3] After performing the RGEs analysis for each model, we conclude that unless we input extra terms to the minimal NCG 2HDM potential, none of these models can enhance the Higgs boson mass prediction. Besides, the three models present an accidental $U(1)$ symmetry, due to the absence of terms proportional to μ_{12}^2 and λ_5, which explicitly break such phase symmetry between the two doublets. This predicts a massless pseudoscalar involving a domain wall problem which is ruled out by current experimental data. In view of this, we investigate the validity of the NCG 2HDM Type-II when both parameters μ_{12}^2 and λ_5 are taken different from zero. In that case, we found points in the parameter space defined by μ_{12}^2 and λ_5 which are in agreement with the experimental Higgs boson mass and that also satisfy the

[3]The 2HDM Type-I is not possible to obtain in NCG unless we extend the minimal model to include an extra fermion, like the right-handed neutrino.

phenomenological constrains imposed by the 'alignment limit'[4].

We replicate the aforementioned construction for the NCG 2HDM in presence of a right-handed neutrino (per family) together with a singlet scalar field (coming from a Majorana mass term). In that case, it is possible to get the correct Higgs boson mass value, but again, this results (after electroweak symmetry breaking) in a non-observed massless pseudoscalar field. As before, we built a phenomenologically viable NCG 2HDM Type-II plus one singlet scalar field, but now introducing only the parameter μ_{12}^2.

In our second approach, we make use of the recent efforts made in [17] to find a spectral triple for the non-associative 'Bison algebra' $\mathbb{B}_2$, together with a well-defined scalar sector. The Bison algebra $\mathbb{B}_2$ was introduced in [25] as one of the four 8-dimensional (not necessarily associative) $\mathbb{Z}_2\times\mathbb{Z}_4$-graded algebras. These four algebras share the group of automorphisms $SU(2)\times U(1)$, but only two of them (including $\mathbb{B}_2$) admit an involution operation. In particular, $\mathbb{B}_2$ behaves in such a way that the non-Abelian generators act only on the 'odd' components of the algebra while the Abelian generators act on both components. Furthermore, the Abelian generator contains exactly the hypercharges values as observed for the (one-family) standard model lepton sector just as desired, i.e, the $SU(2)$ symmetry is 'chiral', just as observed in the standard model electroweak sector, with the left handed degrees of freedom being identified with the 'odd' elements of the algebra. Based on this fact, we construct a representation for both leptons and anti-leptons as indicated by charges encoded in the algebra symmetries. We endow this fermion space with an inner product so that we can introduce the notions of Hermitian and unitary operators acting on this. Then, we show a reliable way to put together the remaining spectral triple components, including a Hermitian Dirac operator commuting with the representations of the algebra elements. Hence, giving rise to a new fertile ground for future studies on the scalar sector of this non-associative model.

In Chapter 1, we provide an introduction to the Standard Model of particle physics, including its particle content together with their charges and symmetries. Then, we go through the Higgs mechanism, where we present how the gauge bosons acquire their masses. So, we close this chapter with the main 2HDM generalities, including its wide scalar mass spectrum as well as the phenomenological constraints that should be respected. In Chapter 2, we present the basic ideas of the spectral reconstruction of Riemannian geometry as well as its generalization to finite spaces. Here, we also develop a pedagogical approach to almost-commutative manifolds by employing Krajewski diagrams. In particular, we depict the Glashow-Weinberg-Salam model for the SM leptons in order to set the basis for our work in the last chapter. After outline the most relevant aspects of the spectral action, we conclude this chapter with the basic ideas to reconstruct the NCG axioms from the differential graded $*$-algebra approach. In Chapter 3, we show that the NCG SM allows a Higgs sector with more than one Higgs doublet. After determining the most general form for the Yukawa interaction and the scalar potential, we will proceed to construct the Lepton Specific, Flipped, and Type-II 2HDMs. Next, we inquire about their low energy phenomenology by using the high scale NCG boundary conditions to run the RGEs. Then, we introduce the necessary terms to have a phenomenologically viable NCG 2HDM. We repeat the same steps in the presence of a Majorana right-handed neutrino and a singlet scalar field. In Chapter 4, we present the non-associative Bison algebra model. Then,

[4] This constrain implies that the the gauge bosons massess should be given dominantly (close to 99%) by the 125 GeV Higgs boson mass eigenstate

we proceed to elaborate a (nonassociative) almost commutative spectral triple to put together the electroweak theory for the SM leptons. Before closing with the conclusions of this work, we introduce a ‘twisted’ commutator in order to have a well-defined scalar sector.

Chapter 1

The SM and beyond

In this chapter, we will set the SM background required to understand this work. First, we will give an introduction to the particle content of the SM. Next, we will explain remarkable features about gauge symmetries by illustrating the simplest case: the Abelian symmetry. Later, we will present the details concerning the calculation of the RGEs for the gauge couplings only. Finally, we present the 2HDM generalities to fix the basis for our work in Chapter 4.

1.1 The SM and fundamental particles

The SM is the theory which describes the fundamental particles and its interactions. The particle content of this theory consists of (spin-$\frac{1}{2}$) fermions, (spin-1) gauge bosons, and the (spin-0) Higgs boson. Fermions are fields that transform either under the left-handed $(\frac{1}{2}, 0)$ or the right-handed $(0, \frac{1}{2})$ representation of the Lorentz group [26] (see next section). There are three families (generations) and each one contain 4 different kinds of fermions: two leptons and two quarks (table 1.1).

	Family 1	Family 2	Family 3	Charge
Leptons	e	μ	τ	-1
	ν	ν	ν	0
Quarks	u_1, u_2, u_3	c_1, c_2, c_3	t_1, t_2, t_3	$+\frac{2}{3}$
	d_1, d_2, d_3	s_1, s_2, s_3	b_1, b_2, b_3	$-\frac{1}{3}$

Table 1.1: The SM fermions. Leptons do not carries colour charge unlike quarks.

The gauge bosons are fields that transform under the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group. They mediate the fundamental interactions as shown in table 1.2.

Boson	Interaction
γ	Electromagnetic
$W^{\pm}$	Electroweak
Z	
g	Strong

Table 1.2: The gauge bosons and their corresponding fundamental interaction. γ and g mean for the photon and gluons, respectively. mean for the photon and gluons respectively. There are 8 gluons that act as mediators for the strong interaction.

The theoretical framework to study relativistic fermion and boson fields is the Quantum Field Theory (QFT), where particles can be created and annihilated. A very convenient way to describe a QFT is by means of Lagrangian formalism. In fact, the Lagrangians which describe the elementary particles and their interactions can be constructed from symmetry principles. The most important three examples of a QFT are:

1. Quantum Electrodynamics results by imposing the local gauge principle based on the $U(1)$ Abelian symmetry to the Dirac equation for the electron. As a result, the Maxwell equations are obtained together with a current term associated to the electromagnetic interaction between the electron and the photon.

2. Quantum Chromodynamics results by imposing the local gauge principle based on a non-Abelian $SU(3)$ symmetry to the Dirac equation for quarks. It explains the asymptotic freedom observed in strong interactions and the self-interaction of gauge bosons.

3. Electroweak theory results by imposing the local gauge principle for both Abelian and non-Abelian $U(1) \times SU(2)$ symmetry to the Dirac equation for the SM fermions. The non-Abelian symmetry, in this case, forbids the mass terms for the fermions, while the local gauge invariance, forbids the mass terms for the gauge bosons. Thus, electroweak symmetry force all particles to be massless. To be consistent with the spectrum of known fermions and bosons, four real scalar fields without mass are organized in what is known as the Higgs doublet. One of them acquires a vacuum expectation value, so spontaneously breaks the symmetry and generates mass for all fermions. Through the same mechanism the other three scalar fields can explain the masses for the $W^{\pm}$ and Z gauge bosons, whereas the photon (together with the neutrinos) remain massless.

Except by the neutrinos (which are chiral particles), the SM fermions can be decomposed in right-handed and left-handed spinor components. In table 1.3 we have shown the transformation properties for the first generation of fermions under the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$. The sub-indices in th gauge group mean for colour, left (chiral) and hypercharge symmetries respectively.

	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$
L	1	**2**	$-\frac{1}{2}$
Q	**3**	**2**	$+\frac{1}{6}$
e_R	1	1	-1
u_R	**3**	1	$+\frac{2}{3}$
d_R	**3**	1	$-\frac{1}{3}$

Table 1.3: Transformation properties for the first generation of the SM fermions. Here $L = (\nu_L \; e_L)^{\mathrm{T}}$ and $Q = (u_L \; d_L)^{\mathrm{T}}$ are the $SU(2)_L$ doublets. The last column are the SM hypercharges.

The hypercharge values in table 1.3 can be obtained from the (Electro-Magnetic) charges (Q_{EM}) in table 1.1 by means of the Gellman Gell-Mann–Nishijima formula

$$Y = Q_{\mathrm{EM}} - I_3, \tag{1.1}$$

where $I_3 = \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$ is the diagonal generator of $SU(2)_L$. There

$$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}, \qquad \text{and} \qquad Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \tag{1.2}$$

are the $SU(2)_L$ doublets.

For instance, for the lepton doublet L we have $Q_{\mathrm{EM}} = \begin{pmatrix} 0 & \\ & -1 \end{pmatrix}$ and so we get

$$Y_{\nu_L, e_L} = \begin{pmatrix} 0 & \\ & -1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \\ & -\frac{1}{2} \end{pmatrix}. \tag{1.3}$$

1.2 Gauge symmetries

In this section, we will develop two of the main physical gauge theories. First we briefly explain the generalities of the Lorentz symmetry and why it is required when building physical theories. explaining the procedure to build a gauge-invariant action.

1.2.1 Lorentz invariance

Let us think in an experiment that spans a short time and undertaken in a small free-falling laboratory. In such a case, the Einstein's equivalence principle states that the effect of the gravity can be dropped so that the laws of physics remain identical to those observed by an inertial (in absence of gravity) laboratory in Minkowski spacetime. Henceforth, this means that, in a local neighborhood, space-time posses Lorentz invariance. Therefore, Einstein's observation tells us that gravitation is a gauge theory of the group $SO(1,3)$, the first nonabelian gauge theory ever proposed [27].

In order to obtain the $SO(3,1)$ generators, let us first consider the angular momentum operators which are the generators of the rotation group $SO(3)$, and satisfy

$$[J^i, J^j] = i\epsilon_{ijk} J^k, \tag{1.4}$$

so in components we have that

$$J^k = i\epsilon_{ijk} x^i \partial^j.$$

Next, we define a matrix representation of angular momentum operators by contracting the components with the anti-symmetric tensor ϵ_{ijk} as follows

$$\begin{aligned} J^{lm} := \epsilon_{lmk} J^k &= i\epsilon_{lmk}\epsilon_{ijk} x^i \partial^j \\ &= i(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi}) x^i \partial^j \\ &= i(x^l \partial^m - x^m \partial^l). \end{aligned}$$

Then, there are three generators. If we now generalize the last to 4 dimensions, there appear three additional generators J^{01}, J^{02}, J^{03}, corresponding to the Lorentz boosts

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu). \tag{1.5}$$

Furthermore, the six generators of $SO(3,1)$ satisfy

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\mu\rho}), \tag{1.6}$$

where $g^{\mu\nu}$ = diag(-1,1,1,1) is the metric tensor for the Minkowski space-time.

The Lorentz symmetry reveals the space-time homogeneity and isotropy. Then, all the Lagrangians that one constructs be requested to be invariant under such gauge symmetry.

1.2.2 $U(1)$ symmetry

Let us consider a free Dirac fermion field (like the electron) given by the wave function $\psi(x) \in \Gamma(M, E)$, where $x \in M$ means for space-time points and E for a $U(1)$-bundle whose fibers are sections of the tensor product between some spinor bundle and an associated bundle. From quantum mechanics, we know that the complex number $\psi(x)$ (which is just the number whose square give us the relative probability of finding the object at x) is completely defined by a normalized state in a complex Hilbert space. Then, there is an implicit phase factor $e^{i\theta}$ of freedom in the definition of such state

$$\parallel \psi(x) \parallel = \parallel e^{i\theta}\psi(x) \parallel .$$

Next, lets suppose we decide to make an arbitrary phase change of the wave function at each space-time point. If this phase change is global, that is, if the phase change associated with the angle θ is the same at any space-time point, this change will not destroy the delicate balance between kinetic energy and potential energy in the Scrödinger equation. In summary

$$\begin{aligned} \theta = \text{constant} \quad &\Leftrightarrow \quad \text{global phase change}, \\ \theta = \theta(x) \quad &\Leftrightarrow \quad \text{local phase change}. \end{aligned}$$

Moreover, to be consistent with the causality principle of the special relativity, all the interactions are required to be invariant under independent (local) change of phases at all space-time points [28]. This means that, for a (local) transformation of the form

$$\psi(x) \to e^{i\theta(x)}\psi(x)\,, \tag{1.7}$$

the free Dirac Lagrangian

$$\mathcal{L}_0 = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi \tag{1.8}$$

is not invariant, since

$$\partial_\mu\psi \to \partial_\mu(e^{i\theta(x)}\psi) = e^{i\theta(x)}[i\partial_\mu\theta(x) + \partial_\mu]\psi,$$

does not preserve the primitive form. Therefore, unless we change the ordinary derivative ∂_μ, to compensate the term coming from the derivative of $e^{i\theta(x)}$, the free Lagrangian in Eq. (1.8) will not be invariant. Thus, we should replace ∂_μ by the covariant derivative defined by

$$D_\mu = \partial\mu + X_\mu, \tag{1.9}$$

where the new term introduced must have an index μ like that of normal derivative and should transforms as follows

$$X_\mu \to X_\mu - i\partial_\mu\theta(x)\,.$$

Here, it is convenient to redefine X_μ in terms of a new spin-1 (since $\partial_\mu\theta$ has a Lorentz index [29]) field and appropriate constants

$$A_\mu := \frac{1}{iq}X_\mu. \tag{1.10}$$

Hence, the covariant derivative can be conveniently written as

$$D_\mu = \partial\mu + iqA_\mu, \tag{1.11}$$

where the new field should transforms as follows

$$A_\mu \to A_\mu - \frac{1}{q}\partial_\mu\theta(x), \tag{1.12}$$

where q acts as the generator of the group $U(1)$ while $\theta(x)$ is the local transformation parameter.

Consequentially, we are able to define a Lagrangian which is invariant under local phase (or gauge $U(1)$) transformations as follows

$$\begin{aligned}\mathcal{L} &= i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi \\ &= \mathcal{L}_0 + qA_\mu\bar{\psi}\gamma^\mu\psi.\end{aligned} \tag{1.13}$$

In addition, one can note that the principle of local (gauge) invariance has generated an interaction between the Dirac spinor (electron) and the gauge field A_μ (photon), which is the Quantum Electrodynamics (QED) vertex (see section 2.4).

From a different point of view, one may note that the electron's energy-momentum appears in the phase of its wave function as follows

$$\psi(x) \propto e^{i(Et-\boldsymbol{p}\cdot\boldsymbol{x})}. \tag{1.14}$$

Then, the transformation in Eq. (1.7) would change the energy and the momentum of the electron. So, there should be a (new) field that compensates these changes to ensure energy-momentum conservation for the entire system.

The only remaining ingredient, to have an authentically propagating A_μ field, is a gauge invariant kinetic term of the form

$$\mathcal{L}_k = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{1.15}$$

where $F^{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength. Note also that a mass term $m^2 A^\mu A_\mu$, is forbidden because it would violate gauge invariance.

Summarizing, one could enunciate the gauge principle as follows: the invariance of the Lagrangian under local transformations implies the existence of one gauge field (the photon) corresponding to each generator of the gauge group ($U(1)$ in this case).

1.3 The Higgs mechanism

In this section we will go though the procedure by which gauge bosons acquire their masses.

1.3.1 Abelian Higgs model

By what we learned in the last section, the photon should be massless as required by the $U(1)$ symmetry. Then, one would like to know if is there another possibility to give mass to the photon (just like in superconductors)

Let us consider a complex scalar field ϕ with charge $-e$ which couples to the photon. In this case

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - V(\phi), \tag{1.16}$$

so, if we look for the most general renormalizable potential invariant under $\phi(x) \to e^{-ie\eta(x)}\phi(x)$, then we get

$$V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2, \tag{1.17}$$

where we assume that $\lambda > 0$, otherwise an unbounded potential from below would implies that there not exists a state of minimum energy [30]. The scalar mass term μ^2 will split the theory in two possibilities as it can be either $\mu^2 < 0$ or $\mu^2 > 0$.

For the first case $\mu^2 < 0$, the potential preserves the Lagrangian's symmetries as shown in figure 1.1. Then, we have a massless photon and a charged scalar field ϕ with mass μ.

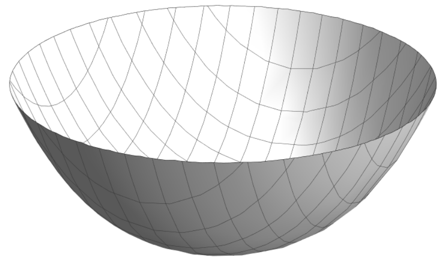


Figure 1.1: Scalar potential with $\mu^2 < 0$

The further option $\mu^2 > 0$, (figure 1.2) lead us to a minimum energy state (which breaks the gauge symmetry) given by

$$\langle \phi \rangle = \sqrt{\frac{-\mu^2}{2\lambda}} := \frac{v}{\sqrt{2}}, \tag{1.18}$$

where $\langle \phi \rangle$ is known as the vacuum expectation value (VEV) of ϕ.

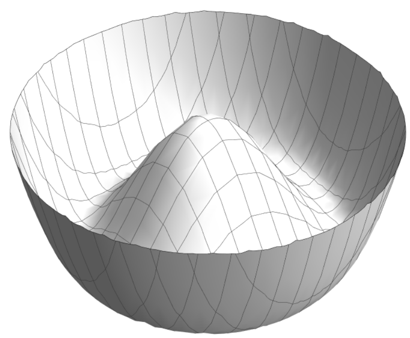


Figure 1.2: Scalar potential with $\mu^2 > 0$

Without loss of generality, the direction of the vacuum can be chosen to lie along the direction of the real axis of ϕ. Next, we parametrized ϕ in polar form (splitting its real and imaginary parts) as

$$\phi := e^{i\frac{\chi}{v}} \frac{(v+h)}{\sqrt{2}}, \tag{1.19}$$

where $\chi(x)$ and $h(x)$ are real fields which don not acquire any VEV's. Then, the gauge transformation for the field in polar coordinates states that

$$\phi(x) \to e^{i\theta(x)} e^{i\frac{\chi(x)}{v}} \left(\frac{v + h(x)}{\sqrt{2}} \right). \tag{1.20}$$

Next, if we choose $\eta(x) = -\frac{\chi(x)}{ev}$, which is called the 'unitary gauge', then we get

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^\dagger D^\mu\phi - \left(-\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2\right) \\ &\to -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\frac{1}{2}\left[\partial_\mu h + iqA_\mu(v+h)\right]\left[\partial_\mu h - iqA_\mu(v+h)\right] + \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{4}\lambda(v+h)^4 \\ &= -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\partial^\mu h\partial_\mu h + \frac{1}{2}q^2A^\mu A_\mu(v+h)^2 + \frac{1}{2}\mu^2(v+h)^2 - \frac{1}{4}\lambda(v+h)^4,\end{aligned} \tag{1.21}$$

so this Lagrangian contains the following terms

$$\mathcal{L} \supseteq \frac{1}{2}q^2v^2A^\mu A_\mu + q^2vA^\mu A_\mu h + \frac{1}{2}q^2A^\mu A_\mu h^2. \tag{1.22}$$

Then, as a consequence of the spontaneously symmetry breaking the gauge field A_μ has acquires mass

$$m_A = qv. \tag{1.23}$$

The mechanism by means the gauge bosons acquire masses from an gauge invariant Lagrangian is known as the Brout–Englert–Higgs [31, 1].

Finally, it is important to examine what happened with the two initial scalar degrees of freedom. From Eqs.(1.21) and (1.18), we see that there is one scalar particle h that acquires mass, which is called the Higgs boson. For this part, the other one is absorbed by gauge field (to give their masses) as a longitudinal mode and is known as the Goldstone boson.

1.3.2 Gauge bosons masses

Let us consider now the gauge symmetry $SU(2) \times U(1)$. Then

$$\begin{aligned}W_\mu = \frac{1}{2}W_\mu^a\tau_a &= \frac{1}{2}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}W_\mu^1 + \frac{1}{2}\begin{pmatrix} & -i \\ i & \end{pmatrix}W_\mu^2 + \frac{1}{2}\begin{pmatrix} 1 & \\ & -1\end{pmatrix}W_\mu^3 \\ &= \frac{1}{2}\begin{pmatrix} W_\mu^3 & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 + iW_\mu^2 & -W_\mu^3\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & -W_\mu^3\end{pmatrix}\end{aligned} \tag{1.24}$$

where we have defined $W_\mu^+ := \sqrt{2}W_\mu^1 - iW_\mu^2$, and $W_\mu^- := \sqrt{2}W_\mu^1 + iW_\mu^2$. On the other hand, the covariant derivative is now given by

$$\begin{aligned}\mathcal{D}_\mu &= \partial_\mu - i\frac{g_2}{2}W_\mu^a\tau_a - ig_1Y_\Phi B_\mu \\ &= \begin{pmatrix} \partial_\mu - i\frac{g_2}{2}W_\mu^3 - ig_1YB_\mu & -i\frac{1}{\sqrt{2}}g_2W\mu^+ \\ -i\frac{1}{\sqrt{2}}g_2W\mu^- & \partial_\mu + i\frac{g_2}{2}W_\mu^3 - ig_1YB_\mu\end{pmatrix}.\end{aligned} \tag{1.25}$$

Next, to calculate $\mathcal{D}_\mu \Phi$, where $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ is a scalar doublet with hypercharge 1, we make use of the unitary gauge as before to get

$$\mathcal{D}_\mu \Phi = \mathcal{D}_\mu \begin{pmatrix} 0 \\ \frac{v+h}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\frac{i}{\sqrt{2}} g_2 W_\mu^+ (v+h) \\ \partial_\mu h + i \left(\frac{g_2}{2} W_\mu^3 - g_1 Y B_\mu \right) (v+h) \end{pmatrix}.$$

Then, the scalar kinetic Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{kin} =& \left(\mathcal{D}_\mu \Phi\right)^\dagger \mathcal{D}_\mu \Phi \\ =& \left(\tfrac{i}{\sqrt{2}} g_2 W^{\mu -} (v+h),\ \partial^\mu h - i(\tfrac{1}{\sqrt{2}} g_2 W_3^\mu - g_1 Y B^\mu) \right) \begin{pmatrix} -\frac{i}{\sqrt{2}} g_2 W_\mu^+ (v+h) \\ \partial_\mu h + i \left(\frac{g_2}{2} W_\mu^3 - g_1 Y B_\mu \right) (v+h) \end{pmatrix} \\ =& \frac{1}{2} \Bigg(\underbrace{\partial^\mu h \partial_\mu h + \frac{1}{2} g_2^2 W^{\mu -} W_\mu{}^+ (v+h)^2}_{\mathcal{L}_{\mathrm{WBH}}} + \underbrace{\frac{1}{2} g_2 W_3^\mu - g_1 Y B_\mu}_{\mathcal{L}_{\mathrm{ZAH}}} \Bigg), \end{aligned} \tag{1.26}$$

so, for the charged bosons $W^\pm$ we have

$$\begin{aligned} \mathcal{L}_{\mathrm{WBH}} &= \frac{1}{4} g_2^2 W^{\mu -} W_\mu{}^+ (v+h)^2 \\ &\supseteq \frac{1}{4} g_2^2 v^2 W^{\mu -} W_\mu{}^+, \end{aligned} \tag{1.27}$$

which implies that the masses for the charged gauge bosons are

$$M_{W^\pm} = \frac{1}{2} g_2 v. \tag{1.28}$$

Next, for the Z and the photon we have

$$\begin{aligned} \mathcal{L}_{\mathrm{ZAH}} &= \frac{1}{2} \left(\frac{1}{2} g_2 W_3^\mu - g_1 Y B^\mu \right)^2 (v+h)^2 \\ &= \frac{1}{2} \left(\frac{1}{4} g_2^2 W_3^\mu W_\mu^3 - \frac{1}{2} g_1 g_2 Y W_3^\mu B^\mu + \frac{1}{4} g_1^2 Y^2 B^\mu B_\mu \right)^2 (v+h)^2 \\ &= \frac{1}{8} \left(W_3^\mu, \quad B^\mu \right) \begin{pmatrix} g_2^2 & -g_1 g_2 \\ -g_1 g_2 & g_1^2 \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix} (v+h)^2 \\ &= \frac{1}{8} \left(Z^\mu, \quad A^\mu \right) \begin{pmatrix} m_Z^2 & \\ & m_A^2 \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} (v+h)^2, \end{aligned} \tag{1.29}$$

where

$$\begin{aligned} m_Z^2 &= \frac{1}{2} \left(g_1^2 + g_2^2 + \sqrt{(g_1^2 - g_2^2)^2 + 4 g_1^2 g_2^2} \right) \\ &= g_1^2 + g_2^2, \end{aligned} \tag{1.30}$$

and

$$
\begin{aligned}
m_A^2 &= \frac{1}{2}\left(g_1^2+g_2^2-\sqrt{(g_1^2-g_2^2)^2+4g_1^2g_2^2}\right)\\
&= 0.
\end{aligned}
\tag{1.31}
$$

Also, we have that

$$
\tan(2\theta_W)=\frac{-2g_1g_2}{g_2^2-g_1^2} \quad\Rightarrow\quad \cos\theta_W=g_2, \quad \text{and} \quad \sin\theta_W=g_1. \tag{1.32}
$$

Then

$$
\begin{aligned}
\mathcal{L}_{\text{ZAH}} &= \frac{1}{8}(g_1^2+g_2^2)(v+h)^2Z^\mu Z_\mu\\
&= \frac{1}{2}\frac{g_2^2}{4}(1+\tan^2\theta_W)(v+h)^2Z^\mu Z_\mu\\
&\supseteq \frac{1}{2}\left(\frac{g_2 v}{2\cos\theta_W}\right)^2 Z^\mu Z_\mu,
\end{aligned}
\tag{1.33}
$$

so the Z boson mass is given by

$$
M_Z=\frac{g_2 v}{2\cos\theta_W}. \tag{1.34}
$$

1.4 Beyond the SM: 2-Higgs doublets model

One of the most popular extensions to the standard model is the so called two Higgs doublets model (2HDM). In this model, rather than just one Higgs boson, there are two Higgs fields $\Phi_1=\begin{pmatrix}\phi_1^+\\ \phi_1^0\end{pmatrix}$ and $\Phi_2=\begin{pmatrix}\phi_2^+\\ \phi_2^0\end{pmatrix}$ which are doublets of the $SU(2)\times U(1)$ gauge group. The enlarged Yukawa sector for one family may be written as [32]

$$
\begin{aligned}
-\mathcal{L}_Y &= Q_L\Phi_1 y_u^i u_R + Q_L\Phi_1 y_d^i d_R + L_L\Phi_1 y_e^i e_R\\
&\quad + Q_L\Phi_2 y_u^i u_R + Q_L\Phi_2 y_d^i d_R + L_L\Phi_2 y_e^i e_R + \text{h.c.}.
\end{aligned}
\tag{1.35}
$$

This general 2HDM Yukawa interaction terms can lead to flavor changing neutral currents (FCNC) mediated by extra neutral scalars at tree level. To illustrate that, note that the neutral Higgs scalars ϕ will mediate FCNC of the form $\bar{u}u'\phi$ to tree level, where u and u' are two different up-type quarks. Current experimental efforts have shown that this kind of processes are very suppressed in the nature [33, 34, 35]. To avoid FCNC to tree level, it is necessary that all fermions with the same quantum numbers couple to one and the same Higgs doublet, as specified in table 1.4. This can be reached by implementing a $\mathbb{Z}_2$ symmetry of 'parity' [36], so that $\Phi_1\to-\Phi_1$, and $\Phi_2\to+\Phi_2$. So, by taking as a convention $u_R\to+u_R$, and depending on the parity of the remaining SM fermions we have the following four conserving flavour models:

1. When $e_R \to -e_R$, and $d_R \to +d_R$, we get the so called **Lepton Specific** model, which is defined by the following Yukawa Lagrangian

$$-\mathcal{L}_Y = Q_L\Phi_2 y_u^i u_R + Q_L\Phi_2 y_d^i d_R + L_L\Phi_1 y_e^i e_R. \tag{1.36}$$

2. For the converse case $e_R \to +e_R$, and $d_R \to -d_R$, we obtain the **Flipped** model, which is defined by the following Yukawa terms

$$-\mathcal{L}_Y = Q_L\Phi_2 y_u^i u_R + Q_L\Phi_1 y_d^i d_R + L_L\Phi_2 y_e^i e_R. \tag{1.37}$$

3. Next, when $e_R \to +e_R$, and $d_R \to +d_R$, we have the **Type-I** model. This is determined by the Lagrangian

$$-\mathcal{L}_Y = Q_L\Phi_2 y_u^i u_R + Q_L\Phi_2 y_d^i d_R + L_L\Phi_2 y_e^i e_R. \tag{1.38}$$

4. The last option is $e_R \to -e_R$, and $d_R \to -d_R$, so we arrive to the **Type-II** model. Its characteristic Yukawa Lagrangian is given by

$$-\mathcal{L}_Y = Q_L\Phi_2 y_u^i u_R + Q_L\Phi_1 y_d^i d_R + L_L\Phi_1 y_e^i e_R. \tag{1.39}$$

Model	u_R	d_R	e_R
Type I	Φ_2	Φ_2	Φ_2
Type II	Φ_2	Φ_1	Φ_1
Lepton-specific	Φ_2	Φ_2	Φ_1
Flipped	Φ_2	Φ_1	Φ_2

Table 1.4: Models with flavour conservation. We have used the convention where u_R always couples to Φ_2.

The most general CP-conserving potential for the 2HDM [37, 38] is given by

$$\begin{aligned} V = &\mu_1^2|\Phi_1|^2 + \mu_2^2|\Phi_2|^2 - \mu_{12}^2\left[(\Phi_1^\dagger\Phi_2) + (\Phi_2^\dagger\Phi_1)\right] + \frac{\lambda_1}{2}|\Phi_1|^4 + \frac{\lambda_2}{2}|\Phi_2|^4 \\ &+ \lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) + \lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) + \frac{\lambda_5}{2}\left[(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2\right], \end{aligned} \tag{1.40}$$

where μ_{21}^2 means for the conjugate of μ_{12}^2. For further use, we will write explicitly the mixed terms as follows:

1. The λ_4 coefficient is given by

$$\begin{aligned} (\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) &= \left[\begin{pmatrix}\phi_1^-, & \phi_1^{0*}\end{pmatrix}\begin{pmatrix}\phi_1^+ \\ \phi_1^0\end{pmatrix}\right]\left[\begin{pmatrix}\phi_2^-, & \phi_2^{0*}\end{pmatrix}\begin{pmatrix}\phi_2^+ \\ \phi_2^0\end{pmatrix}\right] \\ &= (\phi_1^-\phi_1^+ + \phi_1^{0*}\phi_1^0)(\phi_2^-\phi_2^+ + \phi_2^{0*}\phi_2^0) \\ &= \phi_1^-\phi_1^+\phi_2^-\phi_2^+ + \phi_1^-\phi_1^+\phi_2^{0*}\phi_2^0 + \phi_1^{0*}\phi_1^0\phi_2^-\phi_2^+ + \phi_1^{0*}\phi_1^0\phi_2^{0*}\phi_2^0, \end{aligned} \tag{1.41}$$

2. The λ_3 coefficient is given by

$$(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) = \left[\begin{pmatrix}\phi_1^-, & \phi_1^{0*}\end{pmatrix}\begin{pmatrix}\phi_2^+\\ \phi_2^0\end{pmatrix}\right]\left[\begin{pmatrix}\phi_2^-, & \phi_2^{0*}\end{pmatrix}\begin{pmatrix}\phi_1^+\\ \phi_1^0\end{pmatrix}\right]$$
$$= (\phi_1^-\phi_2^+ + \phi_1^{0*}\phi_2^0)(\phi_2^-\phi_1^+ + \phi_2^{0*}\phi_1^0)$$
$$= \phi_1^-\phi_2^+\phi_2^-\phi_1^+ + \phi_1^-\phi_2^+\phi_2^{0*}\phi_1^0 + \phi_1^{0*}\phi_2^0\phi_2^-\phi_1^+ + \phi_1^{0*}\phi_2^0\phi_2^{0*}\phi_1^0. \quad (1.42)$$

3. The coefficient of λ_5 is given by

$$(\Phi_1^\dagger\Phi_2)^2 + (\Phi_2^\dagger\Phi_1)^2 = \left[\begin{pmatrix}\phi_1^-, & \phi_1^{0*}\end{pmatrix}\begin{pmatrix}\phi_2^+\\ \phi_2^0\end{pmatrix}\right]^2 + \left[\begin{pmatrix}\phi_2^-, & \phi_2^{0*}\end{pmatrix}\begin{pmatrix}\phi_1^+\\ \phi_1^0\end{pmatrix}\right]^2$$
$$= (\phi_1^-\phi_2^+ + \phi_1^{0*}\phi_2^0)^2 + (\phi_2^-\phi_1^+ + \phi_2^{0*}\phi_1^0)^2$$
$$= (\phi_1^-\phi_2^+)^2 + (\phi_1^{0*}\phi_2^0)^2 + 2\phi_1^{0*}\phi_2^0\phi_1^-\phi_2^+ + \text{h.c.}$$

The difference between the Eqs. (1.41) and (1.42) will be very useful in chapter 4 and it is given by

$$(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) - (\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1) = \phi_1^-\phi_1^+\phi_2^{0*}\phi_2^0 + \phi_1^{0*}\phi_1^0\phi_2^-\phi_2^+$$
$$- \phi_1^-\phi_2^+\phi_2^{0*}\phi_1^0 - \phi_1^{0*}\phi_2^0\phi_2^-\phi_1^+. \quad (1.43)$$

When $\mu_1^2 < 0$ and $\mu_2^2 < 0$ both fields acquire vacuum expectation values (VEV's) v_1 and v_2, which can be taken (by an unitary gauge transformation) as real (uncharged)

$$\langle\Phi_1\rangle_0 = \begin{pmatrix}0\\ \frac{v_1}{\sqrt{2}}\end{pmatrix}, \quad \langle\Phi_2\rangle_0 = \begin{pmatrix}0\\ \frac{v_2}{\sqrt{2}}\end{pmatrix}, \quad (1.44)$$

which are related to the total vacuum expectation value $v = \sqrt{v_1^2 + v_2^2} \approx 246$ GeV by

$$v_1 = v\cos\beta, \qquad \text{and} \qquad v_1 = v\sin\beta.$$

So we are able to express the VEV's ratio by

$$\tan\beta = \frac{v_2}{v_1}.$$

It is always possible to choose the phases of the scalar doublet fields so that both v_1 and v_2 are positive. Note also that it defines the free parameter β, which we will take in the range $0 \leq \beta \leq \frac{\pi}{2}$ [39]. Taking into account these definitions, the potential in Eq. (1.40) should satisfy the minimum conditions

$$\mu_1^2 = \mu_{12}^2\tan\beta - \frac{\lambda_1}{2}v_1^2 - \frac{\lambda_{345}}{2}v_2^2,$$
$$\mu_2^2 = \mu_{12}^2\cot\beta - \frac{\lambda_2}{2}v_2^2 - \frac{\lambda_{345}}{2}v_1^2. \quad (1.45)$$

with $\lambda_{345} := \lambda_3 + \lambda_4 + \lambda_5$.

After electro-weak symmetry breaking, a total of eight scalar degrees of freedom appears. Three of them are the Goldstone modes $G^\pm$ and G^0, which are absorbed to get the masses for SM gauge bosons $W^\pm$ and Z. The remaining are two charged Higgs bosons $H^\pm$, and three more: h, H (both uncharged and CP-even) and A (CP-odd) [40]

$$\Phi_1 = \begin{pmatrix} \phi_1^+ \\ \frac{\rho_1 + i\eta_1 + v_1}{\sqrt{2}} \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \phi_2^+ \\ \frac{\rho_2 + i\eta_2 + v_2}{\sqrt{2}} \end{pmatrix}, \tag{1.46}$$

where $\Re(\phi_i^0) = \rho_i + v_i$ and $\Im(\phi_i^0) = \eta_i$, for $i = 1, 2$.

1.4.1 Scalar masses and mixings

Charged sector

By using the minimum conditions (1.45), we may eliminate μ_1^2 and μ_2^2 from the potential (1.40). Now, by making $\Phi_i \to \begin{pmatrix} \phi_i^+ \\ \frac{v_i}{\sqrt{2}} \end{pmatrix}$ we can expand the potential to get

$$\begin{aligned} V = &\left(\mu_{12}^2 \tan\beta - \frac{\lambda_1}{2} v_1^2 - \frac{\lambda_{345}}{2} v_2^2\right)\left(\phi_1^+\phi_1^- + \frac{v_1^2}{2}\right) + \frac{\lambda_1}{2}\left(\phi_1^+\phi_1^- + \frac{v_1^2}{2}\right)^2 \\ &+ \left(\mu_{12}^2 \tan\beta - \frac{\lambda_2}{2} v_2^2 - \frac{\lambda_{345}}{2} v_1^2\right)\left(\phi_2^+\phi_2^- + \frac{v_2^2}{2}\right) + \frac{\lambda_2}{2}\left(\phi_2^+\phi_2^- + \frac{v_2^2}{2}\right)^2 \\ &- \mu_{12}^2\left[\left(\phi_2^+\phi_1^- + \frac{v_1 v_2}{2}\right) + \left(\phi_2^+\phi_1^- + \frac{v_1 v_2}{2}\right)\right] \\ &+ \lambda_3\left(\phi_1^+\phi_1^- + \frac{v_1^2}{2}\right)\left(\phi_2^+\phi_2^- + \frac{v_2^2}{2}\right) + \lambda_4\left(\phi_2^-\phi_1^+ + \frac{v_1 v_2}{2}\right)\left(\phi_2^+\phi_1^- + \frac{v_1 v_2}{2}\right) \\ &+ \frac{\lambda_5}{2}\left[\left(\phi_2^+\phi_1^- + \frac{v_1 v_2}{2}\right)^2 + \left(\phi_2^-\phi_1^+ + \frac{v_1 v_2}{2}\right)^2\right]. \end{aligned}$$

Then, the interesting potential terms are

$$
\begin{aligned}
V \supseteq & \left(\mu_{12}^2 \tan\beta - \frac{\lambda_1}{2} v_1^2 - \frac{\lambda_{345}}{2} v_2^2\right) \phi_1^+ \phi_1^- + \frac{\lambda_1}{2} \phi_1^+ \phi_1^- v_1^2 \\
& + \left(\mu_{12}^2 \cot\beta - \frac{\lambda_2}{2} v_2^2 - \frac{\lambda_{345}}{2} v_1^2\right) \phi_2^+ \phi_2^- + \frac{\lambda_2}{2} \phi_2^+ \phi_2^- v_2^2 \\
& - \mu_{12}^2 \left(\phi_2^+ \phi_1^- + \phi_2^- \phi_1^+\right) + \lambda_3 \left(\phi_1^+ \phi_1^- \frac{v_2^2}{2} + \phi_2^+ \phi_2^- \frac{v_1^2}{2}\right) \\
& + \lambda_4 \left(\phi_2^- \phi_1^+ + \phi_2^- \phi_1^+\right) \frac{v_1 v_2}{2} + \lambda_5 \left(\phi_2^+ \phi_1^- + \phi_2^- \phi_1^+\right) \frac{v_1 v_2}{2} \\
& = \mu_{12}^2 \left(\tan\beta(\phi_1^+ \phi_1^-) + \cot\beta(\phi_2^+ \phi_2^-) - \left(\phi_2^+ \phi_1^- + \phi_2^- \phi_1^+\right)\right) \\
& - \frac{\lambda_{45}}{2} v_1 v_2 \left[-\left(\phi_2^+ \phi_1^- + \phi_2^- \phi_1^+\right) + \phi_1^+ \phi_1^- \tan\beta + \phi_2^+ \phi_2^- \cot\beta\right],
\end{aligned}
\tag{1.47}
$$

where, we have defined $\lambda_{45} = \lambda_4 + \lambda_5$. Thus, the mass-squared matrix for the charged fields $\phi_1^\pm$ and $\phi_2^\pm$ can be diagonalized and the angle β is the rotation angle that performs that diagonalization. Therefore we have that

$$
\begin{aligned}
V \supset & \left(\mu_{12}^2 - \frac{\lambda_{45}}{2} v_1 v_2\right) \begin{pmatrix} \phi_1^- & \phi_2^- \end{pmatrix} \begin{pmatrix} \tan\beta & -1 \\ -1 & \cot\beta \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix} \\
= & \left(\mu_{12}^2 - \frac{\lambda_{45}}{2} v_1 v_2\right) \begin{pmatrix} G^\pm & H^\mp \end{pmatrix} \begin{pmatrix} 0 & \\ & \tan\beta + \cot\beta \end{pmatrix} \begin{pmatrix} G^\pm \\ H^\pm \end{pmatrix},
\end{aligned}
\tag{1.48}
$$

where the square-mass eigenstates are given by

$$
\begin{pmatrix} G^+ \\ H^+ \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \phi_1^+ \\ \phi_2^+ \end{pmatrix}
\tag{1.49}
$$

So we have a zero eigenvalue corresponding to the charged Goldstone boson $G^\pm$ which is eaten by the $W^\pm$. The another eigenvalue is

$$
\begin{aligned}
m_{H^\pm}^2 &= \left(\mu_{12}^2 - \frac{\lambda_{45}}{2} v_1 v_2\right) (\tan\beta + \cot\beta) \\
&= \left(\mu_{12}^2 - \frac{\lambda_{45}}{2} v_1 v_2\right) \frac{v_1^2 + v_2^2}{v_1 v_2} \\
&= \left(\frac{\mu_{12}^2}{v_1 v_2} - \frac{\lambda_{45}}{2}\right) v^2.
\end{aligned}
\tag{1.50}
$$

CP-even sector

Now for the real-uncharged fields ρ_1 and ρ_2 we make $\Phi_i \to \begin{pmatrix} 0 \\ \frac{\rho_i + v_i}{\sqrt{2}} \end{pmatrix}$, $i = 1, 2$, and taking into account Eq. (1.45) we can expand Eq.(1.40) to get

$$\begin{aligned} V = &\left(\mu_{12}^2 \tan\beta - \frac{\lambda_1}{2} v_1^2 - \frac{\lambda_{345}}{2} v_2^2\right) \left(\frac{\rho_1 + v_1}{\sqrt{2}}\right)^2 + \frac{\lambda_1}{2} \left(\frac{\rho_1 + v_1}{\sqrt{2}}\right)^4 \\ &+ \left(\mu_{12}^2 \tan\beta - \frac{\lambda_2}{2} v_2^2 - \frac{\lambda_{345}}{2} v_1^2\right) \left(\frac{\rho_2 + v_2}{\sqrt{2}}\right)^2 + \frac{\lambda_2}{2} \left(\frac{\rho_2 + v_2}{\sqrt{2}}\right)^4 \\ &- 2\mu_{12}^2 \left(\frac{\rho_1 + v_1}{\sqrt{2}}\right) \left(\frac{\rho_2 + v_2}{\sqrt{2}}\right) + \lambda_{345} \left(\frac{\rho_1 + v_1}{\sqrt{2}}\right)^2 \left(\frac{\rho_2 + v_2}{\sqrt{2}}\right)^2, \end{aligned}$$

so we have

$$\begin{aligned} V \supseteq &\left(\mu_{12}^2 \tan\beta - \frac{\lambda_1}{2} v_1^2 - \frac{\lambda_{345}}{2} v_2^2\right) \frac{\rho_1^2}{2} + \frac{\lambda_1}{2} \frac{6\rho_1^2 v_1^2}{4} \\ &+ \left(\mu_{12}^2 \cot\beta - \frac{\lambda_2}{2} v_2^2 - \frac{\lambda_{345}}{2} v_1^2\right) \frac{\rho_2^2}{2} + \frac{\lambda_2}{2} \frac{6\rho_2^2 v_2^2}{4} \\ &- 2\mu_{12}^2 \frac{\rho_1 \rho_2}{2} + \lambda_{345} \frac{\rho_1^2 v_2^2 + 4\rho_1 v_1 \rho_2 v_2 + \rho_2^2 v_2^2}{4} \\ &= \mu_{12}^2 \tan\beta \frac{\rho_1^2}{2} + \mu_{12}^2 \cot\beta \frac{\rho_2^2}{2} + \frac{\lambda_1}{2} v_1^2 \rho_1^2 + \frac{\lambda_2}{2} v_2^2 \rho_1^2 \\ &- \mu_{12}^2 \rho_1 \rho_2 + \lambda_{345} \rho_1 v_1 \rho_2 v_2 \\ &= \left(\mu_{12}^2 \tan\beta + \lambda_1 v_1^2\right) \frac{\rho_1^2}{2} + \left(\mu_{12}^2 \cot\beta + \lambda_2 v_2^2\right) \frac{\rho_2^2}{2} \\ &+ 2(-\mu_{12}^2 + \lambda_{345} v_1 v_2) \frac{\rho_1 \rho_2}{2}. \end{aligned}$$

Then, the mass-squared matrix of the real-uncharged fields ρ_1 and ρ_2 can be diagonalized as follows

$$\begin{aligned} V \supset &\frac{1}{2} \begin{pmatrix} \rho_1 & \rho_2 \end{pmatrix} \begin{pmatrix} \mu_{12}^2 \tan\beta + \lambda_1 v_1^2 & -\mu_{12}^2 + \lambda_{345} v_1 v_2 \\ -\mu_{12}^2 + \lambda_{345} v_1 v_2 & \mu_{12}^2 \cot\beta + \lambda_2 v_2^2 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \\ = &\frac{1}{2} \begin{pmatrix} H & h \end{pmatrix} \begin{pmatrix} m_H^2 & \\ & m_h^2 \end{pmatrix} \begin{pmatrix} H \\ h \end{pmatrix}. \end{aligned}$$

where the diagonalization is reached by a rotation in term s of the mixing angle α as

$$\begin{pmatrix} m_H^2 & \\ & m_h^2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \mathcal{M}^2 \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \tag{1.51}$$

where

$$\mathcal{M}^2 = \begin{pmatrix} \mathcal{M}_{11}^2 & \mathcal{M}_{12}^2 \\ \mathcal{M}_{12}^2 & \mathcal{M}_{22}^2 \end{pmatrix} = \begin{pmatrix} \mu_{12}^2 \tan\beta + \lambda_1 v_1^2 & -\mu_{12}^2 + \lambda_{345} v_1 v_2 \\ -\mu_{12}^2 + \lambda_{345} v_1 v_2 & \mu_{12}^2 \cot\beta + \lambda_2 v_2^2 \end{pmatrix},$$

and so, the mass (square) eigenstates h and H are attained by

$$\begin{aligned} H &= \rho_1 \cos\alpha + \rho_2 \sin\alpha, \qquad & h &= \rho_2 \cos\alpha - \rho_1 \sin\alpha, \\ \rho_1 &= H \cos\alpha - h \sin\alpha, \qquad & \rho_2 &= H \sin\alpha + h \cos\alpha, \end{aligned} \tag{1.52}$$

corresponding to the eigenvalues

$$m_{H,h}^2 = \frac{1}{2}\left(\mathcal{M}_{11}^2 + \mathcal{M}_{22}^2 \pm \sqrt{(\mathcal{M}_{11}^2 - \mathcal{M}_{22}^2)^2 + 4\mathcal{M}_{12}^4}\right), \tag{1.53}$$

The diagonalization angle is given by

$$\tan(2\alpha) = \frac{2\mathcal{M}_{12}^2}{\mathcal{M}_{11}^2 - \mathcal{M}_{22}^2}. \tag{1.54}$$

CP-odd sector

Finally, to get the mass terms for the pseudoscalar field A, we expand the potential (1.40) around $\Phi_i \to \begin{pmatrix} 0 \\ \frac{i\eta_i + v_i}{\sqrt{2}} \end{pmatrix}$, $i = 1, 2$,

$$\begin{aligned} V = &\left(\mu_{12}^2 \tan\beta - \frac{\lambda_1}{2} v_1^2 - \frac{\lambda_{345}}{2} v_2^2\right) \left|\left(\frac{i\eta_1 + v_1}{\sqrt{2}}\right)\right|^2 + \frac{\lambda_1}{2}\left(\frac{\eta_1^2 + v_1^2}{2}\right)^2 \\ &+ \left(\mu_{12}^2 \tan\beta - \frac{\lambda_2}{2} v_2^2 - \frac{\lambda_{345}}{2} v_1^2\right) \left|\left(\frac{i\eta_2 + v_2}{\sqrt{2}}\right)\right|^2 + \frac{\lambda_2}{2}\left(\frac{\eta_2^2 + v_2^2}{2}\right)^2 \\ &- \mu_{12}^2 \left[\left(\frac{-i\eta_1 + v_1}{\sqrt{2}}\right)\left(\frac{i\eta_2 + v_2}{\sqrt{2}}\right) + \left(\frac{i\eta_1 + v_1}{\sqrt{2}}\right)\left(\frac{-i\eta_2 + v_2}{\sqrt{2}}\right)\right] \\ &+ \lambda_{34}\left(\frac{\eta_1^2 + v_1^2}{\sqrt{2}}\right)\left(\frac{\eta_2^2 + v_2^2}{\sqrt{2}}\right) \\ &+ \frac{\lambda_5}{2}\left[\left(\frac{-i\eta_1 + v_1}{\sqrt{2}}\right)^2 \left(\frac{i\eta_2 + v_2}{\sqrt{2}}\right)^2 + \left(\frac{i\eta_1 + v_1}{\sqrt{2}}\right)^2 \left(\frac{-i\eta_2 + v_2}{\sqrt{2}}\right)^2\right]. \end{aligned}$$

So, after electroweak symmetry breaking the total potential includes the following mass terms

$$
\begin{aligned}
V \supseteq & \left(\mu_{12}^2 \tan\beta - \frac{\lambda_1}{2} v_1^2 - \frac{\lambda_{345}}{2} v_2^2\right) \frac{\eta_1^2}{2} + \frac{\lambda_1}{2} \frac{\eta_1^2 v_1^2}{2} \\
& + \left(\mu_{12}^2 \cot\beta - \frac{\lambda_2}{2} v_2^2 - \frac{\lambda_{345}}{2} v_1^2\right) \frac{\eta_2^2}{2} + \frac{\lambda_2}{2} \frac{\eta_2^2 v_2^2}{2} \\
& - \mu_{12}^2 \eta_1 \eta_2 + \frac{\lambda_{34}}{4} \left(\eta_1^2 v_2^2 + \eta_2^2 v_1^2\right) + \lambda_5 \eta_1 \eta_2 v_1 v_2 \\
& = \mu_{12}^2 \tan\beta \frac{\eta_1^2}{2} + \mu_{12}^2 \cot\beta \frac{\eta_2^2}{2} - \mu_{12}^2 \eta_1 \eta_2 \\
& + \lambda_5 \left(\eta_1 \eta_2 v_1 v_2 - \frac{v_1^2 \eta_2^2}{4} - \frac{v_2^2 \eta_1^2}{4}\right) \\
& = \frac{\mu_{12}^2}{2} \left(\tan\beta \eta_1^2 + \cot\beta \eta_2^2 - 2\eta_1 \eta_2\right) \\
& - \frac{\lambda_5}{2} \left(-2\eta_1 \eta_2 v_1 v_2 - \frac{v_1^2 \eta_2^2}{2} - \frac{v_2^2 \eta_1^2}{2}\right).
\end{aligned}
\tag{1.55}
$$

Then, the mass-squared matrix for the CP-odd η_1 and η_2 is given by

$$
\begin{aligned}
V \supset & \frac{1}{2} \left(\mu_{12}^2 - \lambda_5 v_1 v_2\right) \begin{pmatrix} \eta_1 & \eta_2 \end{pmatrix} \begin{pmatrix} \tan\beta & -1 \\ -1 & \cot\beta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \\
= & \frac{1}{2} \left(\mu_{12}^2 - \lambda_5 v_1 v_2\right) \begin{pmatrix} G^0 & A \end{pmatrix} \begin{pmatrix} 0 & \\ & \tan\beta + \cot\beta \end{pmatrix} \begin{pmatrix} G^0 \\ A \end{pmatrix},
\end{aligned}
\tag{1.56}
$$

where the square-mass eigenstates are given by

$$
\begin{pmatrix} G^0 \\ A \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{1.57}
$$

Therefore, we have a zero eigenvalue corresponding to the massless Goldston boson G^0 which gives mass to the vector boson Z, and the other one is given by

$$
\begin{aligned}
m_A^2 &= \left(\mu_{12}^2 - \lambda_5 v_1 v_2\right) (\tan\beta + \cot\beta) \\
&= \left(\mu_{12}^2 - \lambda_5 2 v_1 v_2\right) \frac{v_1^2 + v_2^2}{v_1 v_2} \\
&= \left(\frac{\mu_{12}^2}{v_1 v_2} - \lambda_5\right) v^2.
\end{aligned}
\tag{1.58}
$$

1.4.2 Interaction of the CP-even fields with the gauge bosons

By using equations (1.46) and (1.52), the interaction of the gauge bosons with the uncharged scalar fields h and H is given by

$$\begin{aligned}\mathcal{L} \supseteq & \left(\left(\frac{g_2}{4}\right)^2 W_\mu^- W^{\mu+} + \frac{1}{2}\left(\frac{g_2}{4\cos\theta_W}\right)^2 Z_\mu Z^\mu\right)(2\rho_1 v_1 + 2\rho_2 v_2) \\ = & \left(\left(\frac{g_2}{4}\right)^2 W_\mu^- W^{\mu+} + \frac{1}{2}\left(\frac{g_2}{4\cos\theta_W}\right)^2 Z_\mu Z^\mu\right) 2v_1\left(\rho_1 + \rho_2\tan\beta\right) \\ = & \left(\left(\frac{g_2}{4}\right)^2 W_\mu^- W^{\mu+} + \frac{1}{2}\left(\frac{g_2}{4\cos\theta_W}\right)^2 Z_\mu Z^\mu\right) \frac{2v_1\left(\rho_1\cos\beta + \rho_2\sin\beta\right)}{\cos\beta} \\ = & \left(\left(\frac{g_2}{4}\right)^2 W_\mu^- W^{\mu+} + \frac{1}{2}\left(\frac{g_2}{4\cos\theta_W}\right)^2 Z_\mu Z^\mu\right) 2v\left(\rho_1\cos\beta + \rho_2\sin\beta\right) \\ = & \left(\left(\frac{g_2}{4}\right)^2 W_\mu^- W^{\mu+} + \frac{1}{2}\left(\frac{g_2}{4\cos\theta_W}\right)^2 Z_\mu Z^\mu\right) 2v\left(H\cos(\alpha-\beta) - h\sin(\alpha-\beta)\right). \end{aligned} \tag{1.59}$$

In this way, the coupling of the gauge bosons ($V = W^\pm, Z$) with the CP-even scalar fields H and h are given by the factors $C_V^H := \cos(\alpha-\beta)$ and $C_V^h := -\sin(\alpha-\beta)$ repectively. As required by the so called 'alignment limit', the SM Higgs boson h should be approximately aligned with the direction of the scalar field vacuum expectation values [38, 41, 42], so that the gauge bosons $W^\pm$ and Z dominantly acquire their masses from only one Higgs doublet. In that case, the constrain $C_V^h \to 1$ must be satisfied.

1.4.3 Renormalization group equations

Here, we present the renormalization group equations (RGEs) for the 2HDM, which describe the behaviour for the gauge, Yukawa, and scalar couplings as functions of the energy scale.

Gauge couplings RGEs

The gauge couplings beta functions for two Higgs doublets can be obtained from Eq. (A.17) in Appendix A, and are given by

$$\begin{aligned} 16\pi^2\beta_{g_3} &= -7g_3^3, \\ 16\pi^2\beta_{g_2} &= -3g_2^3, \\ 16\pi^2\beta_{g_1} &= 7g_1^3, \end{aligned} \tag{1.60}$$

satisfying the low energy boundary conditions at the Z boson mass scale

$$g_1(91.19) = 0.36, \quad g_2(91.19) = 0.65, \quad g_3(91.19) = 1.2. \tag{1.61}$$

The behaviour of the gauge couplings in Eqs. (1.60) as function of the scale are shown in figure 1.3.

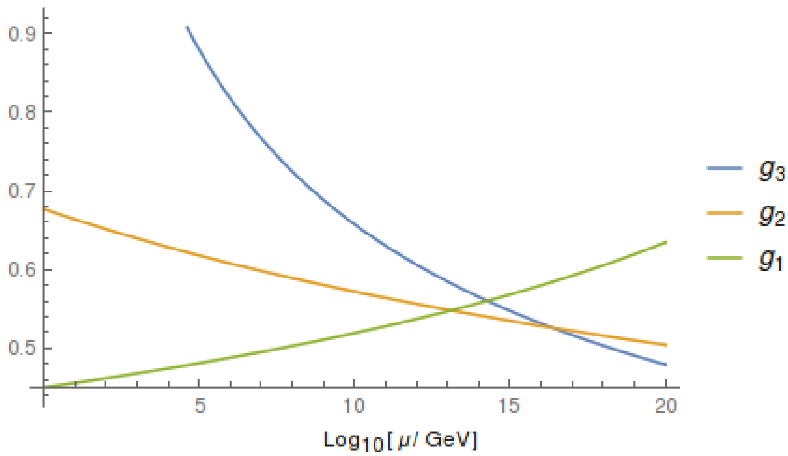


Figure 1.3: Gauge couplings running for the two Higgs doublet model. There is no matching point for the three couplings at any scale just like in the minimal SM.

RGEs for the scalar couplings

The quartic potential terms can be re-written in the general form [43]

$$V = -\frac{1}{4} f_{ijkl} \phi_i \phi_j \phi_k \phi_l. \tag{1.62}$$

We will not to go through the details on how to get the 1-loop RGEs for the scalar couplings. Instead, by defining the the differential operator [44] $\beta := \frac{\mathrm{d}}{\mathrm{d}\ \ln\mu} = \mu\frac{\mathrm{d}}{\mathrm{d}\mu}$, we start from the very general formula given by [37]

$$\begin{aligned}
16\pi^2 \beta_{\Lambda^{jl}_{km}} = & \sum_{p=1=q}^{2} 2\left(2\Lambda^{jp}_{kq}\Lambda^{ql}_{pm} + \Lambda^{jp}_{kq}\Lambda^{lq}_{pm} + \Lambda^{jp}_{qk}\Lambda^{ql}_{pm} + \Lambda^{jl}_{pq}\Lambda^{pq}_{km} + \Lambda^{jq}_{pm}\Lambda^{pl}_{kq}\right) \\
& + \sum_{p=1}^{2} T_{kp}\Lambda^{jl}_{pm} + T_{mp}\Lambda^{jl}_{kp} + T^*_{jp}\Lambda^{pl}_{km} + T^*_{lp}\Lambda^{jp}_{km} \\
& - 4\mathrm{Tr}(Y^\dagger_{je}Y_{ke}Y^\dagger_{le}Y_{me} + 3Y^\dagger_{jd}Y_{kd}Y^\dagger_{ld}Y_{md} + 3Y^\dagger_{ku}Y_{ju}Y^\dagger_{mu}Y_{lu} + 3Y^\dagger_{jd}Y_{lu}Y^\dagger_{mu}Y_{kd} \\
& + 3Y^\dagger_{ku}Y_{md}Y^\dagger_{ld}Y_{ju} - 3Y^\dagger_{jd}Y_{lu}Y^\dagger_{ku}Y_{md} - 3Y^\dagger_{mu}Y_{kd}Y^\dagger_{ld}Y_{ju}),
\end{aligned} \tag{1.63}$$

where $j,k,l,m \in \{1,2\}$ and $T_{ij} = \mathrm{Tr}(Y_{ie}Y^\dagger_{je} + 3Y_{ie}Y^\dagger_{ju} + 3Y_{id}Y^\dagger_{jd})$, and also one has $\Lambda^{11}_{11} = \lambda_1$, $\Lambda^{22}_{22} = \lambda_2$, $\Lambda^{12}_{12} = \Lambda^{21}_{21} = \lambda_3$, $\Lambda^{12}_{21} = \Lambda^{21}_{12} = \lambda_4$, $\Lambda^{11}_{22} = \Lambda^{22*}_{11} = \lambda_5$.

1.4.4 Topological objects in the 2HDM

Let us now to have a closer look to models which involve a degenerate vacua like the 2HDM. If the manifold of such a degenerate vacua is disconnected such that its homotopy group is non-trivial, then sheet-like topological defects, the so-called 'domain walls', are formed at the boundaries of the different degenerate vacua during the symmetry breaking phase transition [45]. Phase transitions producing domain walls occur at a finite rate and so, the fields can select different vacua in causally disconnected regions of space. This divides the universe into 'domains', where the interfaces between them are the domain walls. The walls have a tension under which they collapse as quickly as causality permits. This property results in an undesirable fate for the Universe where domain walls can be present in nature, in disagreement to current observations. The energy density of matter and radiation both scale proportionally to $(\text{time})^{-2}$ in their respective epochs of domination. However, domain wall energy density scales proportionally to $(\text{time})^{-1}$. This means that domain walls will come to dominate the universe at late times. This is the so-called domain wall problem. So, if a theory predicts the existence of cosmic domain walls some constraints must be placed such that they become unstable and collapsed before our current universe or, at least, such that domination occurs after present day [46].

In the 2HDM we can found many 'accidental' symmetries for the potential (1.40), like $\mathbb{Z}_2$, CP or $U(1)$, which may appear by some special choices of the parameters in the potential (1.40). If, for example, we only take real values for the parameters μ_{12}^2 and λ_5, then the potential (1.40) becomes 'CP-invariant'. When we allow the presence of terms which explicitly break such symmetries, the degeneracy of the vacua is removed and the scalar potential contains so-called true and false vacua. The true vacuum is the global minimum of the potential while the false vacuum is a local minimum with higher energy. The energy difference between these vacua produces a pressure on the domains of false vacuum causing the domain walls to collapse when this pressure becomes comparable to the surface tension of the walls. Therefore, the domain wall problem could be eliminated in this scenario if domain wall are sufficiently short-lived so that they do not survive long enough to dominate the energy density of the universe.

Now, we look out to the 2HDM potential (1.40), and investigate its behavior under the $U(1)$ global symmetry defined by

$$\Phi_1 \to e^{i\theta_1}\Phi_1, \quad \Phi_2 \to e^{i\theta_2}\Phi_2. \tag{1.64}$$

Note that the terms proportional to μ_{12}^2 and λ_5 are the only sources of breaking of the global $U(1)$ symmetry because

$$(e^{i\theta_1}\Phi_1)^\dagger(e^{i\theta_2}\Phi_2) = e^{i(\theta_2-\theta_1)}\Phi_1^\dagger\Phi_2. \tag{1.65}$$

In the special case when both μ_{12}^2 and λ_5 are zero, the potential (1.40) acquires an exact global phase $U(1)$ symmetry. When the VEVs v_1 and v_2 both acquire non vanishing values, then, after spontaneous symmetry breaking, this new symmetry preserves the $U(1)_{EM}$ as in the SM

$$SU(2)_L \times U(1)_Y \times U(1) \to U(1)_{EM}. \tag{1.66}$$

As it is pointed out in [47, 48], in this case the vacum manifold is not simply connected, since the homotopy group is nontrivial

$$\pi_1 \left(\frac{SU(2) \times U(1)_Y \times U(1)}{U(1)_{EM}} \right) \neq 1, \tag{1.67}$$

and so, resulting in topological stable vortices. As a consequence of the $U(1)$ symmetry breaking, a Nambu-Goldstone field appears. In this case, such a longitudinal mode is identified with the massless (see Eq. (1.58)) CP-odd pseudoscalar field A, which is ruled out. Therefore, in order to have a phenomenologically viable model, the parameters μ_{12}^2 and λ_5 should be taken different from zero.

Chapter 2

Noncommutative geometry

Conne's noncommutative geometry [49, 50, 51, 52, 53, 54], reconstructs Riemannian geometry by the algebraic concept of the spectral triple. When one turns the attention into noncommutative algebras, it results in a generalization of Riemann's geometry concepts to spaces with a finite number of elements. In this chapter, we start by introducing the main ideas of this type of geometry. Then we focus on almost-commutative geometry to build particle physics models. After introducing Krajewski diagrams, we will give the example of the Weinberg-Salam model in NCG to finally close with the most relevant aspects about the spectral action principle.

2.1 Commutative manifolds

In this section, I will explain how the usual topological and geometric notions on a manifold M can be replaced by spectral data in terms of operators on a Hilbert space.

Let us consider a finite-dimensional spin-manifold M and let $C^\infty(M)$ be the set of complex-valued coordinate (infinitely differentiable) functions. Then, we define the following operations:

- $(f+g)(x) = f(x) + g(x)$,
- $(f \cdot g)(x) = f(x) \cdot g(x)$,
- $(f.g)^*(x) = (g^* f^*)(x)$,

with these operations $C^\infty(M)$ becomes a $*$-algebra, commutative and associative.

Now, let us consider the spinor bundle $S \to M$, and among all its associated sections $\psi : M \to S$, we choose our spinor fields as given by those which are smooth and square-integrable $\psi \in L^2(M,S)$. Then, this sets up in a Hilbert space and the algebra $C^\infty(M)$ acts on it as follows:

$$(\hat{f} \cdot \psi)(x) = f(x) \cdot \psi(x), \tag{2.1}$$

where we have used the 'hat' to denote the representation of $f \in C^\infty(M)$ on $L^2(M,S)$. From Eq. (2.1) we can deduce that this representation is just the identity.

Consider now the (Euclidean) Dirac gamma matrices γ^μ given by

$$\gamma^0 = \begin{pmatrix} & 1_2 \\ 1_2 & \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} & -i\sigma_j \\ i\sigma_j & \end{pmatrix}, \tag{2.2}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{2.3}$$

From the spin (Levi-Civita) connection ∇^S on the bundle S we are able to build the curved Dirac operator $\not{D} := -i\gamma^\mu \nabla^S_\mu$. This is a first-order differential operator acting on the spinor fields, and satisfying the following

$$\nabla^S_\mu(\hat{f}\psi) = \hat{f}\nabla^S_\mu(\psi) + \partial_\mu(\hat{f})\psi, \tag{2.4}$$

for all $f \in C^\infty(M, \mathbb{C})$ and $\psi \in L^2(M, S)$.

Now, we define the usual chirality operator by

$$\gamma_5 := \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1_2 & \\ & 1_2 \end{pmatrix},$$

and the (anti-linear) charge conjugation operator (that interchanges particles and antiparticles) which is given by the product between the real γ matrices [55] in Eq. (2.2) as follows

$$J_M := \gamma^0\gamma^2 \circ \mathrm{cc} = \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & 1 \\ & & -1 & \end{pmatrix} \circ \mathrm{cc},$$

where 'cc ' means for complex conjugation . It is straightforward to show the consistency of the following relations

$$J_M^2 = -\mathbb{I}, \qquad J_M \not{D} = \not{D} J_M, \qquad J_M\gamma_5 = \gamma_5 J_M,$$

and as we will discuss in the next section, they define a KO-dimension = 4.

Another important fact is the boundedness of the commutator $[\not{D}, \hat{f}]$, which would allow the introduction of metric information for the spectral triple and to have well defined gauge bosons. Taking into account the Leibniz rule in Eq. (2.4) we get

$$\begin{aligned} [\not{D}, \hat{f}]\psi &= -i\gamma^\mu\nabla^S_\mu(\hat{f}\psi) + i\hat{f}\gamma^\mu\nabla^S_\mu(\psi) \\ &= -i\gamma^\mu\hat{f}\nabla^S_\mu(\psi) - i\gamma^\mu\partial_\mu(\hat{f})\psi + i\hat{f}\gamma^\mu\nabla^S_\mu\psi \\ &= -i[\gamma^\mu, \hat{f}]\nabla^S_\mu(\psi) - \underbrace{i\gamma^\mu\partial_\mu(\hat{f})\psi}_{\text{bounded}}, \end{aligned} \tag{2.5}$$

with $f \in C^\infty(M,\mathbb{C})$ and $\psi \in L^2(M,S)$. As the domain of f is bounded, then $\partial_\mu f \in C^\infty(M)$ will be bounded on the same domain. Now, by making use of Eq. (2.1) we have that $\hat{f} = f1_4$, so we get

$$\begin{aligned}[\gamma^\mu, \hat{f}] &= [\gamma^\mu, f1_4] \\ &= [\gamma^\mu, 1_4]f \\ &= 0.\end{aligned}$$

Hence, the commutator

$$[\not{D}, \hat{f}] = -i\gamma^\mu \partial_\mu(\hat{f})\psi, \tag{2.6}$$

is bounded. This fact, is necessary in order to have a a well notion of defined geodesic distance as we now see. Let us consider the Riemann's metric where the infinitesimal length element is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{2.7}$$

Thus, it is possible to define the geodesic distance between any pair of (space-time) points $x, y \in M$ as the infimum of the path lengths from x to y [8]

$$d_g(x,y) = \inf_\gamma \int_x^y ds. \tag{2.8}$$

Then, to extrapolate metric information to spectral geometry, the Eq. (2.8) is replaced by

$$d_D(x,y) = \sup\{|f(x) - f(y)| : f \in C^\infty(M,\mathbb{C}), ||[\not{D}, \hat{f}]|| \le 1\}. \tag{2.9}$$

Since the commutator in Eq. (2.6) is given by the Clifford multiplication of the gradient ∇f, it follows that its (operator) norm on $L^2(M,S)$ is given by

$$||[\not{D}, \hat{f}]|| = \sup_{x \in M} ||\nabla f||. \tag{2.10}$$

In this sence, the Dirac operator encodes the metric information on a spectral triple.

Therefore, we can defined the 'canonical' spectral triple associated to a Riemannian compact spin manifold M as given by

$$\{C^\infty(M,\mathbb{C}), L^2(M,S), \not{D}, J_M, \gamma_5\}.$$

In an analogous way, we may define the even spectral triple $\{A, H, D, J, \gamma\}$, where A is an $*$-algebra, which is represented by linear operators acting on the Hilbert space H, and D is a Hermitian operator on H. The remaining operators (also acting on H) are the invertible anti-linear J and the grading γ which decomposes the Hilbert space H into two eigenspaces.

Finally, we close by stating (without proof) the Connes reconstruction theorem:

Theorem. Consider an (even) spectral triple $\{A, H, D, J, \gamma\}$ whose algebra A is commutative. Then there exists a compact Riemannian spin manifold M (of even dimension), whose spectral triple $\{C^\infty(M,\mathbb{C}), L^2(M,S), \not{D}\}$ coincides with$\{A, H, D, J, \gamma\}$[8, 56].

Henceforth we will focus only on 4-dimensional spin-manifolds M such that it describes the space-time structure and the fermionic spinor fields.

2.2 Geometry from noncommutative algebra

In our previous definition of spectral triple, we will replace $C^\infty(M)$ for any associative and not necessarily commutative $*$-algebra A. In particular, we restrict ourselves to the case of a finite dimensional matrix algebra A. In this case, the Hilbert space H is a finite dimensional one, and the algebra is represented on H by

$$\begin{aligned} \rho : & A \to B(H) \\ & a \mapsto \rho(a), \end{aligned}$$

which in turn defines the following maps

$$\begin{aligned} \rho_L(a) : & H \to H \\ & h \mapsto \rho_L(a)h := ah \end{aligned} \qquad \text{and} \qquad \begin{aligned} \rho_R(a) : & H \to H \\ & h \mapsto \rho_R(a)h := ha. \end{aligned}$$

The Dirac operator D will be an Hermitian matrix. The chirality γ and charge conjugation J operators will be given by unitary and anti-unitary matrices, respectively, acting also on H.

The following relations should be satisfied

$$\gamma^2 = \mathbb{I}, \quad \gamma^* = \gamma^{-1}, \quad [\gamma, \rho(a)] = 0, \quad \{\gamma, D\} = 0, \quad J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J, \tag{2.11}$$

where the symbols $\epsilon, \epsilon', \epsilon'' \in \{1, -1\}$ are introduced in order to define the KO-dimension[1] (signature) modulo 8 as shown in table 2.1.

	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Table 2.1: Mod 8 KO-signature table.

If we replace $J \to J\gamma$ [58, 17], then, for the even signature cases, we get the values shown in table 2.2.

	0	2	4	6
ϵ	1	1	-1	-1
ϵ'	-1	-1	-1	-1
ϵ''	1	-1	1	-1

Table 2.2: Alternative selection for even KO-signature after $J \to J\gamma$.

Note that in both cases, a KO -dimension 6 will requires $\epsilon'' = -1 \Leftrightarrow \{J, \gamma\} = 0$ and $\epsilon = \epsilon' = -1 \Leftrightarrow (J^2 = 1 \wedge [J, D] = 0) \vee (J^2 = -1 \wedge \{J, D\} = 0)$

[1]The name 'KO' comes from the Bott periodicity theorems for real K-theory based on real vector bundles [57].

There are two remaining axioms that should be satisfied. These are known as the order zero and the order one conditions. In order to avoid the massless photon condition[2] we will consider the order two condition [59]

$$\text{Order zero condition:} \quad [\rho(a), J\rho(b^*)J^*] = 0, \tag{2.12a}$$
$$\text{Order one condition:} \quad [[D, \rho(a)], J\rho(b^*)J^*] = 0, \tag{2.12b}$$
$$\text{Order two condition:} \quad [[D, \rho(a)], J[D, \rho(b^*)]J^*] = 0. \tag{2.12c}$$

In terms of the left and right action operators, defined by

$$L_x(h) = xh, \quad \text{and} \quad R_x(h) = hx, \quad \text{for} \quad x \in A, \quad \text{and} \quad h \in H, \tag{2.13}$$

we can re-write the order conditions as follows

$$\text{Order zero:} \quad [L_x, R_y] = 0, \tag{2.14a}$$
$$\text{Order one:} \quad [[D, L_x], JL_{y^*}J^*] = 0, \tag{2.14b}$$
$$\text{Order two:} \quad [[D, L_x], J[D, L_y]^*J^*] = 0. \tag{2.14c}$$

In particular, for a nonassociative algebra (see Chapter 4) we have

$$\begin{aligned} [L_x, R_y]\, h &= L_x R_y h - R_y L_x h \\ &= x(hy) - (xh)y \\ &\neq 0, \end{aligned}$$

which means that (at least) the order zero axiom must be reinterpreted when working with nonassociative algebras.

2.3 Almost-commutative manifolds

Given two spectral triples $\{A_a, H_a, D_a; J_a, \gamma_a\}$ and $\{A_b, H_b, D_b; J_b, \gamma_b\}$, the product between them is defined by

- $A_a \otimes A_b$
- $H_a \otimes H_b$
- $D_a \otimes 1_b + \gamma_a \otimes D_b$
- $J_a \otimes J_b$
- $\gamma_a \otimes \gamma_b$

[2]In order to ensure that the photon remains massless, then the condition $[D_F, \rho(\alpha)] = 0$ for $\alpha = (a, \mathrm{diag}\{a, a^*\}, 0) \in \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ can be imposed [9]

A very special case, is when the canonical spectral triple and the one associated to any finite space are tensor together (figure 2.1). This is called ‘almost-commutative manifold’ and is the main structure in NCG to get particle physics models. The classification for such kind of spaces are possible thanks to Krajewski diagrams.

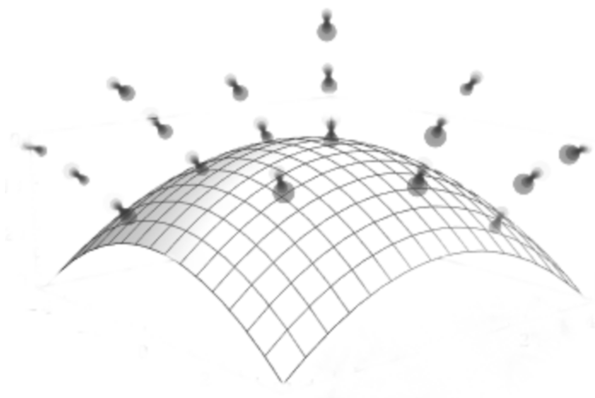


Figure 2.1: Almost-commutative manifold depicted as ‘finite’ fibers attached to the continuous space-time (base manifold).

2.3.1 Krajewski diagrams

Krajewski diagrams make possible the classification of almost commutative geometries [60, 61, 62, 63, 64, 65].

Let us consider the algebra $M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$, with N sumands. A Krajewski diagram for this algebra is defined by a network going from $N \times N$ to $2N \times 2N$ nodes (figure 2.2).

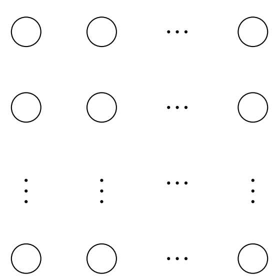


Figure 2.2: A network with a maximum of $2N \times 2N$ nodes.

Any couple of nodes sharing either the same row or the same column may be joined by a simple arrow. So ‘diagonal’ arrows are forbidden. The extremes of any arrow (starting or end points) will define the algebra representations, interpreted as the fermion degrees of freedom. The arrows by itself will define the the mass matrices and each one is interpreted as one block in the diagonal-block matrix $\mathcal{M}$ (which conforms the Dirac operator). A node should be either a source or a sink but not both at the same time. From any node, it can diverge or converge to a maximum of two arrows. Double arrows are allowed, and by the irreducibility of the representation [61, pag 7], one of its extremes should carry a 'double multiplicity' whereas the other extreme should have a 'single multiplicity'. To represent such a double multiplicity we can paint the node in black whereas the single one remains white. We can also add one extra row and column to stand for double multiplicity as shown in figure 2.3

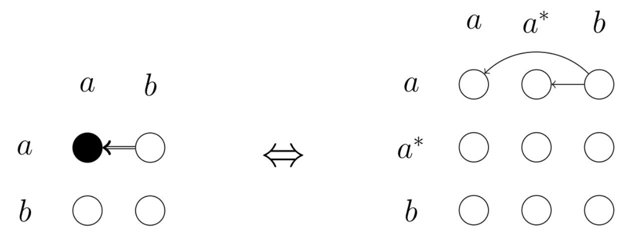


Figure 2.3: Equivalence between Krajewski diagrams with a double arrow.

The nodes will mean for both fermions and anti-fermions. If the arrows diverge from (converge to) it, then the node will be interpreted as left-handed (right-handed) algebra representations or fermions. For anti-fermions if the arrows diverge from (converge to) it, then the node will correspond to a right-handed (left-handed) anti-fermions.

Horizontal (vertical) arrows will mean for mass matrices whose dimension are given by taking the product between the multiplicity of the column (row) labeling its starting point with the multiplicity of the column (row) labeling its end point. Then, take the tensor product to the right (left) with the multiplicity labeling the row (column) where the arrow is located.

Examples

Let us consider the simple algebra $M_A(\mathbb{C})\oplus M_B(\mathbb{C})$. Then, we will show how to get the representation and Dirac operator associated to a specific Krajewski diagram. We will label each column (row) with an algebra element in black (blue) as well as with the corresponding algebra multiplicity in blue (black). To find the fermion (anti-fermion) representation we will use the labels in black (blue). The green and red colours will help us as a guide to calculate the representations.

1. Consider the Krajewski diagram depicted in figure 2.4.

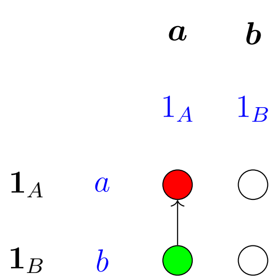


Figure 2.4: Krajewski diagram 1.

Then, the fermion and anti-fermion algebra representation is given by

$$\begin{aligned} \rho_L &= \left(a \otimes 1_B\right), & \rho_R &= \left(a \otimes 1_A\right), \\ \rho_L^C &= \left(b \otimes 1_A\right), & \rho_R^C &= \left(a \otimes 1_A\right). \end{aligned} \tag{2.15}$$

Meanwhile, the Dirac matrix blocks are given by

$$\mathcal{M} = \left(1_A \otimes M_{B\times A}\right) \tag{2.16}$$

2. For the same algebra of the last example, let us consider the Krajewski diagram shown in figure 2.5.

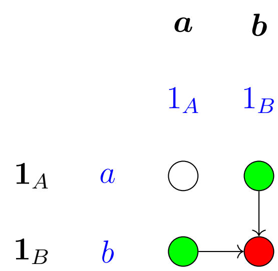


Figure 2.5: Krajewski diagram 2.

The corresponding representation and Dirac matrix are

$$\begin{aligned} \rho_L &= \begin{pmatrix} a \otimes 1_B & \\ & b \otimes 1_A \end{pmatrix}, & \rho_R &= \left(b \otimes 1_B\right), \\ \rho_L^C &= \begin{pmatrix} b \otimes 1_A & \\ & a \otimes 1_B \end{pmatrix}, & \rho_R^C &= \left(b \otimes 1_B\right). \end{aligned} \tag{2.17}$$

Next, the Dirac matrix blocks are given by

$$\mathcal{M} = \left(M_{A\times B} \otimes \mathbf{1}_B \quad 1_B \otimes M'_{A\times B}\right) \tag{2.18}$$

3. The Krajewski diagram in figure 2.6

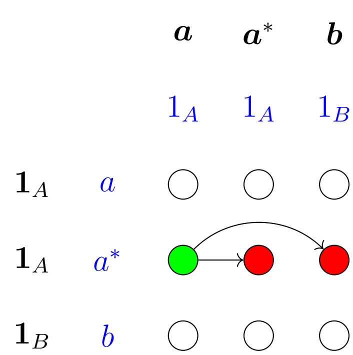


Figure 2.6: Krajewski diagram 3.

The corresponding representation is given by

$$
\begin{aligned}
\rho_L &= \begin{pmatrix} a \otimes 1_A \end{pmatrix}, \qquad & \rho_R &= \begin{pmatrix} a^* \otimes 1_A & \\ & b \otimes 1_A \end{pmatrix}, \\
\rho_L^C &= \begin{pmatrix} a^* \otimes 1_A \end{pmatrix}, \qquad & \rho_R^C &= \begin{pmatrix} a^* \otimes 1_A & \\ & a^* \otimes 1_B \end{pmatrix}.
\end{aligned} \tag{2.19}
$$

While the Dirac matrix blocks are then

$$
\mathcal{M} = \begin{pmatrix} M_{A\times A} \otimes \mathbf{1}_A, & M'_{A\times B} \otimes \mathbf{1}_A \end{pmatrix} \tag{2.20}
$$

2.3.2 Finite space for the SM

In terms of almost commutative manifolds, the SM is given by the tensor product between the canonical spectral triple with the one associated to finite algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. The corresponding Krajewski diagram is depicted in figure 2.7, [61, 66].

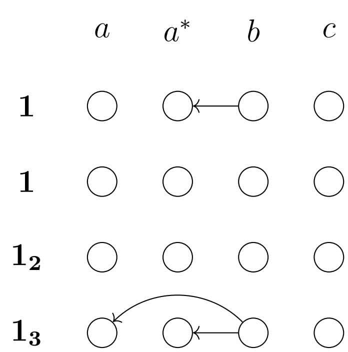


Figure 2.7: The minimal SM Krajewski diagram.

This diagram enable us to read off the representation ρ and the Dirac operator D_F as follows

$$
\rho = \left(\begin{array}{cc|cc} \rho_L & & & \\ & \rho_R & & \\ \hline & & \rho_L^C & \\ & & & \rho_R^C \end{array}\right), \qquad D_F = \left(\begin{array}{cc|cc} & \mathcal{M} & & \\ \mathcal{M}^\dagger & & & \\ \hline & & & \mathcal{M}^* \\ & & \mathcal{M}^T & \end{array}\right), \tag{2.21}
$$

where the symbols "$\dagger$" and "$*$" are the transpose conjugate and complex conjugation respectively. The blocks are given explicitly by

$$
\begin{aligned}
\rho_L &= \begin{pmatrix} b & \\ & b \otimes 1_3 \end{pmatrix}, \qquad & \rho_R &= \begin{pmatrix} a & & \\ & a \otimes 1_3 & \\ & & a^* \otimes 1_3 \end{pmatrix}, \\
\rho_L^C &= \begin{pmatrix} a 1_2 & \\ & 1_2 \otimes c \end{pmatrix}, \qquad & \rho_R^C &= \begin{pmatrix} a & & \\ & 1 \otimes c & \\ & & 1 \otimes c \end{pmatrix},
\end{aligned} \tag{2.22}
$$

and

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} & \\ & \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \otimes 1_3 \end{pmatrix}, \tag{2.23}$$

where $a, m_i, n_{ij} \in \mathbb{C}$, $b \in \mathbb{H}$ and $c \in M_3(\mathbb{C})$. We define the chirality operator by

$$\gamma_F = \left(\begin{array}{cc|cc} \gamma_L & & & \\ & \gamma_R & & \\ \hline & & \gamma_L^c & \\ & & & \gamma_R^c \end{array}\right), \tag{2.24}$$

where the left and right components on the particle basis are given by

$$\gamma_L = \begin{pmatrix} +1_2 & \\ & +1_2 \otimes 1_3 \end{pmatrix},$$

$$\gamma_R = \begin{pmatrix} -1 & & \\ & -1 \otimes 1_3 & \\ & & -1 \otimes 1_3 \end{pmatrix} = \begin{pmatrix} -1_2 & \\ & -1_2 \otimes 1_3 \end{pmatrix} = -\gamma_L,$$

and the antiparticle side is given by $\gamma_L^c = -\gamma_L$ and $\gamma_R^c = \gamma_L$. It is possible to show that the following relations are satisfied

$$\gamma^2 = 1_{30}, \qquad \gamma\rho = \rho\gamma, \qquad \gamma D_F = -D_F \gamma\,,$$

The charge conjugation matrix is given by

$$J = \left(\begin{array}{cc|cc} & & J_1 & \\ & & & J_1' \\ \hline J_2 & & & \\ & J_2' & & \end{array}\right) \circ \mathrm{cc}, \tag{2.25}$$

where[3] $J_1, J_2, J_1', J_2' \in \{\gamma_L, -\gamma_L\}$ should be selected such that the relation $J^2 = \epsilon 1_{30}$ is satisfied for $\epsilon \in \{-1, 1\}$. Its commutation relation with γ is given by

$$J\gamma = \left(\begin{array}{cc|cc} & & J_1\gamma_R^c & \\ & & & J_1'\gamma_L^c \\ \hline J_2\gamma_L & & & \\ & J_2'\gamma_R & & \end{array}\right) \circ \mathrm{cc} = \left(\begin{array}{cc|cc} & & -J_1\gamma_L & \\ & & & J_1'\gamma_L \\ \hline J_2\gamma_L & & & \\ & -J_2'\gamma_L & & \end{array}\right) \circ \mathrm{cc},$$

[3]Notice that it implies that $J_1, J_2, J_1', J_2', \gamma_L$, always commute.

$$\gamma J = \left(\begin{array}{cc|cc} & & \gamma_L J_1 & \\ & & & \gamma_R J_1' \\ \hline \gamma_R^c J_2 & & & \\ & \gamma_L^c J_2' & & \end{array}\right) \circ * = \left(\begin{array}{cc|cc} & & \gamma_L J_1 & \\ & & & -\gamma_L J_1' \\ \hline -\gamma_L J_2 & & & \\ & \gamma_L J_2' & & \end{array}\right) \circ \mathrm{cc}.$$

So we conclude that $J\gamma = \epsilon'' \gamma J$, for $\epsilon'' = -1$.

Now the product between D_F and J is given by

$$D_F J = \left(\begin{array}{cc|cc} & & & \mathcal{M} J_1 \\ & & \mathcal{M}^\dagger J_1' & \\ \hline & \mathcal{M}^* J_2 & & \\ \mathcal{M}^T J_2' & & & \end{array}\right) \circ \mathrm{cc},$$

$$J D_F = \left(\begin{array}{cc|cc} & & & J_1 \mathcal{M} \\ & & J_1' \mathcal{M}^\dagger & \\ \hline & J_2 \mathcal{M}^* & & \\ J_2' \mathcal{M}^T & & & \end{array}\right) \circ \mathrm{cc}.$$

So, for $D_F J = \epsilon' J D_F$ we have that $\epsilon' = 1$. Here, we make the identification to 1-generation of the standard model fermions

$$\begin{array}{ccccccc} b & \to & L, & \qquad & a \otimes 1_2 & \to & \overline{L}, \\ b \otimes 1_3 & \to & Q, & & 1_2 \otimes c & \to & \overline{Q}, \\ a & \to & e_R, & & a & \to & \overline{e_R}, \\ a \otimes 1_3 & \to & u_R, & & 1 \otimes c & \to & \overline{u_R}, \\ a^* \otimes 1_3 & \to & d_R, & & 1 \otimes c & \to & \overline{d_R}\,, \end{array}$$

where we have used the SM fields defined in table 1.3. The bar over the symbols denotes antiparticles. The finite Hilbert space of fermions is given by

$$H = \underbrace{H_L}_{\{L,Q\}} \oplus \underbrace{H_R}_{\{e_R,u_R,d_R\}} \oplus \underbrace{\overline{H_L}}_{\{\overline{L},\overline{Q}\}} \oplus \underbrace{\overline{H_R}}_{\{\overline{e_R},\overline{u_R},\overline{d_R}\}.}$$

Fermion quadrupling problem

When we take the tensor product between the canonical spectral triple and the finite space that we have already constructed, one finds that each fermion degree of freedom appears four times [67, 68]. To see this, let us consider only the electron in the finite space. Then

$$\mathcal{H} = \underbrace{L^2(M,S)}_{\text{Dirac spinors}} \otimes \left(\underbrace{H_L}_{e_L} \oplus \underbrace{H_R}_{e_R} \oplus \underbrace{\overline{H_L}}_{\overline{e_L}} \oplus \underbrace{\overline{H_R}}_{\overline{e_R}} \right), \tag{2.26}$$

where it is evident the quadrupling of the fermion particles. Then, if we split $L^2(M,S)$ according to γ^5, we have

$$\mathcal{H} = \Bigg(\underbrace{L^2(M,S)^+}_{\text{Right-Weyl spinor}} \oplus \underbrace{L^2(M,S)^-}_{\text{Left-Weyl spinor}} \Bigg) \otimes \Bigg(\underbrace{H_L}_{e_L} \oplus \underbrace{H_R}_{e_R} \oplus \underbrace{\overline{H_L}}_{\overline{e_L}} \oplus \underbrace{\overline{H_R}}_{\overline{e_R}} \Bigg), \tag{2.27}$$

and so, for the Weyl spinors ξ_L and η_R we have that $\Psi \in \mathcal{H}$ is given by

$$\Psi = \begin{pmatrix} \xi_L \\ \eta_R \end{pmatrix} \otimes (e_L + e_R + \overline{e_L} + \overline{e_R}) = \begin{pmatrix} \xi_L \otimes e_L + \xi_L \otimes e_R + \xi_L \otimes \overline{e_L} + \xi_L \otimes \overline{e_R} \\ \eta_R \otimes e_L + \eta_R \otimes e_R + \eta_R \otimes \overline{e_L} + \eta_R \otimes \overline{e_R} \end{pmatrix}. \tag{2.28}$$

Then, the total Hilbert space is given by

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-, \tag{2.29}$$

where the subspace $\mathcal{H}^-$ does not possess a definite chirality. Hence, the total Hilbert space is reduced to the set of 'classical fermions' [53]

$$\mathcal{H} \to \mathcal{H}^+ = \{\Psi \in \mathcal{H} : \gamma\Psi = \Psi\}, \tag{2.30}$$

and so, decreasing the fermion degrees of freedom to the half. Then $\Psi \in \mathcal{H}^+$is given by

$$\Psi = \xi_L \otimes (e_L + \overline{e_R}) \oplus \eta_R \otimes (e_R + \overline{e_L}). \tag{2.31}$$

Still doubling the number of fermions. Then, following [69], we need to impose the condition

$$J\Psi = \Psi. \tag{2.32}$$

Hence, taking into mind that $J(\xi \otimes e_L) = J_M\xi \otimes \overline{e_L}$ we get

$$\begin{aligned} J_M\xi_L \otimes (\overline{e_L} + e_R) \oplus J_M\eta_R \otimes (\overline{e_R} + e_L) &= \xi_L \otimes (e_L + \overline{e_R}) \oplus \eta_R \otimes (e_R + \overline{e_L}) \\ \Rightarrow \quad J_M\xi_L = \eta_R \quad &\wedge \quad J_M\eta_R = \xi_L, \end{aligned} \tag{2.33}$$

which give us the correct number of degrees of freedom.

An alternative possibility to obtain Eq. (2.32), is by defining an antisymmetric bilinear form $\langle J\Psi', D_\omega\Psi\rangle$ for any $\Psi', \Psi \in \mathcal{H}^+$ and then restricting it by [53, 70]

$$\langle J\Psi', D_\omega\Psi\rangle \to \langle J\Psi, D_\omega\Psi\rangle. \tag{2.34}$$

Regardless which one of the two options is select, the KO-dimension should be 6.

Then, in order to work on KO-dimension 6 one should take $J_1 = J_2$ and $J_1' = J_2'$ in Eq. (2.25) such that $J^2 = 1 \Rightarrow \epsilon = 1$.

2.3.3 Symmetries and gauge fields

Given any pair of spectral triples $\{A_a, H_a, D_a; J_a, \gamma_a\}$ and $\{A_b, H_b, D_b; J_b, \gamma_b\}$, we say that they are unitary equivalent if there exists an operator $U : H_a \to H_b$ such that the diagrams on figure 2.8 commute.

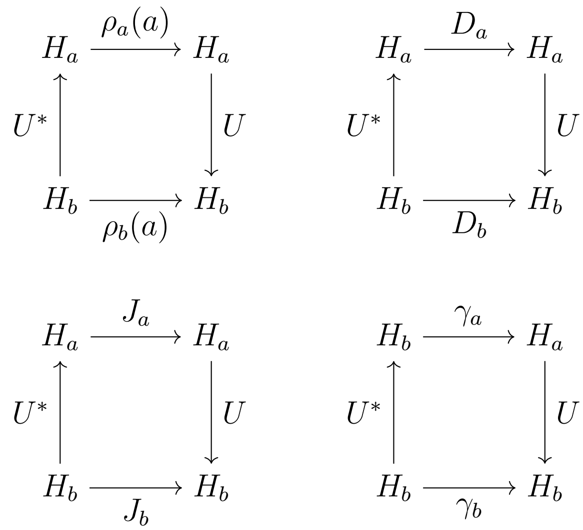


Figure 2.8: The commutativity of the diagrams mean that $U\rho_a(a)U^* = \rho_b(a)$, $UD_aU^* = D_b$, $UJ_aU^* = J_b$, and $U\gamma_aU^* = \gamma_b$.

In particular, if we take $U = \rho(u)J\rho(u)J^*$, for some unitary $u \in A$, we can prove that

$$\{A, H, D; J, \gamma\} \approxeq \{A, H, UDU^*; J, \gamma\}, \tag{2.35}$$

with a new representation given by

$$U\rho(a)U^* = u\rho(a)u^*,$$

corresponding to an *inner* automorphism. Moreover, if α is an automorphism then it is inner when

$$\alpha(a) = uau^*, \quad \text{for some } u \in A.$$

We define the gauge group $\mathfrak{G}$, associated to a spectral triple, by

$$\mathfrak{G} := \{U = \rho(u)J\rho(u)J^* | u \in \mathcal{U}(A)\}.$$

Then, given the map $f : \mathcal{U}(A) \to \mathfrak{G}$ defined by $\rho(u) \mapsto \rho(u)J\rho(u)J^*$, $\mathfrak{G}$ is isomorphic to $\mathcal{U}(A)/\mathcal{U}(A_J)$, where $A_J = \{a \in A | \rho(a) = J\rho(a^*)J^*\}$.

Now, let us consider $U = \rho(u)J\rho(u)J^*$ for any spectral triple, then, the procedure to change to a new (equivalent) spectral triple as in Eq. (2.35) defines the 'fluctuation' of the Dirac operator and is given by

$$\begin{aligned} D \to \quad UDU^* &= \rho(u)J\rho(u)J^*DJ\rho(u^*)J^*\rho(u^*) \\ &= D + \rho(u)[D, \rho(u^*)] + \epsilon'\rho(u)[D, \rho(u^*)]J^*. \end{aligned}$$

Thus, for an almost-commutative manifold with Dirac operator given by

$$D = \not{D} \otimes 1_{30} + \gamma_5 \otimes D_F,$$

there are two sorts of fluctuations:

1. $\hat{f}' \otimes \rho'(a)[\not{D} \otimes 1_{30}, \hat{f} \otimes \rho(a)] \Rightarrow$ Gauge fields.
2. $\hat{f}' \otimes \rho'(a)[\gamma_5 \otimes D_F, \hat{f} \otimes \rho(a)] \Rightarrow$ Higgs field,

with $\hat{f} \otimes \rho(a) \in C^\infty(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}))$.

In particular, the gauge fields $A_\mu = -ix'\partial_\mu x$ corresponding to the fluctuation term $x'[\not{D} \otimes 1_{32}, x]$, can be written as follows

$$A_\mu = \left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right)$$

Given $x = (a, b, c)$ and $y = (a', b', c')$, and the $U(1)$, $SU(2)$, $U(3)$ gauge fields

$$\Lambda_\mu = -ia'\partial_\mu a, \quad Q_\mu = -ib'\partial_\mu b, \quad \text{and} \quad V_\mu = -ic'\partial_\mu c$$

the blocks of A_μ can be written as

$$A_1 = \begin{pmatrix} Q_\mu & & & & \\ & Q_\mu \otimes 1_3 & & & \\ & & -\Lambda_\mu & & \\ & & & \Lambda_\mu \otimes 1_3 & \\ & & & & -\Lambda_\mu \otimes 1_3 \end{pmatrix},$$

acting on the particle basis $\{\nu, e_L, u_L, d_L, e_R, u_R, d_R\}$, and

$$A_2 = \begin{pmatrix} \Lambda_\mu \otimes 1_2 & & & \\ & V_\mu \otimes 1_2 & & \\ & & \Lambda_\mu & \\ & & & V_\mu \otimes 1_2 \end{pmatrix},$$

acting on the antiparticle basis $\{\overline{\nu}, \overline{e_L}, \overline{u_L}, \overline{d_L}, \overline{e_R}, \overline{u_R}, \overline{d_R}\}$. In order to make the reduction $U(3) \rightarrow SU(3)$, we should impose $\mathrm{Tr}(A_\mu) = 0$. Because Q_μ is trace-less then $\mathrm{Tr}(V_\mu) = -\Lambda_\mu$, and so we can change $V\mu$ by the traceless $SU(3)$ field $V_\mu \rightarrow -({V_\mu}^* - \frac{1}{3}\Lambda_\mu \otimes 1_3)$.

Next, we define $\delta_\mu \equiv A_\mu - J_F A_\mu J_F^\dagger$ so

$$\delta_\mu = \left(\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right) - \left(\begin{array}{c|c} A_2^\dagger & \\ \hline & A_1^\dagger \end{array}\right),$$

where the action on the particle basis is given by

$$A_1 - A_2^\dagger = \begin{pmatrix} Q_\mu - \Lambda_\mu 1_2 & & & & \\ & V_\mu 1_2 + (Q_\mu + \frac{1}{3}\Lambda_\mu 1_2)\otimes 1_3 & & & \\ & & -2\Lambda_\mu & & \\ & & & V_\mu + \frac{4}{3}\Lambda_\mu \otimes 1_3 & \\ & & & & V_\mu - \frac{2}{3}\Lambda_\mu \otimes 1_3 \end{pmatrix}, \tag{2.36}$$

and the action on the antiparticle basis is $A_2 - A_1^\dagger = -(A_1 - A_2^\dagger)$. In this way, for the particle basis $\{L, Q, e_R, u_R, d_R\}$ one can identify the SM fermion's hypecharges given by the factors to the left of the field Λ_μ [70, 71].

Note also that δ_μ corresponds to the derivations[4] of an associative algebra

$$\delta_x = L_x - R_{x^*}, \tag{2.37}$$

where x is an anti-Hermitian element of the algebra. Furthermore, we have introduced the left and right action operators which are related by

$$R_x = J L_{x^*} J^*. \tag{2.38}$$

2.3.4 The Glashow-Weinberg-Salam Model

The electroweak theory of Glashow-Weinberg-Salam has been described with great consistency by means of Connes' noncommutative geometry [70]. In this model, the fermion space is given by:

$$H = H_l \oplus H_{l^\dagger}, \tag{2.39}$$

where the space of leptons is given by $H_l = \{\nu_R, e_R, \nu_L, e_L\}$ and the space of anti-leptons is given by $H_{l^\dagger} = \{(\nu_R)^\dagger, (e_R)^\dagger, (\nu_L)^\dagger, (e_L)^\dagger\}$. The input algebra is given by $A = \mathbb{C} \oplus \mathbb{H}$, which is an associative algebra and is represented on H by:

$$L_{(a,b)} = \begin{pmatrix} b_a & & & \\ & b & & \\ & & a1_2 & \\ & & & a1_2 \end{pmatrix}, \tag{2.40}$$

where $a \in \mathbb{C}$, $b \in \mathbb{H}$ and $b_a = \begin{pmatrix} a & \\ & \overline{a} \end{pmatrix}$. Meanwhile the grading and real structure operator are given by:

$$\gamma = \begin{pmatrix} -1_2 & & & \\ & 1_2 & & \\ & & 1_2 & \\ & & & -1_2 \end{pmatrix}, \qquad \text{and} \qquad J = \begin{pmatrix} & 1_4 \\ 1_4 & \end{pmatrix} \circ \text{cc}. \tag{2.41}$$

[4]We will introduce the concept of algebra derivations in section 2.5.

The right action operator is given by Eq. (2.38)

$$R_{(a,b)} = JL^*_{(a,b)}J^{-1} = \begin{pmatrix} a1_2 & & & \\ & a1_2 & & \\ & & b_a & \\ & & & b \end{pmatrix}.$$

Furthermore, the anti-Hermitian elements of $\mathbb{C} \oplus \mathbb{H}$ are $i, i\sigma_1, i\sigma_2, i\sigma_3$, where σ_i are the Pauli matrices defined in Eq. (2.3). Then, from Eq. (2.37) we can write the algebra derivations as follows

$$\delta_0 := L_{(i,0)} - R_{(i,0)}, \qquad \delta_1 := L_{(0,i\sigma_1)} - R_{(0,i\sigma_1)},$$
$$\delta_2 := L_{(0,i\sigma_2)} - R_{(0,i\sigma_2)}, \qquad \delta_3 := L_{(0,i\sigma_3)} - R_{(0,i\sigma_3)},$$

and thus, we have that

$$\delta_0 = \begin{pmatrix} 0 & & & & & & & \\ & -2i & & & & & & \\ & & -i & & & & & \\ & & & -i & & & & \\ & & & & 0 & & & \\ & & & & & 2i & & \\ & & & & & & i & \\ & & & & & & & i \end{pmatrix}, \qquad \delta_1 = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & i & & & & \\ & & i & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & -i \\ & & & & & & -i & 0 \end{pmatrix},$$

$$\delta_2 = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & 0 & 1 \\ & & & & & & -1 & 0 \end{pmatrix}, \qquad \delta_3 = \begin{pmatrix} 0 & & & & & & & \\ & 0 & & & & & & \\ & & i & 0 & & & & \\ & & 0 & -i & & & & \\ & & & & 0 & & & \\ & & & & & 0 & & \\ & & & & & & -i & 0 \\ & & & & & & 0 & i \end{pmatrix}.$$

Here, one can identified the eigenvalues of δ_0 as the correct hypercharges for the leptons and anti-leptons $\{\nu_R, e_R, \nu_L, e_L, (\nu_R)^\dagger, (e_R)^\dagger, (\nu_L)^\dagger, (e_L)^\dagger\}$.

2.4 Spectral action

The dynamics for the bosons of the theory is described by the spectral action which is defined by

$$S_B := \mathrm{Tr}\left(f(\frac{D_\omega}{\Lambda}) \right). \tag{2.42}$$

The geometry of a manifold may be recovered from the spectrum of the Dirac operator by means of the heat kernel coefficients [72] and since D^2_ω is a generalized Laplacian (see [70, proposition 3.1]), i.e. $D_\omega^2 = \Delta^E - Q$, for some bundle $E \to M$ and section $Q \in \Gamma(\mathrm{End}(E))$, we can expand S_B by using the heat kernel methods (see appendix B)

$$S_B = \mathrm{Tr}\left(f(\frac{D_\omega}{\Lambda})\right) \approx 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2) + f_0 a_4(D_\omega^2) + O(\Lambda^{-1}), \tag{2.43}$$

where $f : \mathbb{R} \to \mathbb{R}$ is required to be even, positive and such that $f \to 0$ when the cutoff parameter $\Lambda \to \infty$, for $j \in \{2, 4\}$ $f_j = \int_0^\infty f(v)v^{j-1}dv$ and $f_0 = f(0)$. The coefficients $a_k(D_\omega^2)$ are given by

$$a_k(D_\omega^2) = \int_M a_k(x, D_\omega^2)\sqrt{|g|}d^4x, \tag{2.44}$$

where $a_k(x, D_\omega^2)$ are the Seeley-DeWitt coefficients and its explicit form is well know [10, 70]. In particular, they contain the full kinetic and potential bosonic Lagrangian terms as follows

$$\begin{aligned} a_2(x, D_\omega^2) &\supseteq -\frac{1}{4\pi^2}\mathrm{Tr}(\Phi^2), \\ a_4(x, D_\omega^2) &\supseteq -\frac{1}{8\pi^2}\left(\frac{1}{3}\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) + \mathrm{Tr}[(D_\mu\Phi)^\dagger D^\mu\Phi] + \mathrm{Tr}(\Phi^4)\right), \end{aligned}$$

where

$$\mathrm{Tr}(F^{\mu\nu}F_{\mu\nu}) = 80\mathrm{Tr}(\Lambda_{\mu\nu}\Lambda^{\mu\nu}) + 24\mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + 24\mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}),$$

and the curvature of the fields are given by

$$\begin{aligned} \Lambda_{\mu\nu} &= \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu, \\ Q_{\mu\nu} &= \partial_\mu Q_\nu - \partial_\nu Q_\mu + i[Q_\mu, Q_\nu], \\ V_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu + i[V_\mu, V_\nu]. \end{aligned}$$

Hence, the gauge sector is described by the action

$$S_B \supseteq \frac{f_0}{\pi^2}\int_M (\frac{10}{3}\mathrm{Tr}(\Lambda_{\mu\nu}\Lambda^{\mu\nu}) + \mathrm{Tr}(Q_{\mu\nu}Q^{\mu\nu}) + \mathrm{Tr}(V_{\mu\nu}V^{\mu\nu}))\sqrt{|g|}d^4x.$$

Next, we redefine the gauge bosons by means of

$$\Lambda_\mu := \frac{g_1}{2}Y_\mu, \qquad Q_\mu^a := \frac{g_2}{2}W_\mu^a, \qquad \text{and} \qquad V_\mu^i := \frac{g_3}{2}G_\mu^i.$$

Then, we impose the following normalization

$$\frac{f_0}{2\pi^2}g_3^2 = \frac{f_0}{2\pi^2}g_2^2 = \frac{f_0}{2\pi^2}\frac{5}{3}g_1^2 = \frac{1}{4}, \tag{2.45}$$

which corresponds to the usual grand unified relation for the gauge couplings [71]. So we get the following high energy condition

$$\frac{f_0}{\pi^2} = \frac{1}{2g^2}, \tag{2.46}$$

where g is the unified gauge coupling. Despite the fact that we do not have a matching point for the gauge couplings at high energies (neither for the minimal SM nor for the 2HDM) as shown in figure 1.3, we will assume the existence of such unification scale U.

2.5 The differential graded $*$-algebra approach to NCG

Here we introduce the basic ideas to generalize NCG to non-associative algebras by introducing the concept of differential graded $*$-algebras. In particular, we focus on reproduce the order axioms we saw in section 2.2 as well as to quickly review the symmetries in this scenario.

2.5.1 Shifting $A \to \Omega A$

Given any (not necessarily associative) algebra over a field $\mathbb{F}$, we define for any element $a \in A$ the formal symbols $\delta(a)$, so that the following properties are satisfied

$$\begin{aligned} \delta(\alpha a) &= \alpha\delta(a), \quad \alpha \in \mathbb{F} \\ \delta(a_1 + a_2) &= \delta(a_1) + \delta(a_2) \\ \delta(a_1 a_2) &= \delta(a_1)a_2 + a_1\delta(a_2), \end{aligned} \tag{2.47}$$

where $a_1, a_2 \in A$. In analogy to the notion of dual space in Riemannian geometry, these 'differentials' define the algebra $\Omega^1 A$ of one-forms, which in turn give rise to the graded algebra $\Omega A = \Omega^0 A \oplus \Omega^1 A \oplus \Omega^2 A \oplus \cdots$, where $\Omega^0 A = A$ and $\Omega^n A$ consists of linear combinations containing n differetials $\delta(a)$. This algebra is graded in the sense that if $\omega_n \in \Omega^n A$ and $\omega_m \in \Omega^m A$, then it is true that $\omega_n \omega_m \in \Omega^{n+m} A$. The δ's may be interpreted as linear maps $\delta : \Omega^0 \to \Omega^1$ and this notion can be extended to define the linear map $d : \Omega^m A \to \Omega^{m+1} A$, so that $d(a) = \delta(a)$ and the graded Leibniz rule is satisfied

$$d(\omega_n \omega_m) = d(\omega_n)\omega_m + (-1)^{n+m}\omega_n d(\omega_m). \tag{2.48}$$

Then, we have that $d^2(\omega) = 0$, for all $\omega \in \Omega A$ and the pair $(\Omega A, d)$ is known as a differential graded algebra. If in addition we have that $\delta(a^*) = \pm\delta(a)^*$, such a couple is called a differential graded $*$-algebra.

2.5.2 Shifting $H \to \Omega A \oplus H$

We start by introducing the Eilenberg's notion of bi-representation of A into a bi-module H by equipping the space $A \oplus H$ with the following bi-linear product

$$(a_1 + h_1)(a_2 + h_2) := a_1 a_2 + a_1 h_2 + a_2 h_1, \quad \text{for } a_1, a_2 \in A \text{ and } h_1, h_2 \in H, \tag{2.49}$$

where $a_1 a_2 \in A$ is the product inherit from A and $a_1 h_2, a_2 h_1 \in H$ are given by module multiplication. Let us now to extend the algebra involution to $* : A \oplus H \to A \oplus H$, so that $(a + h)^* = a^* + Jh$, where J is the usual anti-linear (invertible) charge-conjugation operator on H. In particular, note that when the algebra is associative, then it is satisfied $(a_1 h)a_2 = a_1(ha_2)$, which implies that $R_{a_2} L_{a_1}(h) = L_{a_1} R_{a_2}(h)$, and so, it recovers the order zero condition in Eq. (2.14a).

Now, we will extrapolate the above ideas in order to find a bi-representation of ΩA on H, by defining the new algebra $\Omega A \oplus H$ with the product

$$(\omega + h)(\omega' + h') := \omega\omega' + \omega h' + \omega' h, \quad \text{for } \omega, \omega' \in \Omega A \text{ and } h, h' \in H, \tag{2.50}$$

where $\omega\omega' \in A$ is the product inherit from A, while and $\omega h' \in H$ and $\omega' h \in H$ are bilinear products that define the left-action and right-action of ΩA on H. Because the elements of ΩA is generated by linear combinations of elements $a \in A$ and the formal symbols $d(a)$, then it is necessary to introduce a representation of the map $d : \Omega^0 A \to \Omega^1 A$. This is just the work that the Dirac operator D does by defining

$$d(a) = Dah - aDh, \ \text{ for } a \in A, \text{and } h \in H. \tag{2.51}$$

The left action of $d(a)$ is given by

$$L_{\rho(d(a))} = [D, L_a]. \tag{2.52}$$

In the special case when $\Omega A \oplus H$ is an associative algebra, it should be satisfied that

$$(\omega, \omega', \omega'') = 0, \ \ , (\omega, \omega', h) = 0, \ \ , (\omega, h, \omega') = 0, \ \ , (h, \omega, \omega') = 0, \tag{2.53}$$

where $\omega, \omega' \in \Omega A$ and $h \in H$. Notice that for $\omega, \omega' \in \Omega^0 A$ into the third relation in Eq.(2.53) we go back to the order zero condition. When $\omega = d(a) \in \Omega^1 A$ and $\omega' = a' \in A$, the third relation in Eq.(2.53) is given by

$$(d(a)h)a' = d(a)(ha') \Leftrightarrow [J(a')^* J^*, [D, a]]h = 0 \tag{2.54}$$

which corresponds to the order one condition in Eq.(2.14b). Let us now to see what happen when $\omega = d(a) \in \Omega^1 A$ and $\omega' = d(a') \in \Omega^1 A$, in such a case, the same relation in Eq.(2.53) give us that

$$(d(a)h)d(a') = d(a)(hd(a')) \Leftrightarrow [J[D, a']^* J^*, [D, a]]\, h = 0, \tag{2.55}$$

which is just the order two condition pointed out in Eq.(2.14c).[5]

[5]However, there are some subtleties as to whether this should be the commutator or the anti-commutator [59].

2.5.3 Symmetries for the differential graded $*$-algebra

In order to describe the symmetries in this setting, we start by defining the automorphisms of a $*$-algebra as the invertible linear map $\alpha : A \to A$ that preserve the product and the $*$-operation

$$\begin{aligned}\alpha(aa') &= \alpha(a)\alpha(a'),\\ \alpha(a^*) &= (\alpha a)^*,\end{aligned} \tag{2.56}$$

where $a, a' \in A$. In the same way, we can define the $*$-algebra derivations so that

$$\begin{aligned}\delta(aa') &= \delta(a)a' + a\delta(a'),\\ \delta(a^*) &= \pm(\delta a)^*.\end{aligned} \tag{2.57}$$

The derivations form a Lie algebra which is denoted $\mathrm{Der}(A)$, with Lie product given by $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$. We stand out that the derivations are the infinitesimal generators of the automorphisms so that $\alpha(a) = e^{\delta}(a)$, then, when α is infinitesimally close to the identity map we can write it simply as $\alpha = \mathbf{1} + \delta$.

Similarly, for the differential graded $*$-algebra $\Omega A \oplus H$ we define its automorphisms as the invertible linear maps $\alpha : \Omega A \oplus H \to \Omega A \oplus H$, that preserve the product as well as the grading and the $*$-operation so that

$$\begin{aligned}\alpha &= \alpha \oplus \alpha_1 \oplus \alpha_2 \oplus \cdots\\ \alpha[(\omega + h)^*] &= \alpha[(\omega + h)]^*\\ \alpha[(\omega + h)(\omega' + h')] &= \alpha(\omega + h)\alpha(\omega' + h'),\end{aligned} \tag{2.58}$$

for $\alpha_n : \Omega^n A \to \Omega^n A$ and, following the notation used if [16], $\alpha_\infty : H \to H$.

Therefore, we can extrapolate the last conditions to its infinitesimal generators (the derivations) which, instead of preserve the product (the last of the equations (2.58)), they should satisfy the Leibniz rule (the last of the equations (2.47)). Now, because A is an associative finite-dimensional $*$-algebra, its derivations are given by $\delta_x = L_x - R_x$ (fon an anti-hermitian element $a = -a^* \in A$), then we can translate this to a derivation in $\Omega A \oplus H$ by taking $\delta_n = L_x - R_{x^*}$ getting so the SM gauge group, as it was shown in section 2.3.3.

Finally, we just want to mention that $\delta_n = L_x - R_{x^*} + T_n$ is a more general extension to the derivations on the associative algebra A, where the linear operators $T_n : \Omega^n A \to \Omega^n A$ may be non-zero for $n \geq 1$, provided that they satisfy

$$T_{n+m}(\omega_n \omega_m) = (T_n \omega_n)\omega_m + \omega_n(T_m \omega_m), \tag{2.59}$$

$$T_n(\omega_n^*) = (T_n \omega_n)^*, \tag{2.60}$$

which give rise to an extra $U(1)_{B-L}$ gauge symmetry, as explained in [16].

Chapter 3

Two Higgs doublets on NCG

In this chapter, we will show that in a more general way the NCG SM allows a number of Higgs doublets equal to the number of Yukawa couplings. First, we will proceed to write the most general form for the Yukawa interaction and the scalar Lagrangian including the kinetic and potential terms. Once we have done this, we will present the minimal SM together its low energy phenomenology. Next, we will repeat an identical procedure for the for the Lepton-specific, Flipped, and Type II 2HDMs[1] with and without right-handed neutrino.

As we will see, NCG imposes constrains in the form of the scalar potential (and the Yukawa couplings) to high energy, which can be used as boundary conditions for the RGEs analysis. The mixing angle β (introduced in section 1.4) should be adjusted so that the mass of the top quark is close to its experimental value of 173 GeV.

3.1 Many Higgs doublets on NCG

From here on, we will only handle with one SM generation. In particular, we will focus on the third SM family. So the symbols y_e, y_d, and y_u will mean for the tau, bottom, and top Yukawa couplings, respectively. The 'Higgs' content of the theory is given by the terms of the form $f'\otimes\rho'[\gamma_5\otimes D_F, f\otimes\rho] = \gamma_5 \otimes \rho'[D_F, \rho]$, with $f \in C^\infty(M)$ and $\rho \in A_F$. In particular, the finite part is given by

$$\Phi = D_F + \rho'[D_F, \rho] + \epsilon' J\rho'[D_F, \rho]J^\dagger, \tag{3.1}$$

where the commutator $[D_F, \rho]$ is given by

$$[D_F, \rho] = \left(\begin{array}{cc|cc} & \rho_L\mathcal{M} - \mathcal{M}\rho_R & & \\ \rho_R\mathcal{M}^\dagger - \mathcal{M}^\dagger\rho_L & & & \\ \hline & & & \rho_L^C\mathcal{M}^* - \mathcal{M}^*\rho_R^C \\ & & \rho_R^C\mathcal{M}^T - \mathcal{M}^T\rho_L^C & \end{array}\right).$$

[1] For the Type I model, one of the Higgs doublets appears only into the potential but not in the Yukawa interaction. However, in NCG both the Yukawa interaction and the potential terms come from the same element: the fluctuated Dirac operator. So it is not possible to put this a model in the framework of the minimal NCG SM.

So after multiplying it by ρ', we can write

$$\rho'[D_F,\rho] := \left(\begin{array}{cc|cc} & \Omega_1 & & \\ \Omega_1' & & & \\ \hline & & & \Omega_2 \\ & & \Omega_2' & \end{array}\right) = \left(\begin{array}{cc|cc} & \Omega_1 & & \\ \Omega_1^\dagger & & & \\ \hline & & & 0 \\ & & 0 & \end{array}\right), \tag{3.2}$$

where

$$\begin{aligned} \Omega_1 &= \rho_L'(\rho_L\mathcal{M} - \mathcal{M}\rho_R), & \Omega_2 &\equiv \rho_L^{'C}(\rho_L\mathcal{M}^* - \mathcal{M}^*\rho_R^C) \\ \Omega_2' &= \rho_R^{'}(\rho_R\mathcal{M}^\dagger - \mathcal{M}^\dagger\rho_L), & \Omega_2' &\equiv \rho_R^{'C}(\rho_R^C\mathcal{M}^T - \mathcal{M}^T\rho_L^C). \end{aligned}$$

By using equations (2.22) and (3.53), it is straightforward to show that $\Omega_2 = \Omega_2' = 0$. From the Hermiticity of the (fluctuated) Dirac operator, it should satisfy that $\Omega_1' = \Omega_1^\dagger$, and then it is enough to calculate Ω_1. Hence, let us write

$$\Omega_1 := \begin{pmatrix} \begin{pmatrix} m_1' \\ m_2' \end{pmatrix} & \\ & \begin{pmatrix} n_{11}' & n_{12}' \\ n_{21}' & n_{22}' \end{pmatrix} \otimes 1_3 \end{pmatrix}, \tag{3.3}$$

where we have introduced the complex numbers $m_i', n_{ij}' \in \mathbb{C}$ which conform the complex doublets given by

$$\begin{aligned} \begin{pmatrix} m_1' \\ m_2' \end{pmatrix} &= b'(b - a1_2)\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \\ \begin{pmatrix} n_{11}' \\ n_{21}' \end{pmatrix} &= b'(b - a1_2)\begin{pmatrix} n_{11} \\ n_{21} \end{pmatrix}, \\ \begin{pmatrix} n_{12}' \\ n_{22}' \end{pmatrix} &= b'(b - a^*1_2)\begin{pmatrix} n_{12} \\ n_{22} \end{pmatrix}. \end{aligned} \tag{3.4}$$

Now, from equations (2.25) and (3.2) we have

$$J\rho'[D_F,\rho]J^\dagger = \left(\begin{array}{cc|cc} & 0 & & \\ 0 & & & \\ \hline & & & \Omega_1^\dagger \\ & & \Omega_1'^\dagger & \end{array}\right). \tag{3.5}$$

So, taking into account $\epsilon' = 1$ and inserting the Dirac operator in Eq. (2.21) and (3.5) on Eq (3.1) we get

$$\Phi = \left(\begin{array}{cc|cc} & \mathcal{M} + \Omega_1 & & \\ \mathcal{M}^\dagger + \Omega_1^\dagger & & & \\ \hline & & & \mathcal{M}^* + \Omega_1^\dagger \\ & & \mathcal{M}^T + \Omega_1 & \end{array}\right),$$

where

$$
\mathcal{M} + \Omega_1 = \begin{pmatrix} \begin{pmatrix} m_1 + m_1' \\ m_2 + m_2' \end{pmatrix} & \\ & \begin{pmatrix} n_{11} + n_{11}' & n_{12} + n_{12}' \\ n_{21} + n_{21}' & n_{22} + n_{22}' \end{pmatrix} \otimes 1_3 \end{pmatrix}.
$$

So in general, we have three different Higgs fields, one for each fermion sector.

Next, we define

$$
\begin{aligned}
y_e^* \Theta_e &:= y_e^* \begin{pmatrix} (\phi_e)_1 \\ (\phi_e)_2 \end{pmatrix} \equiv \begin{pmatrix} m_1 + m_1' \\ m_2 + m_2' \end{pmatrix} \\
y_u^* \Theta_u &:= y_u^* \begin{pmatrix} (\phi_u)_1 \\ (\phi_u)_2 \end{pmatrix} \equiv \begin{pmatrix} n_{11} + n_{11}' \\ n_{21} + n_{21}' \end{pmatrix} \\
y_d^* \Theta_d &:= y_d^* \begin{pmatrix} (\phi_d)_1 \\ (\phi_d)_2 \end{pmatrix} \equiv \begin{pmatrix} n_{12} + n_{12}' \\ n_{22} + n_{22}' \end{pmatrix},
\end{aligned}
$$

then, the total scalar sector of the model is given by the fluctuation of the Dirac operator

$$
\Phi = \left(\begin{array}{ccccc|ccccc}
 & & y_e^*\Theta_e & & & & & & & \\
 & & & y_u^*\Theta_u & y_d^*\Theta_d & & & & & \\
y_e\Theta_e^\dagger & & & & & & & & & \\
 & y_u\Theta_u^\dagger & & & & & & & & \\
 & y_d\Theta_d^\dagger & & & & & & & & \\
\hline
 & & & & & & & y_e\Theta_e^* & & \\
 & & & & & & & & y_u\Theta_u^* & y_d\Theta_d^* \\
 & & & & & y_e^*\Theta_e^T & & & & \\
 & & & & & & y_u^*\Theta_u^T & & & \\
 & & & & & & y_d^*\Theta_d^T & & &
\end{array}\right). \tag{3.6}
$$

Note that we have omitted the 1_3 factors on the quark sector.

3.1.1 Yukawa Interaction

To avoid extra fermionic degrees of freedom, we should consider fermions of the form $\Psi \in L^2(S,M)^+ \otimes H_L \oplus L^2(S,M)^- \otimes H_R$,

$$
\begin{aligned}
\Psi = \boldsymbol{e}_L \otimes e_L + \boldsymbol{e}_R \otimes e_R &+ \overline{\boldsymbol{e}_L} \otimes \overline{e_L} + \overline{\boldsymbol{e}_R} \otimes \overline{e_R} \\
+ \boldsymbol{\nu}_L \otimes \nu \qquad\qquad &+ \overline{\boldsymbol{\nu}_L} \otimes \overline{\nu} \\
+ \boldsymbol{u}_L \otimes u_L + \boldsymbol{u}_R \otimes u_R &+ \overline{\boldsymbol{u}_L} \otimes \overline{u_L} + \overline{\boldsymbol{u}_R} \otimes \overline{u_R} \\
+ \boldsymbol{d}_L \otimes d_L + \boldsymbol{d}_R \otimes d_R &+ \overline{\boldsymbol{d}_L} \otimes \overline{d_L} + \overline{\boldsymbol{d}_R} \otimes \overline{d_R},
\end{aligned}
$$

where the bold symbols mean for Weyl spinors. The Yukawa interaction for the SM fermions is given by the symmetric scalar product $\langle J\Psi, (\gamma_5 \otimes \Phi)\Psi\rangle$, where

$$\begin{aligned} J\Psi = & J_M \boldsymbol{e}_L \otimes \overline{e_L} + J_M \boldsymbol{e}_R \otimes \overline{e_R} + J_M \overline{\boldsymbol{e}_L} \otimes e_L + J_M \overline{\boldsymbol{e}_R} \otimes e_R \\ & + J_M \boldsymbol{\nu}_L \otimes \overline{\nu} \qquad\qquad\qquad + J_M \overline{\boldsymbol{\nu}_L} \otimes \nu \\ & + J_M \boldsymbol{u}_L \otimes \overline{u_L} + J_M \boldsymbol{u}_R \otimes \overline{u_R} + J_M \overline{\boldsymbol{u}_L} \otimes u_L + J_M \overline{\boldsymbol{u}_R} \otimes u_R \\ & + J_M \boldsymbol{d}_L \otimes \overline{d_L} + J_M \boldsymbol{d}_R \otimes \overline{d_R} + J_M \overline{\boldsymbol{d}_L} \otimes d_L + J_M \overline{\boldsymbol{d}_R} \otimes d_R, \end{aligned}$$

and taking into account that

$$\gamma_5 \begin{pmatrix} \boldsymbol{\xi_L} \\ \boldsymbol{\xi_R} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\xi_L} \\ \boldsymbol{\xi_R} \end{pmatrix},$$

we have that

$$\begin{aligned} (\gamma_5 \otimes \Phi)\Psi = & -\boldsymbol{e}_L \otimes \Phi e_L + \boldsymbol{e}_R \otimes \Phi e_R + \overline{\boldsymbol{e}_L} \otimes \Phi \overline{e_L} - \overline{\boldsymbol{e}_R} \otimes \Phi \overline{e_R} \\ & - \boldsymbol{\nu}_L \otimes \Phi \nu \qquad\qquad\qquad + \overline{\boldsymbol{\nu}_L} \otimes \Phi \overline{\nu} \\ & - \boldsymbol{u}_L \otimes \Phi u_L + \boldsymbol{u}_R \otimes \Phi u_R + \overline{\boldsymbol{u}_L} \otimes \Phi \overline{u_L} - \overline{\boldsymbol{u}_R} \otimes \Phi \overline{u_R} \\ & - \boldsymbol{d}_L \otimes \Phi d_L + \boldsymbol{d}_R \otimes \Phi d_R + \overline{\boldsymbol{d}_L} \otimes \Phi \overline{d_L} - \overline{\boldsymbol{d}_R} \otimes \Phi \overline{d_R}. \end{aligned}$$

Next, the action of Φ on the basis elements of the finite Hilbert space is given by equation (3.6) as follows

$$\begin{aligned} \Phi \nu &= y_e (\phi_e)_1^* e_R, & \Phi \overline{\nu} &= y_e^* (\phi_e)_1 \overline{e_R}, \\ \Phi e_L &= y_e (\phi_e)_2^* e_R, & \Phi \overline{e_L} &= y_e^* (\phi_e)_2 \overline{e_R}, \\ \Phi u_L &= y_u (\phi_u)_1^* u_R + y_d (\phi_d)_1^* d_R, & \Phi \overline{u_L} &= y_u^* (\phi_u)_1 \overline{u_R} + y_d^* (\phi_d)_1 \overline{d_R}, \\ \Phi d_L &= y_u (\phi_u)_2^* u_R + y_d (\phi_d)_2^* d_R, & \Phi \overline{d_L} &= y_u^* (\phi_u)_2 \overline{u_R} + y_d^* (\phi_d)_2 \overline{d_R}, \\ \Phi e_R &= y_e^* (\phi_e)_1 \nu \quad + y_e^* (\phi_e)_2 e_L, & \Phi \overline{e_R} &= y_e (\phi_e)_1^* \overline{\nu} \quad + y_e (\phi_e)_2^* \overline{e_L}, \\ \Phi u_R &= y_u^* (\phi_u)_1 u_L + y_u^* (\phi_u)_2 d_L, & \Phi \overline{u_R} &= y_u (\phi_u)_1^* \overline{u_L} + y_u (\phi_u)_2^* \overline{d_L}, \\ \Phi d_R &= y_d^* (\phi_d)_1 u_L + y_d^* (\phi_d)_2 d_L, & \Phi \overline{d_R} &= y_d (\phi_d)_1^* \overline{u_L} + y_d (\phi_d)_2^* \overline{d_L}. \end{aligned}$$

Now, by making use of the symmetry of the bilinear form $\langle J_M \boldsymbol{\xi}_R, \boldsymbol{\eta}_R\rangle = \langle J_M \boldsymbol{\eta}_R, \boldsymbol{\xi}_R\rangle$, we obtain

$$\begin{aligned} \frac{1}{2}\langle J\Psi, (\gamma_5 \otimes \Phi)\Psi\rangle = & - \langle J_M \overline{\boldsymbol{e}_R}, y_e((\phi_e)_2^* \boldsymbol{e}_L + (\phi_e)_1^* \boldsymbol{\nu}_L)\rangle \\ & + \langle J_M \boldsymbol{e}_R, y_e^*((\phi_e)_2 \overline{\boldsymbol{e}_L} + (\phi_e)_1 \overline{\boldsymbol{\nu}_L})\rangle \\ & - \langle J_M \overline{\boldsymbol{u}_R}, y_u((\phi_u)_1^* \boldsymbol{u}_L + (\phi_u)_2^* \boldsymbol{d}_L)\rangle \\ & + \langle J_M \boldsymbol{u}_R, y_u^*((\phi_u)_1 \overline{\boldsymbol{u}_L} + (\phi_u)_2 \overline{\boldsymbol{d}_L})\rangle \\ & - \langle J_M \overline{\boldsymbol{d}_R}, y_d((\phi_d)_1^* \boldsymbol{u}_L + (\phi_d)_2^* \boldsymbol{d}_L)\rangle \\ & + \langle J_M \boldsymbol{d}_R, y_d^*((\phi_d)_1 \overline{\boldsymbol{u}_L} + (\phi_d)_2 \overline{\boldsymbol{d}_L})\rangle. \end{aligned} \tag{3.7}$$

Fermion	Charge	Anti-fermion	Charge
$\boldsymbol{e_L}, \boldsymbol{e_R}$	-1	$(\boldsymbol{e_L})^\dagger, (\boldsymbol{e_R})^\dagger$	$+1$
$\boldsymbol{u_L}, \boldsymbol{u_R}$	$+\frac{2}{3}$	$(\boldsymbol{u_L})^\dagger, (\boldsymbol{u_R})^\dagger$	$-\frac{2}{3}$
$\boldsymbol{d_L}, \boldsymbol{d_R}$	$-\frac{1}{3}$	$(\boldsymbol{e_L})^\dagger, (\boldsymbol{e_R})^\dagger$	$+\frac{1}{3}$

Table 3.1: The standard model lepton an quark charges

Taking into account the electromagnetic charges for the fermions as expressed by table 3.1, and by Eq. (3.7) we should select the scalar field's charges as indicated in table 3.2

Scalar field	Charge	Anti-particle	Charge
$(\phi_e)_1$	$+1$	$(\phi_e)_1^*$	-1
$(\phi_e)_2$	0	$(\phi_e)_2^*$	0
$(\phi_u)_1$	0	$(\phi_u)_1^*$	0
$(\phi_u)_2$	-1	$(\phi_u)_2^*$	$+1$
$(\phi_d)_1$	$+1$	$(\phi_d)_1^*$	-1
$(\phi_d)_2$	0	$(\phi_d)_2^*$	0

Table 3.2: Allowed charge selection for the scalar fields

3.1.2 Kinetic terms

The kinetic terms for the scalar sector will be given by $\mathrm{Tr}[(D_\mu\Phi)(D^\mu\Phi)]$, where $D_\mu\Phi = \partial_\mu\Phi + i[\delta_\mu, \Phi]$. We may express it by

$$D_\mu\Phi = \left(\begin{array}{c|c} \Delta & \Pi \\ \hline \Pi^* & \Delta^* \end{array}\right), \tag{3.8}$$

where $\Pi = 0$ (in this case) and

$$\Delta = \begin{pmatrix} & & y_e^* D_\mu\Theta_e & & \\ & & & y_u^* D_\mu\Theta_u & y_d^* D_\mu\Theta_d \\ y_e D_\mu{\Theta_e}^* & & & & \\ & y_u D_\mu{\Theta_u}^* & & & \\ & y_d D_\mu{\Theta_d}^* & & & \end{pmatrix},$$

where we have introduced the covariant derivatives given by

$$D_\mu\Theta_e = \begin{pmatrix} \partial_\mu(\phi_e)_1 + i[(Q_\mu^3 + \Lambda_\mu)(\phi_e)_1 + Q_\mu^+(\phi_e)_2] \\ \partial_\mu(\phi_e)_2 + i[(-Q_\mu^3 + \Lambda_\mu)(\phi_e)_2 + Q_\mu^-(\phi_e)_1] \end{pmatrix},$$

$$D_\mu\Theta_d = \begin{pmatrix} \partial_\mu(\phi_d)_1 + i[(Q_\mu^3 + \Lambda_\mu)(\phi_d)_1 + Q_\mu^+(\phi_d)_2] \\ \partial_\mu(\phi_d)_2 + i[(-Q_\mu^3 + \Lambda_\mu)(\phi_d)_2 + Q_\mu^-(\phi_d)_1] \end{pmatrix},$$

$$D_\mu\Theta_u = \begin{pmatrix} \partial_\mu(\phi_u)_1 + i[(Q_\mu^3 - \Lambda_\mu)(\phi_u)_1 + Q_\mu^+(\phi_u)_2] \\ \partial_\mu(\phi_u)_2 + i[(-Q_\mu^3 - \Lambda_\mu)(\phi_u)_2 + Q_\mu^-(\phi_u)_1] \end{pmatrix}.$$

In terms of the Pauli matrices in Eq. (2.3), the covariant derivatives are given explicitly by

$$
\begin{aligned}
D_\mu \Theta_e &= \partial_\mu \Theta_e + i Q_\mu^\alpha \sigma^\alpha \Theta_e + i\Lambda_\mu \Theta_e, \\
D_\mu \Theta_d &= \partial_\mu \Theta_d + i Q_\mu^\alpha \sigma^\alpha \Theta_d + i\Lambda_\mu \Theta_d, \\
D_\mu \Theta_u &= \partial_\mu \Theta_u + i Q_\mu^\alpha \sigma^\alpha \Theta_u - i\Lambda_\mu \Theta_u,
\end{aligned}
$$

with

$$
\begin{aligned}
Q_\mu^+ &= Q_\mu^1 - iQ_\mu^2, \\
Q_\mu^- &= Q_\mu^1 + iQ_\mu^2.
\end{aligned}
$$

The action of Eq. (3.8) on the antiparticle basis gives an extra '2' factor. So we can write

$$
\begin{aligned}
\mathrm{Tr}[(D_\mu\Phi)(D^\mu\Phi)] = 4[&|y_e|^2(D_\mu\Theta_e)^\dagger D_\mu\Theta_e + 3|y_d|^2(D_\mu\Theta_d)^\dagger D_\mu\Theta_d \\
&+ 3|y_u|^2(D_\mu\Theta_u)^\dagger D_\mu\Theta_u].
\end{aligned}
$$

Now, taking into account Eq.(2.46), the kinetic terms for the scalar Lagrangian are given by

$$
\begin{aligned}
\mathcal{L}_\mathcal{K} &= \frac{f_0}{8\pi^2}\mathrm{Tr}[(D_\mu\Phi)(D^\mu\Phi)] \\
&= \frac{|y_e|^2}{4g^2}(D_\mu\Theta_e)^\dagger(D_\mu\Theta_e) + \frac{3|y_d|^2}{4g^2}(D_\mu\Theta_d)^\dagger(D_\mu\Theta_d) \\
&+ \frac{3|y_u|^2}{4g^2}(D_\mu\Theta_u)^\dagger(D_\mu\Theta_u). \qquad (3.9)
\end{aligned}
$$

3.1.3 Potential terms

Unless that established otherwise, we assume that the bilinear term in the scalar potential will have negative signs while the quadratic terms will be positive. From Eq. (3.6), we are able to get the quadratic scalar terms as the trace of the fluctuated Dirac operator Φ

$$
\begin{aligned}
\mathrm{Tr}(\Phi^2) = 4[&|y_e|^2(|(\phi_e)_1|^2 + |(\phi_e)_2|^2) + 3|y_d|^2(|(\phi_d)_1|^2 + |(\phi_d)_2|^2) \qquad (3.10) \\
&+ 3|y_u|^2(|(\phi_u)_1|^2 + |(\phi_u)_2|^2)] \\
= 4[&|y_e|^2(\Theta_e^\dagger\Theta_e) + 3|y_d|^2(\Theta_d^\dagger\Theta_d) + 3|y_u|^2(\Theta_u^\dagger\Theta_u)],
\end{aligned}
$$

as well as the quartic terms

$$
\begin{aligned}
\mathrm{Tr}(\Phi^4) = 4[&|y_e|^4(|(\phi_e)_1|^2 + |(\phi_e)_2|^2)^2 + 3|y_d|^4(|(\phi_d)_1|^2 + |(\phi_d)_2|^2)^2 \qquad (3.11) \\
&+ 3|y_u|^4(|(\phi_u)_1|^2 + |(\phi_u)_2|^2)^2 \\
= 4[&|y_e|^4(\Theta_e^\dagger\Theta_e)^2 + 3|y_d|^4(\Theta_d^\dagger\Theta_d)^2 + 3|y_u|^4(\Theta_u^\dagger\Theta_u)^2 \\
&+ 6|y_u|^2|y_d|^2(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d)].
\end{aligned}
$$

So the total scalar potential is given by

$$
\begin{aligned}
V &= -\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) + \frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) \\
&= -\frac{2f_2\Lambda^2}{\pi^2}[|y_e|^2(\Theta_1^\dagger\Theta_1) + 3|y_d|^2(\Theta_d^\dagger\Theta_d) + 3|y_u|^2(\Theta_u^\dagger\Theta_u)] \\
&+ \frac{f_0}{2\pi^2}[|y_e|^4(\Theta_e^\dagger\Theta_e)^2 + 3|y_d|^4(\Theta_d^\dagger\Theta_d)^2 + 3|y_u|^4(\Theta_u^\dagger\Theta_u)^2 \\
&+ 6|y_u|^2|y_d|^2(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d)] \\
&= \frac{1}{2g^2}\Big[-\frac{2f_2\Lambda^2}{f_0}[|y_e|^2(\Theta_e^\dagger\Theta_e) + 3|y_d|^2(\Theta_d^\dagger\Theta_d) + 3|y_u|^2(\Theta_u^\dagger\Theta_u)]\Big] \\
&+ \frac{1}{4g^2}[|y_e|^4(\Theta_e^\dagger\Theta_e)^2 + 3|y_d|^4(\Theta_d^\dagger\Theta_d)^2 + 3|y_u|^4(\Theta_u^\dagger\Theta_u)^2 \\
&+ 6|y_u|^2|y_d|^2(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d)].
\end{aligned} \tag{3.12}
$$

3.2 The SM

In the SM, all of the quarks and charged leptons acquire their masses by coupling to the same Higgs doublet scalar field. From here on, we will assume that Θ_2 is such field, so we get

$$
\Theta_u = \begin{pmatrix} (\phi_u)_1 \\ (\phi_u)_2 \end{pmatrix} = \Theta_d = \begin{pmatrix} (\phi_d)_1 \\ (\phi_d)_2 \end{pmatrix} = \Theta_e = \begin{pmatrix} (\phi_e)_1 \\ (\phi_e)_2 \end{pmatrix} := \Theta_2 = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix}.
$$

Then, from Eq. (3.9) the kinetic term on the scalar Lagrangian is given by

$$
\mathcal{L}_{\mathcal{K}} = \frac{|y_e|^2 + 3|y_d|^2 + 3|y_u|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2), \tag{3.13}
$$

which suggests the following normalization

$$
\Theta_2 \to \frac{2g}{\sqrt{|y_e|^2 + 3|y_d|^2 + 3|y_u|^2}}\Phi_2. \tag{3.14}
$$

Potential terms

Now, we replace the normalized field in Eq. (3.14) into the quadratic scalar potential given by Eq. (3.11), so that we get

$$
\begin{aligned}
-\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &= \frac{1}{2g^2}\left(-\frac{2f_2\Lambda^2}{f_0}(|y_e|^2 + 3|y_d|^2 + 3|y_u|^2)(\Theta_2^\dagger\Theta_2)\right) \\
&= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_2^\dagger\Phi_2\right).
\end{aligned} \tag{3.15}
$$

Taking into account Eq.(1.43), the quartic mixed terms on the potential are given by

$$\begin{aligned}(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d) &= \begin{pmatrix}\varphi_2^- & \varphi_2^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_2^+\\ \varphi_2^0\end{pmatrix}\\ &= \left(\varphi_2^-\varphi_2^{0*} - \varphi_2^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_2^+ - \varphi_2^+\varphi_2^0\right)\\ &= \varphi_2^-\varphi_2^{0*}\varphi_2^0\varphi_2^+ - \varphi_2^-\varphi_2^{0*}\varphi_2^+\varphi_2^0 - \varphi_2^{0*}\varphi_2^-\varphi_2^0\varphi_2^+ + \varphi_2^{0*}\varphi_2^-\varphi_2^+\varphi_2^0\\ &= 0.\end{aligned}$$

Then, the quartic potential terms are given by

$$\begin{aligned}\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &= \frac{1}{4g^2}(|y_e|^4 + 3|y_d|^4 + 3|y_u|^4)(\Theta_2^\dagger\Theta_2)^2\\ &= \frac{4\cdot(|y_e|^4 + 3|y_d|^4 + 3|y_u|^4)g^2}{(|y_e|^2 + 3|y_d|^2 + 3|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2.\end{aligned}$$

In this way, and taking into account $\frac{|y_e|^2}{|y_u|^2} \to 0$ and $\frac{|y_d|^2}{|y_u|^2} \to 0$, we have

$$\begin{aligned}\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &\approxeq \frac{4\cdot 3}{9}g^2(\Phi_2^\dagger\Phi_2)^2\\ &= \frac{4}{3}g^2(\Phi_2^\dagger\Phi_2)^2\end{aligned} \tag{3.16}$$

Then we make the following definition

$$\mu_1^2 = -4\frac{f_2\Lambda^2}{f_0}, \qquad \lambda_2 \approx \frac{4}{3}g^2, \qquad \mu_2^2 = \lambda_1 = \lambda_3 = \lambda_4 = \lambda_5 = 0. \tag{3.17}$$

Therefore, we get a potential of the form

$$V = \mu_1^2(\Phi_2^\dagger\Phi_2) + \lambda_2(\Phi_2^\dagger\Phi_2)^2, \tag{3.18}$$

which corresponds the SM potential, with only one Higgs doublet. The second equation in Eq. (3.17) will be considered as the (high energy) boundary condition for the λ's renormalization group equation. Thus, we might study mass generation after spontaneous symmetry breaking.

Fermionic interaction

In the unitary gauge, the charged components are zero $(\phi_e)_1, (\phi_d)_1, (\phi_u)_2 \to 0$, while the other ones are given by

$$(\phi_u)_1^*, (\phi_e)_2, (\phi_d)_2 \to \frac{2g}{\sqrt{|y_e|^2 + 3|y_d|^2 + 3|y_u|^2}}\left(\frac{\rho_2 + v_2}{\sqrt{2}}\right).$$

Thus, from Eq. (3.7) we should make the following redefinition for the Yukawa couplings

$$y_u := -i\sqrt{\frac{|y_e|^2 + 3|y_d|^2 + 3|y_u|^2}{2}}\frac{m_t}{gv_2}, \tag{3.19a}$$

$$y_e := -i\sqrt{\frac{|y_e|^2 + 3|y_d|^2 + 3|y_u|^2}{2}}\frac{m_e}{gv_2}, \tag{3.19b}$$

$$y_d := -i\sqrt{\frac{|y_e|^2 + 3|y_d|^2 + 3|y_u|^2}{2}}\frac{m_d}{gv_2}. \tag{3.19c}$$

The above relations can be used to impose (high energy) boundary conditions to the SM Yukawa couplings, like the one we will deduce in Eq. (3.22) for the top quark.

Next, by replacing in Eq. (3.7) we get

$$\begin{aligned}\frac{1}{2}\langle J\Psi, (\gamma_5 \otimes \Phi)\Psi\rangle = \; & i\frac{m_e}{v_2}(\rho_2 + v_2)\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \\ & + i\frac{m_d}{v_2}(\rho_2 + v_2)\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle \\ & + i\frac{m_t}{v_2}(\rho_2 + v_2)\langle J_M\overline{\boldsymbol{u}}, \boldsymbol{u}\rangle.\end{aligned}$$

RGEs analysis

Because of the hierarchy of the fermion masses, we will consider the top Yukawa coupling only. So the beta-functions of RGEs for the Yukawa and the (quartic) scalar couplings are given by

$$16\pi^2\beta_{y_u} = y_u\left(-8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_1^2 + \frac{9}{2}y_u^2\right), \tag{3.20}$$

$$16\pi^2\beta_\lambda = 24\lambda^2 - 6y_u^4 + 12y_u^2\lambda - 3(g_1^2 + 3g_2^2)\lambda + \frac{9}{8}g_2^4 + \frac{3}{8}g_1^4 + \frac{3}{4}g_2^2g_1^2. \tag{3.21}$$

Then, by multiplying Eq. (3.19a) times its complex conjugate and since $|y_e| \sim |y_d| \ll |y_u|$, we have that

$$m_t^2 \approx \frac{2g^2v_2^2}{3},$$

and having in mind the standard relation $m_t = \frac{v_2}{\sqrt{2}}y_u$, we obtain

$$y_u \approx \frac{2g}{\sqrt{3}}, \tag{3.22}$$

which is interpreted as the boundary condition for the RGE (3.20). On other hand, the relation for λ in Eq. (3.17) is going to be the boundary condition for the RGE (3.21).

Next, after fixing an unification scale of $\sim 10^{16}$ GeV and an unified gauge coupling value of $g = 0.5$, we carry out the running of the RGEs in Eqs.(3.20) and (3.21) to find the low energy values for both y_u and λ, as shown in table 3.3. In figure 3.1 we have depicted the running for λ. Hence, we might use these values to calculate the top quark and Higgs boson masses[2] (see table 3.4) by replacing in the following equations

$$m_t = \frac{v}{\sqrt{2}} y_u(m_t), \tag{3.23a}$$

$$m_h = v\sqrt{2\lambda(m_h)}, \tag{3.23b}$$

where $v = 246$ GeV.

	10^{16}GeV	10^2GeV
y_u	0.58	1.1
λ	0.33	0.3

Table 3.3: High and low energy values for the Yukawa and scalar couplings.

	Mass
m_t	185
m_h	189

Table 3.4: Mass spectrum for the minimal NCG SM. The mass values are given in GeV.

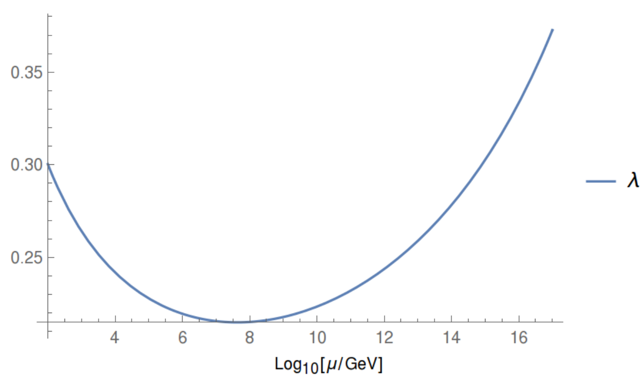


Figure 3.1: Standard model quartic coupling running.

3.3 Two-Higgs-doublet model

In this section, we will go through the construction of three of the four 2HDMs that suppress tree-level FCNC, and which are allowed on NCG: the Lepton-Specific, Flipped, and Type-II models. First, we show the relations that should satisfy the doublets we have found in Eq. (3.6) so that we get the desired model. Then, we will use Eqs. (3.9) to normalize the scalar fields to get the physical ones to subsequently replace them into the potential in Eq. (3.12) to get the NCG boundary conditions for the 2HDM renormalization group equations. Finally, we redefine the Yukawa couplings from Eq. (3.7) in order to compute the explicit form of the interaction between the the SM fermions to the CP-even uncharged scalar fields.

[2]These values coincide with the one obtained in [71, pag 221], with no right-handed neutrino, for a unification scale of $\approx 9.92 \times 10^{16}$ GeV, and $g = 0.49$.

3.3.1 Lepton-Specific

To start our study of the 2-Higgs doublet model, we use the convention where the quarks type up can only couple to the field Φ_2, so we define

$$\Theta_u = \begin{pmatrix} (\phi_u)_1 \\ (\phi_u)_2 \end{pmatrix} := \widetilde{\Theta}_2 = \begin{pmatrix} \varphi_2^{0*} \\ -\varphi_2^- \end{pmatrix}.$$

In the lepton specific model the charged leptons couple to the first doublet, while the quarks couple to the second one. Therefore, we define

$$\Theta_d = \begin{pmatrix} (\phi_d)_1 \\ (\phi_d)_2 \end{pmatrix} := \Theta_2 = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix},$$
$$\Theta_e = \begin{pmatrix} (\phi_e)_1 \\ (\phi_e)_2 \end{pmatrix} := \Theta_1 = \begin{pmatrix} \varphi_1^+ \\ \varphi_1^0 \end{pmatrix}.$$

From Eq. (3.9) we can write the kinetic Lagrangian as

$$\mathcal{L}_{\mathcal{K}} = \frac{|y_e|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{3(|y_d|^2 + |y_u|^2)}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2). \tag{3.24}$$

In order to get a standard two-Higgs doublets model Lagrangian with kinetic terms of the form $\mathcal{L}_{\mathcal{K}} = (D_\mu\Phi_1)^\dagger D^\mu\Phi_1 + (D_\mu\Phi_2)^\dagger D^\mu\Phi_2$, we redefine the scalar fields by setting

$$\Theta_1 \to \frac{2g}{\sqrt{|y_e|^2}}\Phi_1, \tag{3.25a}$$
$$\Theta_2 \to \frac{2g}{\sqrt{3(|y_d|^2 + |y_u|^2)}}\Phi_2, \tag{3.25b}$$

with $\Phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix}$, for $i = 1, 2$.

Potential terms

Now, we replace the normalized fields in Eqs. (3.25) into the quadratic scalar potential given by Eq. (3.11), so that we get

$$\begin{aligned} -\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &= \frac{1}{2g^2}\left[-\frac{2f_2\Lambda^2}{f_0}\left(|y_e|^2(\Theta_1^\dagger\Theta_1) + (3|y_d|^2 + 3|y_u|^2)(\Theta_2^\dagger\Theta_2)\right)\right] \\ &= -\frac{4f_2\Lambda^2}{f_0}\left[(\Phi_1^\dagger\Phi_1) + (\Phi_2^\dagger\Phi_2)\right]. \end{aligned} \tag{3.26}$$

Furthermore, the mixed-quartic terms are zero (but no for the other models)

$$
\begin{aligned}
(\Theta_d^\dagger \Theta_u)(\Theta_u^\dagger \Theta_d) &= \begin{pmatrix} \varphi_2^- & \varphi_2^{0*} \end{pmatrix} \begin{pmatrix} \varphi_2^{0*} \\ -\varphi_2^- \end{pmatrix} \cdot \begin{pmatrix} \varphi_2^0 & -\varphi_2^+ \end{pmatrix} \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix} \\
&= \left(\varphi_2^- \varphi_2^{0*} - \varphi_2^{0*} \varphi_2^-\right) \left(\varphi_2^0 \varphi_2^+ - \varphi_2^+ \varphi_2^0\right) \\
&= \varphi_2^- \varphi_2^{0*} \varphi_2^0 \varphi_2^+ - \varphi_2^- \varphi_2^{0*} \varphi_2^+ \varphi_2^0 - \varphi_2^{0*} \varphi_2^- \varphi_2^0 \varphi_2^+ + \varphi_2^{0*} \varphi_2^- \varphi_2^+ \varphi_2^0 \\
&= 0.
\end{aligned}
\tag{3.27}
$$

So, we have the following quartic terms by

$$
\begin{aligned}
\frac{f_0}{8\pi^2} \mathrm{Tr}(\Phi^4) &= \frac{1}{4g^2} \left[|y_e|^4 (\Theta_1^\dagger \Theta_1)^2 + (3|y_d|^4 + 3|y_u|^4)(\Theta_2^\dagger \Theta_2)^2 \right] \\
&= 4g^2 (\Phi_1^\dagger \Phi_1)^2 + \frac{4 \cdot 3(|y_d|^4 + |y_u|^4) g^2}{9(|y_d|^2 + |y_u|^2)^2} (\Phi_2^\dagger \Phi_2)^2.
\end{aligned}
\tag{3.28}
$$

For the third SM generation, y_u and y_d are the top and bottom Yukawa couplings, and it is satisfied that $|y_d| \ll |y_u|$, then

$$
\mu_1^2 = -4 \frac{f_2 \Lambda^2}{f_0} = \mu_2^2, \qquad \frac{\lambda_1}{2} = 4g^2, \qquad \frac{\lambda_2}{2} \approx \frac{4}{3} g^2, \tag{3.29}
$$

and $\mu_{12}^2, \lambda_3, \lambda_4, \lambda_5 \to 0$. Then, by replacing it into the potential Eq. (1.40) we get

$$
V = \mu_1^2 (\Phi_1^\dagger \Phi_1) + \mu_2^2 (\Phi_2^\dagger \Phi_2) + \frac{\lambda_1}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_2}{2} (\Phi_2^\dagger \Phi_2)^2. \tag{3.30}
$$

We can note that the last potential is a very constrained version of the most general 2HDM potential described in section 3. Furthermore, equation (3.29) will be used as the 'unification' boundary conditions imposed by NCG to the scalar couplings RGEs for the Lepton Specific model.

Fermionic interaction

Next, we make the following redefinition for the Yukawa couplings

$$
\begin{aligned}
y_e &:= -i \sqrt{\frac{|y_e|^2}{2}} \frac{m_e}{g v_1}, \\
y_u &:= -i \sqrt{\frac{3(|y_d|^2 + |y_u|^2)}{2}} \frac{m_t}{g v_2}, \\
y_d &:= -i \sqrt{\frac{3(|y_d|^2 + |y_u|^2)}{2}} \frac{m_d}{g v_2},
\end{aligned}
\tag{3.31}
$$

and inserting this into Eq. (3.7) we get the Yukawa interaction

$$\begin{aligned}\frac{1}{2}\langle J\Psi, (\gamma_5 \otimes \Phi)\Psi\rangle = {} & i\frac{m_e}{v_1}(\rho_1 + v_1)\langle J_M \overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \\ & + i\frac{m_t}{v_2}(\rho_2 + v_2)\langle J_M \overline{\boldsymbol{u}}, \boldsymbol{u}\rangle \\ & + i\frac{m_d}{v_2}(\rho_2 + v_2)\langle J_M \overline{\boldsymbol{d}}, \boldsymbol{d}\rangle .\end{aligned}$$

Now, by using again Eq.(1.52) the interaction of fermions with the CP-even uncharged scalar fields is given by

$$\begin{aligned}\mathcal{L} \supseteq {} & i\frac{m_d}{v_1}\langle J_M \overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \rho_1 + i\left(\frac{m_t}{v_2}\langle J_M \overline{\boldsymbol{u}}, \boldsymbol{u}\rangle + \frac{m_d}{v_2}\langle J_M \overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\rho_2 \\ = {} & i\frac{m_e}{v}\langle J_M \overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \frac{\rho_1}{\cos\beta} + i\left(\frac{m_t}{v}\langle J_M \overline{\boldsymbol{u}}, \boldsymbol{u}\rangle + \frac{m_d}{v}\langle J_M \overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{\rho_2}{\sin\beta} \\ = {} & i\frac{m_e}{v}\langle J_M \overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \frac{H\cos\alpha - h\sin\alpha}{\cos\beta} \\ & + i\left(\frac{m_t}{v}\langle J_M \overline{\boldsymbol{u}}, \boldsymbol{u}\rangle + \frac{m_d}{v}\langle J_M \overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{H\sin\alpha + h\cos\alpha}{\sin\beta} \\ = {} & i\left[\frac{m_e}{v}\langle J_M \overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \frac{\cos\alpha}{\cos\beta} + \left(\frac{m_t}{v}\langle J_M \overline{\boldsymbol{u}}, \boldsymbol{u}\rangle + \frac{m_d}{v}\langle J_M \overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{\sin\alpha}{\sin\beta}\right] H \\ & + i\left[-\frac{m_e}{v}\langle J_M \overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \frac{\sin\alpha}{\cos\beta} + \left(\frac{m_t}{v}\langle J_M \overline{\boldsymbol{u}}, \boldsymbol{u}\rangle + \frac{m_d}{v}\langle J_M \overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{\cos\alpha}{\sin\beta}\right] h. \end{aligned} \tag{3.32}$$

Then, the coupling between the quarks and the SM Higgs boson h is affected by the factor $C^h_{u,d} = \frac{\cos\alpha}{\sin\alpha}$, which should be closed to 1 [38], in accordance with the alignment limit aforementioned in section 1.4.2. By its part, the coupling of the SM fermios to the (non SM) Higgs component H is unsuppressed [73, 74]. The coupling between h and the charged leptons involves the factor $C^h_e = \frac{\sin\alpha}{\cos\beta}$, which we are not going to take it into account in our phenomenological analysis.

3.3.2 Flipped

For this model, d_R couples to Φ_1 and e_R to Φ_2, so we define

$$\Theta_d = \begin{pmatrix}(\phi_d)_1 \\ (\phi_d)_2\end{pmatrix} := \Theta_1 = \begin{pmatrix}\varphi_1^+ \\ \varphi_1^0\end{pmatrix},$$

$$\Theta_e = \begin{pmatrix}(\phi_e)_1 \\ (\phi_e)_2\end{pmatrix} := \Theta_2 = \begin{pmatrix}\varphi_2^+ \\ \varphi_2^0\end{pmatrix}.$$

In this case, the corresponding kinetic Lagrangian is given by

$$\mathcal{L}_{\mathcal{K}} = \frac{3|y_d|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{|y_e|^2 + 3|y_u|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2). \tag{3.33}$$

Therefore, we make the following redefinition

$$\Theta_1 \to \frac{2g}{\sqrt{3|y_d|^2}}\Phi_1, \tag{3.34a}$$

$$\Theta_2 \to \frac{2g}{\sqrt{|y_e|^2+3|y_u|^2}}\Phi_2. \tag{3.34b}$$

Potential terms

The resulting quadratic terms are the same as those given in Eq. (3.26).

$$\begin{aligned}-\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &= \frac{1}{2g^2}\left[-\frac{2f_2\Lambda^2}{f_0}\Big(|y_d|^2(\Theta_1^\dagger\Theta_1)+(|y_e|^2+3|y_u|^2)(\Theta_2^\dagger\Theta_2)\Big)\right]\\ &= -\frac{4f_2\Lambda^2}{f_0}\left[(\Phi_1^\dagger\Phi_1)+(\Phi_2^\dagger\Phi_2)\right],\end{aligned} \tag{3.35}$$

The mixed terms are not zero in this case and can be calculated from Eq.(1.43) as follows

$$\begin{aligned}(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d) &= \begin{pmatrix}\varphi_1^- & \varphi_1^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_1^+\\ \varphi_1^0\end{pmatrix}\\ &= \left(\varphi_1^-\varphi_2^{0*}-\varphi_1^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_1^+-\varphi_2^+\varphi_1^0\right)\\ &= \underbrace{\varphi_1^{0*}\varphi_2^-\varphi_2^+\varphi_1^0+\varphi_1^-\varphi_2^{0*}\varphi_2^0\varphi_1^+}_{\lambda_3}-\underbrace{\varphi_1^-\varphi_2^{0*}\varphi_2^+\varphi_1^0-\varphi_1^{0*}\varphi_2^-\varphi_2^0\varphi_1^+}_{\lambda_4}\\ &= (\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2)-(\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1).\end{aligned} \tag{3.36}$$

Next, the quartic terms are

$$\begin{aligned}\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &= \frac{1}{4g^2}\Big(3|y_d|^4(\Theta_1^\dagger\Theta_1)^2+(|y_e|^4+3|y_u|^4)(\Theta_2^\dagger\Theta_2)^2+6|y_u|^2|y_d|^2(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d)\Big)\\ &= \frac{4}{3}g^2(\Phi_1^\dagger\Phi_1)^2+\frac{4\cdot(|y_e|^4+3|y_u|^4)g^2}{(|y_e|^2+3|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2\\ &\quad+\frac{8g^2|y_u|^2}{|y_e|^2+3|y_u|^2}\Big((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)-(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\Big),\end{aligned} \tag{3.37}$$

and taking into account that $|y_e| \ll |y_u|$, we define

$$\mu_1^2=-4\frac{f_2\Lambda^2}{f_0}=\mu_2^2,\quad \frac{\lambda_1}{2}=\frac{4}{3}g^2,\quad \frac{\lambda_2}{2}\approx\frac{4}{3}g^2,\quad \lambda_3=-\lambda_4\approx\frac{8}{3}g^2,\quad \mu_{12}^2=\lambda_5=0. \tag{3.38}$$

Hence, from Eq. (1.40), we get the flipped NCG potential given by

$$\begin{aligned}V &= \mu_1^2(\Phi_1^\dagger\Phi_1)+\mu_2^2(\Phi_2^\dagger\Phi_2)+\frac{\lambda_1}{2}(\Phi_1^\dagger\Phi_1)^2+\frac{\lambda_2}{2}(\Phi_2^\dagger\Phi_2)^2\\ &\quad+\lambda_3(\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)+\lambda_4(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1).\end{aligned} \tag{3.39}$$

Fermionic interaction

Now, we make the following redefinition for the Yukawa couplings

$$
\begin{aligned}
y_e &:= -i\sqrt{\frac{|y_e|^2+3|y_u|^2}{2}}\frac{m_e}{gv_2},\\
y_u &:= -i\sqrt{\frac{|y_e|^2+3|y_u|^2}{2}}\frac{m_t}{gv_2},\\
y_d &:= -i\sqrt{\frac{3|y_d|^2}{2}}\frac{m_d}{gv_1}.
\end{aligned} \tag{3.40}
$$

Then, to get the Yukawa interaction we replace the last into Eq. (3.7) to get

$$
\begin{aligned}
\frac{1}{2}\langle J\Psi,(\gamma_5\otimes\Phi)\Psi\rangle &= i\frac{m_e}{v_2}(\rho_2+v_2)\langle J_M\overline{\boldsymbol{e}},\boldsymbol{e}\rangle\\
&+i\frac{m_t}{v_2}(\rho_2+v_2)\langle J_M\overline{\boldsymbol{u}},\boldsymbol{u}\rangle\\
&+i\frac{m_d}{v_1}(\rho_1+v_1)\langle J_M\overline{\boldsymbol{d}},\boldsymbol{d}\rangle.
\end{aligned}
$$

By using again Eq.(1.52) the interaction of fermions with the scalars are given by

$$
\begin{aligned}
\mathcal{L} &\supseteq i\left(\frac{m_e}{v_2}\langle J_M\overline{\boldsymbol{e}},\boldsymbol{e}\rangle + i\frac{m_t}{v_2}\langle J_M\overline{\boldsymbol{u}},\boldsymbol{u}\rangle\right)\rho_2+\frac{m_d}{v_1}\langle J_M\overline{\boldsymbol{d}},\boldsymbol{d}\rangle\rho_1\\
&= i\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}},\boldsymbol{e}\rangle + \frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}},\boldsymbol{u}\rangle\right)\frac{\rho_2}{\sin\beta}+i\frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}},\boldsymbol{d}\rangle\frac{\rho_1}{\cos\beta}\\
&= i\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}},\boldsymbol{e}\rangle + \frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}},\boldsymbol{u}\rangle\right)\frac{H\sin\alpha+h\cos\alpha}{\sin\beta}\\
&+i\frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}},\boldsymbol{d}\rangle\frac{H\cos\alpha-h\sin\alpha}{\cos\beta}\\
&= i\left[\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}},\boldsymbol{e}\rangle + \frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}},\boldsymbol{u}\rangle\right)\frac{\sin\alpha}{\sin\beta}+\frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}},\boldsymbol{d}\rangle\frac{\cos\alpha}{\cos\beta}\right]H\\
&+i\left[\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}},\boldsymbol{e}\rangle + \frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}},\boldsymbol{u}\rangle\right)\frac{\cos\alpha}{\sin\beta}-\frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}},\boldsymbol{d}\rangle\frac{\sin\alpha}{\cos\beta}\right]h.
\end{aligned} \tag{3.41}
$$

We see that the coupling between the higgs boson h and the top quark is the same as the found for the Lepton Specific model, so we define this coupling by $C_u^h=\frac{\cos\alpha}{\sin\beta}$. Similarly we introduce the symbol C_d^h to denote the coupling between h and the bottom (type down) quark, which for this particular case is $C_d^h=\frac{\sin\alpha}{\cos\beta}$.

3.3.3 Type-II

For this model, we should make the following choice

$$
\Theta_d=\begin{pmatrix}(\phi_d)_1\\(\phi_d)_2\end{pmatrix}=\Theta_e=\begin{pmatrix}(\phi_e)_1\\(\phi_e)_2\end{pmatrix}:=\Theta_1=\begin{pmatrix}\varphi_1^+\\ \varphi_1^0\end{pmatrix},
$$

Then, the kinetic terms from Eq. (3.9) are

$$\mathcal{L}_{\mathcal{K}} = \frac{|y_e|^2 + 3|y_d|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{3|y_u|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2), \tag{3.42}$$

suggesting the following normalization

$$\Theta_1 \to \frac{2g}{\sqrt{|y_e|^2 + 3|y_d|^2}}\Phi_1, \tag{3.43a}$$

$$\Theta_2 \to \frac{2g}{\sqrt{3}|y_u|}\Phi_2. \tag{3.43b}$$

Potential terms

Again, the resulting quadratic terms are the same as those given in Eqs. (3.26) and (3.35)

$$\begin{aligned} -\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &= \frac{1}{2g^2}\left[-\frac{2f_2\Lambda^2}{f_0}\left((|y_e|^2 + 3|y_d|^2)\Theta_1^\dagger\Theta_1 + 3|y_u|^2\Theta_2^\dagger\Theta_2\right)\right] \\ &= -\frac{4f_2\Lambda^2}{f_0}\left[(\Phi_1^\dagger\Phi_1) + (\Phi_2^\dagger\Phi_2)\right], \end{aligned} \tag{3.44}$$

Taking into account Eq.(1.43), the mixed terms are given by

$$\begin{aligned} (\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d) &= \begin{pmatrix}\varphi_1^- & \varphi_1^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*} \\ -\varphi_2^-\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_1^+ \\ \varphi_1^0\end{pmatrix} \\ &= \left(\varphi_1^-\varphi_2^{0*} - \varphi_1^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_1^+ - \varphi_2^+\varphi_1^0\right) \\ &= \varphi_1^-\varphi_2^{0*}\varphi_2^0\varphi_1^+ - \varphi_1^-\varphi_2^{0*}\varphi_2^+\varphi_1^0 - \varphi_1^{0*}\varphi_2^-\varphi_2^0\varphi_1^+ + \varphi_1^{0*}\varphi_2^-\varphi_2^+\varphi_1^0 \\ &= (\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1). \end{aligned} \tag{3.45}$$

Then, the quartic potential terms are given by

$$\begin{aligned} \frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &= \frac{1}{4g^2}[(|y_e|^4 + 3|y_d|^4)(\Theta_1^\dagger\Theta_1)^2 + 3|y_u|^4(\Theta_2^\dagger\Theta_2)^2 \\ &\quad + 6|y_u|^2|y_d|^2(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d)] \\ &= \frac{4\cdot(|y_e|^4 + 3|y_d|^4)g^2}{(|y_e|^2 + 3|y_d|^2)^2}(\Phi_1^\dagger\Phi_1)^2 + \frac{4g^2}{3}(\Phi_2^\dagger\Phi_2)^2 \\ &\quad + \frac{8g^2|y_d|^2}{|y_e|^2 + 3|y_d|^2}\left((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) - (\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\right). \end{aligned} \tag{3.46}$$

In this way, by defining R_d^e as the rate between $|y_e|^2$ and $|y_d|^2$, we have

$$\begin{aligned} &\mu_1^2 = -4\frac{f_2\Lambda^2}{f_0} = \mu_2^2, \qquad \frac{\lambda_1}{2} = \frac{4((R_d^e)^2 + 3)}{(R_d^e + 3)^2}g^2, \qquad \frac{\lambda_2}{2} \approx \frac{4}{3}g^2, \\ &\lambda_3 \approx \frac{8}{R_d^e + 3}g^2, \qquad \lambda_4 \approx -\frac{8}{R_d^e + 3}g^2, \qquad \mu_{12}^2 = \lambda_5 = 0. \end{aligned} \tag{3.47}$$

Therefore, we get a potential with the same form as the obtained for the flipped model. Despite that, notice that the conditions in Eq. (3.38) (with $R_d^e \to 1$) are not the same as the ones obtained in Eq. (3.47).

Fermionic interaction

Now, we make the following redefinition for the Yukawa couplings

$$\begin{aligned} y_u &:= -i\sqrt{\frac{3}{2}}|y_u|\frac{m_t}{gv_2}, \\ y_e &:= -i\sqrt{\frac{|y_e|^2+3|y_d|^2}{2}}\frac{m_e}{gv_1}, \\ y_d &:= -i\sqrt{\frac{|y_e|^2+3|y_d|^2}{2}}\frac{m_d}{gv_1}, \end{aligned} \tag{3.48}$$

Next, by replacing in Eq. (3.7) we get

$$\begin{aligned} \frac{1}{2}\langle J\Psi, (\gamma_5 \otimes \Phi)\Psi\rangle &= i\frac{m_e}{v_1}(\rho_1 + v_1)\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle \\ &+ i\frac{m_d}{v_1}(\rho_1 + v_1)\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle \\ &+ i\frac{m_t}{v_2}(\rho_2 + v_2)\langle J_M\overline{\boldsymbol{u}}, \boldsymbol{u}\rangle. \end{aligned}$$

Then, by using again Eq.(1.52), the interaction between fermions and scalars is given by

$$\begin{aligned} \mathcal{L} &\supseteq i\left(\frac{m_e}{v_1}\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle + i\frac{m_d}{v_1}\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\rho_1 + \frac{m_t}{v_2}\langle J_M\overline{\boldsymbol{u}}, \boldsymbol{u}\rangle\rho_2 \\ &= i\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle + \frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{\rho_1}{\cos\beta} + i\frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}}, \boldsymbol{u}\rangle\frac{\rho_2}{\sin\beta} \\ &= i\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle + \frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{H\cos\alpha - h\sin\alpha}{\cos\beta} \\ &+ \frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}}, \boldsymbol{u}\rangle\frac{H\sin\alpha + h\cos\alpha}{\sin\beta} \\ &= i\left[\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle + \frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{\cos\alpha}{\cos\beta} + \frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\frac{\sin\alpha}{\sin\beta}\right]H \\ &+ i\left[-\left(\frac{m_e}{v}\langle J_M\overline{\boldsymbol{e}}, \boldsymbol{e}\rangle + \frac{m_d}{v}\langle J_M\overline{\boldsymbol{d}}, \boldsymbol{d}\rangle\right)\frac{\sin\alpha}{\cos\beta} + \frac{m_t}{v}\langle J_M\overline{\boldsymbol{u}}, \boldsymbol{u}\rangle\frac{\cos\alpha}{\sin\beta}\right]h. \end{aligned} \tag{3.49}$$

So, as in the previous cases, the coupling between the Higgs and the top will be given by $C_u^h = \frac{\cos\alpha}{\sin\beta}$, whereas $C_d^h = \frac{\sin\alpha}{\cos\beta}$ is the same as for the Flipped model and so it differs from the one obtained for the Lepton Specific case. It is important to stand out that the the mixing angle β will be restricted by the top quark mass.

3.4 Phenomenology of the NCG 2HDM

Let us start noticing that the main NCG restriction on SM is given by its boundary conditions. However we will see that $\tan\beta \approx 2$ is needed to get a top quark mass of approximately 173 GeV. The Yukawa coupling RGE for the top quark is the same in all the three Lepton Specific, Flipped and Type-II models, which match with the one given in Eq. (3.20). From Eqs. (3.31), (3.40), and (3.48) and taking into account (for the third SM family) that $|y_e| \sim |y_d| \ll |y_u|$, we get again Eq. (3.22) as the common boundary condition for the top Yukawa coupling RGE.

Next, the beta-functions of scalar couplings RGEs can be calculated from Eq. (1.63) and are given by

$$
\begin{aligned}
16\pi^2\beta_{\lambda_1} &= 12\lambda_1^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2|\lambda_5|^2 \\
&\quad + \frac{3}{4}g_1^4 + \frac{3}{2}g_1^2g_2^2 + \frac{9}{4}g_2^4 - 3(g_1^2 + 3g_2^2)\lambda_1, && \text{(3.50a)}\\
16\pi^2\beta_{\lambda_2} &= 12\lambda_2^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2|\lambda_5|^2 \\
&\quad + \frac{3}{4}g_1^4 + \frac{3}{2}g_1^2g_2^2 + \frac{3}{4}g_2^4 - 3(g_1^2 + 3g_2^2 - 4y_u^2)\lambda_2 - 12y_u^4, && \text{(3.50b)}\\
16\pi^2\beta_{\lambda_3} &= (6\lambda_3 + 2\lambda_4)(\lambda_1 + \lambda_2) + 4\lambda_3^2 + 2\lambda_4^2 + 2|\lambda_5|^2 \\
&\quad + \frac{3}{4}g_1^4 - \frac{3}{2}g_1^2g_2^2 + \frac{9}{4}g_2^4 - 3(g_1^2 + 3g_2^2 - 2y_u^2)\lambda_3, && \text{(3.50c)}\\
16\pi^2\beta_{\lambda_4} &= 2\lambda_1\lambda_4 + 2\lambda_2\lambda_4 + 8\lambda_3\lambda_4 + 4\lambda_4^2 + 8|\lambda_5|^2 \\
&\quad +3g_1^2g_2^2 - 3(g_1^2 + 3g_2^2 - 2y_u^2)\lambda_4, && \text{(3.50d)}\\
16\pi^2\beta_{\lambda_5} &= (2\lambda_1 + 2\lambda_2 + 8\lambda_3 + 12\lambda_4)\lambda_5 - 3(g_1^2 + 3g_2^2 - 2y_u^2)\lambda_5, && \text{(3.50e)}
\end{aligned}
$$

Next, we make use of the re-definitions on Eqs. (3.29), (3.38), and (3.47) to impose boundary conditions to the above scalar coupling RGEs. We summarize these boundary conditions in table 3.5.

	Lepton	Flipped	Type-II
y_u	$\frac{2g}{\sqrt{3}}$	$\frac{2g}{\sqrt{3}}$	$\frac{2g}{\sqrt{3}}$
λ_1	$8g^2$	$\frac{8g^2}{3}$	$2g^2$
λ_2	$\frac{8g^2}{3}$	$\frac{8g^2}{3}$	$\frac{8g^2}{3}$
λ_3	0	$\frac{8g^2}{3}$	$2g^2$
λ_4	0	$-\frac{8g^2}{3}$	$-2g^2$
λ_5	0	0	0
μ_{12}^2	0	0	0

Table 3.5: High scale boundary conditions for the scalar and Yukawa couplings.

Assuming an unification scale of $\sim 10^{16}$ GeV and an unified gauge coupling $g = 0.5$, we have fixed $\tan\beta \approx 2.14$ to have a reasonable top mass value:

$$\begin{array}{llll} \text{When} & \tan\beta \to 10.0, & \text{then} \quad m_t \approx 190, & \text{and} \quad m_h \approx 190. \\ \text{When} & \tan\beta \to 2.14, & \text{then} \quad m_t \approx 173, & \text{and} \quad m_h \approx 173. \\ \text{When} & \tan\beta \to 1.00, & \text{then} \quad m_t \approx 137, & \text{and} \quad m_h \approx 140, \end{array}$$

where masses are given in GeV. The charged scalar mass does not depend on $\tan\beta$.

Having fixed all these conditions, we are now in a position to run down to low energies the RGEs in Eqs. (3.50). As reference for our low energy calculations we take the Z boson mass $M_Z = 91.187$ GeV, given by Eq. (1.34).

For the CP-odd field A, we have a null mass (see Eq. (1.55)) because the boundary conditions for μ_{12}^2 and λ_5 are always null.

For the CP-even (uncharged) fields, in particular, for the Lepton Specific model we get the following two different eigenvalues

$$\begin{aligned} M_H^2 &= 5.2 \times 10^3 \ \ \text{GeV}^2 \to M_H = 72 \ \ \text{GeV} \\ M_h^2 &= 3.0 \times 10^4 \ \ \text{GeV}^2 \to M_h \ = 174 \,\text{GeV}. \end{aligned} \tag{3.51}$$

Hence, we can conclude that the SM Higgs boson mass is $m_h = 174$ GeV. The procedure for the remaining two models is exactly the same.

Next by using Eq. (1.50) we can calculate the mass for the charged fields for each one of the three models. The full mass spectrum is shown in table 3.6.

	Lepton-Specific	Flipped	Type-II
m_t	173.5	173.5	173.5
m_h	174	175	175
m_H	72	45.9	48.6
$m_{H^\pm}$	36	118	106
m_A	0	0	0

Table 3.6: Mass spectrum for an unification scale of $\sim 10^{16}$ GeV, $g = 0.5$ and $\tan\beta \approx 2.14$. The values are given in GeV.

If the parameter μ_{12}^2 is zero to some scale, then it will be zero for any scale. So, from (1.56) we obtain $m_A = 0$, as can be appreciated in table 3.6. The Higgs boson mass is still around 174 GeV and it does not overlap with its experimental value.

We saw that the pseudoscalar A becomes massless due to the absence of the terms μ_{12}^2 and λ_5 in the potential in Eq. (1.40). So this potential gets an accidental $U(1)$ symmetry, which amounts to the freedom for independent phase transformations of the two doublets that was discussed in section 1.4.4. Since the VEV of the Higgs fields spontaneously break this symmetry, there appears a Nambu-Goldstone (NG) mode. because the mass for the CP-odd Higgs, m_A, vanishes, it suggest an identification of this pseudoscalar field with the NG mode. But such massless mode does not exist in nature [75, 48].

These models also give rise to stable domain walls, which are sheet-like topological defects that would carry an undesirable fate for the Universe, so they are cosmologically disfavored. This is because domain wall energy per unit volume grows alike to that of the matter and radiation in both the matter and radiation dominated epochs, and at some time will exceed them and come to dominate the Universe energy per unit volume, despite of the energy per unit area of the domain walls [76].

As it was mentioned in section 1.4.4, to have a realistic model, A should acquire mass. This is reached by including $U(1)$ breaking terms on the vacuum structure. This introduction, can lift the general degeneracy observed in the vacuum manifolds associated with domain walls, creating a true and false vacuum. Additionally, the domain wall energy per unit volume suffers a favourable exponential suppression, allowing domain walls to form but not to dominate the Universe by removing the relative growth of the domain wall energy per unit volume to that of the background [45, 77].

The conclusion is that the NCG 2HDM is in conflict with phenomenological constrains unless that parameters like μ_{12}^2 or λ_5 are allowed in the scalar potential. Inspired by the fact that the presence of such terms proportional to μ_{12}^2 and λ_5 not only give mass to the pseudoscalar A but also may forbid the formation of domain walls, we will explore the influence of such terms on the spectral action and so, in the scalar mass spectrum of the NCG 2HDM.

3.5 A viable NCG 2HDM Type-II

In this section we explore the effect to input terms proportional to μ_{12}^2 and λ_5 in our mass spectrum analysis developed in section 3.4. For phenomenological motives we will focus our attention in the Type-II model, and so, from Eq. (3.47) we fix (for $g = 0.5$) the following boundary conditions

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{2}{3}, \quad \lambda_3 = \frac{1}{2}, \quad \text{and} \quad \lambda_4 = -\frac{1}{2}. \tag{3.52}$$

We set $\tan\beta = 2.14$ in order to account for the correct top quark mass (see section 3.4). As a further restriction, we consider only configurations which are in agreement with the experimental Higgs boson mass $m_h \approx 125$ GeV, and satisfying $C_V^h = \sin(\beta - \alpha) \gtrsim 0.99$, as required by the alignment limit [42, 78] .

Hence, taking into account these constrains, in tables 3.7 and 3.8 we have computed the NCG 2HDM type-II scalar mass spectrum for selected values of μ_{12}^2 and λ_5 (identified by color dots) within the region enclosed by $\mu_{12}^2 \geq 10^5$ (GeV)^2 and $-1 \leq \lambda_5 \leq 1$, as it is shown in figure 3.2.

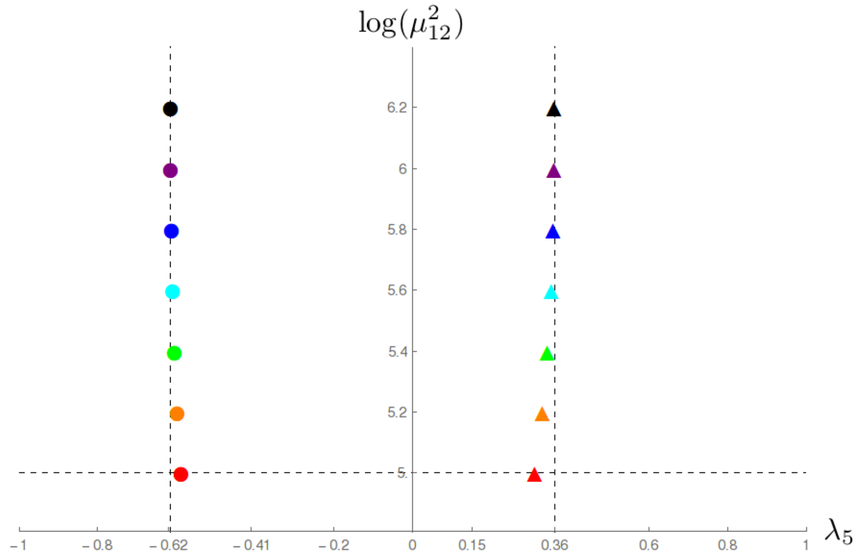


Figure 3.2: The color dots stands for values of λ_5 and the exponent of $\mu_{12}^2 = 10^{\log(\mu_{12}^2)}$ GeV^2 which are in agreement with a Higgs boson mass of approximately 125 GeV.

	m_H[GeV]	m_A[GeV]	$m_{H^\pm}$[GeV]
● (red)	525	500	526
● (orange)	655	635	655
● (green)	819	803	820
● (cyan)	1027	1014	1027
● (blue)	1289	1279	1290
● (purple)	1620	1612	1621
● (black)	2038	2032	2038

Table 3.7: Scalar mass spectrum for the NCG 2HDM type-II with negative values of λ_5 (close to -0.62). The Higgs boson mass is fixed $m_h \approx$ 125 GeV.

	m_H[GeV]	m_A[GeV]	$m_{H^\pm}$[GeV]
▲ (red)	527	519	528
▲ (orange)	656	650	658
▲ (green)	820	815	821
▲ (cyan)	1028	1024	1029
▲ (blue)	1290	1287	1291
▲ (purple)	1621	1619	1622
▲ (black)	2038	2036	2039

Table 3.8: Scalar mass spectrum for the NCG 2HDM type-II with positive values of λ_5 (close to 0.36). The Higgs boson mass is fixed to $m_h \approx$ 125 GeV.

We added to the spectral action (2.42) the non zero parameters μ_{12}^2 and λ_5, so that they satisfy any of the conditions depicted in figure 3.2, then we found that it is possible to have a Higgs boson mass of approximately $m_h = 125$ GeV. We emphasize that the so-called 'decoupling' limit is reached by $\mu_{12}^2 \gtrsim 10^6$ GeV^2 (i.e, the limit where the physical masses of the new Higgs bosons become heavy, $m_{H,A,H^\pm} \gtrsim 1$ TeV, and their effects decouple at low energies). In that case, we found two possibilities: $\lambda_5 \approx -0.62$ as well as $\lambda_5 \approx 0.36$. This can be appreciated by the asymptotic behavior around the dashed vertical lines displayed in figure 3.2.

For the 2HDM type II, the alignment without decoupling is also possible but there is a strong lower limit for the masses of the Higgs-like particles so that $m_{H,H^\pm} \gtrsim 400$ GeV. So, to be in agreement with this phenomenological limit we must take $\mu_{12}^2 \gtrsim 10^5$ GeV^2.

The strength of the couplings between the Higgs boson h to the gauge bosons (C_V^h) and the quark sector[3] (C_u^h and C_d^h) are enlisted in the tables 3.9 and 3.10. The coupling between H and the vector bosons (C_V^H) can be computed directly from C_V^h.

	C_V^h	C_u^h	C_d^h
● (red)	0.998	0.970	1.127
● (orange)	0.999	0.981	1.081
● (green)	1.000	0.988	1.052
● (light blue)	1.000	0.993	1.033
● (blue)	1.000	0.995	1.021
● (purple)	1.000	0.997	1.013
● (black)	1.000	0.998	1.008

Table 3.9: The first column contains the strength of the couplings between h and the vector bosons. The last two columns stands for the coupling of h to the quark sector. The values are obtained for $-0.62 \leq \lambda_5 \leq -0.59$.

	C_V^h	C_u^h	C_d^h
▲ (red)	0.998	0.968	1.134
▲ (orange)	0.999	0.980	1.087
▲ (green)	1.000	0.988	1.055
▲ (light blue)	1.000	0.992	1.035
▲ (blue)	1.000	0.995	1.022
▲ (purple)	1.000	0.997	1.014
▲ (black)	1.000	0.998	1.009

Table 3.10: The first column contains the strength of the couplings between h and the vector bosons. The last two columns stands for the coupling of h to the quark sector. The values are obtained for $0.31 \leq \lambda_5 \leq 0.36$.

We have considered the main theoretical and phenomenological constraints, including the signal strengths of the observed 125 GeV Higgs state at the Large Hadron Collider (LHC), as well as the most recent limits coming from searches for heavy Higgs-like states. Therefore, in order to obtain a viable NCG 2HDM Type-II (together the extra non-zero parameters μ_{12}^2 and λ_5), we observe from tables 3.9 and 3.10 that the alignment limit $C_V^h \approx 1$ is entirely reached for $\mu_{12}^2 \geq 10^5$ (GeV)^2 and $-0.62 \lesssim \lambda_5 \lesssim -0.59$, or $-0.31 \lesssim \lambda_5 \lesssim -0.36$.

It is worth noting that $\lambda_5 = 0$ is still compatible with current Higgs measurements. However, in this case the mass of the top quark is lower than its experimental value $m_t \approx 164$ GeV. For this reason we have chosen non-zero values for λ_5 in order to get $m_t \approx 173$ GeV.

3.6 The NCG 2HDM+σ+ν_R

Among its wide scalar mass spectrum, the 2HDM gave us a Higgs boson mass of ≈ 170 GeV, which coincides with standard NCG predicted value. Inspired by the solution for this shortcoming offered in [15], we will consider an additional Majorana right-handed neutrino to the NCG 2HDM. For this case, the Krajewski diagram is depicted in figure 3.3.

[3] As it was pointed out in section 1.4.2, the couplings between H and the quark sector are unsuppressed from current phenomenological bounds and so we do not take them into account.

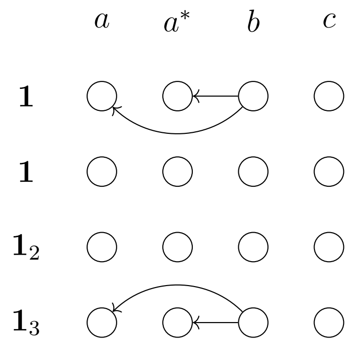


Figure 3.3: Krajewski diagram for the SM with right-handed neutrino.

In that case, there is an extra fermion degree of freedom a^* on the right-handed representation ρ_R (and its corresponding anti-particle state), which corresponds to the right-handed neutrino ν_R. On the Dirac operator side, its blocks acquire the following form

$$\mathcal{M} = \begin{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} & \\ & \begin{pmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{pmatrix} \otimes 1_3 \end{pmatrix}, \tag{3.53}$$

Additionally, in Eq. (2.21) we introduce a Majorana mass term Σ into the off-diagonal block of the Dirac operator. The original goal for inserting such a term was to explain neutrinos light masses through the (Type-I) seesaw mechanism [53, 9, 79, 80]. In general, this term does not survive after the fluctuation procedure given in Eq. (3.1), so it is not a true scalar field. As explained in [**?**], by enlarging the internal (finite) algebra, it is possible to turn this terms into an actual scalar field. For its part, in [16] the reformulation of spectral triples in terms of the 'fused' algebra $\Omega A \oplus H$ discussed in section 2.5.2, address to a new Abelian $B-L$ gauge symmetry which fluctuates a complex scalar field. Furthermore, in [81, 82, 83, 84] new scalar fields have been found by enlarging the fermion sector of the theory[4]. It is not the objective of this thesis to elaborate on this point, so we just assume that the total fluctuated finite Dirac operator is given by

$$\Phi = \left(\begin{array}{cccc|cccc}
 & & y_e^*\Theta_e \;\; y_\nu^*\Theta_\nu & & & & & \\
 & & & y_u^*\Theta_u \;\; y_d^*\Theta_d & & & & \\
\begin{matrix} y_e\Theta_e^\dagger \\ y_\nu\Theta_\nu^\dagger \end{matrix} & & & & & & y_R^*\Sigma & \\
 & \begin{matrix} y_u\Theta_u^\dagger \\ y_d\Theta_d^\dagger \end{matrix} & & & & & & \\
\hline
 & & & & & & y_e\Theta_e^* \;\; y_e\Theta_\nu^* & \\
 & & & & & & & y_u\Theta_u^* \;\; y_d\Theta_d^* \\
 & & y_R\Sigma^* & & \begin{matrix} y_e^*\Theta_e^T \\ y_\nu^*\Theta_\nu^T \end{matrix} & & & \\
 & & & & & \begin{matrix} y_u^*\Theta_u^T \\ y_d^*\Theta_d^T \end{matrix} & &
\end{array}\right). \tag{3.54}$$

[4]The new scalar field that couples the right-handed neutrino to the new fermions, considered in [85], violates the order two axiom, so it is outside of the scope of this work.

Then, the additional Yukawa interaction terms are the following

$$-\frac{1}{2}\langle J\Psi,(\gamma_5\otimes\Phi)\Psi\rangle \supseteq \langle J_M\overline{\boldsymbol{\nu}_R}, y_\nu((\phi_\nu)_1^*\boldsymbol{\nu}_L+(\phi_\nu)_2^*\boldsymbol{e}_L)\rangle+\frac{1}{2}\langle J_M\overline{\boldsymbol{\nu}_R}, y_R^*\Sigma\overline{\boldsymbol{\nu}_R}\rangle + \text{h.c.} \tag{3.55}$$

Moreover, the scalar charges should be chosen as indicated in table 3.11.

Scalar field	Charge	Anti-particle	Charge
$(\phi_\nu)_1$	0	$(\phi_\nu)_1^*$	0
$(\phi_\nu)_2$	-1	$(\phi_\nu)_2^*$	$+1$

Table 3.11: Charge assignment for the new scalar fields

Following Eq. (3.8), the kinetic terms are given by

$$\Delta=\begin{pmatrix} & & y_e^*D_\mu\Theta_e & y_\nu^*D_\mu\Theta_\nu & & \\ & & & & y_u^*D_\mu\Theta_u & y_d^*D_\mu\Theta_d \\ y_eD_\mu\Theta_e^\dagger & & & & & \\ y_\nu D_\mu\Theta_\nu^\dagger & & & & & \\ & y_uD_\mu\Theta_u^\dagger & & & & \\ & y_dD_\mu\Theta_d^\dagger & & & & \end{pmatrix}, \tag{3.56}$$

and the action of Π on the right-handed neutrino (and its antiparticle) is given by multiplication with $\partial_\mu\Sigma$ as follows

$$D_\mu\Phi\Big|_{\Pi}(\overline{\nu_R})=y_R\partial_\mu\Sigma(\nu_R),\quad \text{and}\quad D_\mu\Phi\Big|_{\Pi^*}(\nu_R)=y_R\partial_\mu\Sigma^*(\overline{\nu_R}). \tag{3.57}$$

The new term on Δ (related to Eq. (3.8)) is

$$D_\mu\Theta_\nu=\begin{pmatrix}\partial_\mu(\phi_\nu)_1+i[(Q_\mu^3-\Lambda_\mu)(\phi_\nu)_1+Q_\mu^+(\phi_\nu)_2]\\ \partial_\mu(\phi_\nu)_2+i[(-Q_\mu^3-\Lambda_\mu)(\phi_\nu)_2+Q_\mu^-(\phi_\nu)_1]\end{pmatrix},$$

where

$$D_\mu\Theta_\nu=\partial_\mu\Theta_\nu+iQ_\mu^\alpha\sigma^\alpha\Theta_\nu-i\Lambda_\mu\Theta_\nu. \tag{3.58}$$

Then, the extra kinetic term in the Lagrangian is

$$\frac{f_0}{8\pi^2}\text{Tr}[(D_\mu\Phi)(D^\mu\Phi)]\supseteq\frac{|y_\nu|^2}{4g^2}(D_\mu\Theta_\nu)^\dagger(D_\mu\Theta_\nu)+\frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma. \tag{3.59}$$

From here we may notice that the field Σ should be normalized by

$$\Sigma \to \frac{2\sqrt{2}g}{|y_R|}\sigma. \tag{3.60}$$

Hence, the new quadratic and quartic potential terms are given by

$$\begin{aligned}
-\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &\supseteq -\frac{f_2\Lambda^2}{f_0}\frac{1}{4g^2}\left(4|y_\nu|^2\Theta_\nu^\dagger\Theta_\nu + 2|y_R|^2\Sigma^*\Sigma\right)\\
&= -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2}{4g^2}\Theta_\nu^\dagger\Theta_\nu + \frac{|y_R|^2}{8g^2}\Sigma^*\Sigma\right)\\
&= -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2}{4g^2}\Theta_\nu^\dagger\Theta_\nu + \sigma^*\sigma\right),
\end{aligned} \tag{3.61}$$

$$\begin{aligned}
\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &\supseteq \frac{1}{16g^2}[4|y_\nu|^4(\Theta_\nu^\dagger\Theta_\nu)^2 + 8|y_e|^2|y_\nu|^2(\Theta_\nu^\dagger\Theta_e)({\Theta_e}^\dagger\Theta_\nu)\\
&+ 8|y_\nu|^2|y_R|^2(\Sigma^*\Sigma)(\Theta_\nu^\dagger\Theta_\nu) + 2|y_R|^4(\Sigma^*\Sigma)^2]\\
&= \frac{|y_\nu|^4}{4g^2}(\Theta_\nu^\dagger\Theta_\nu)^2 + \frac{|y_e|^2|y_\nu|^2}{2g^2}(\Theta_\nu^\dagger\Theta_e)({\Theta_e}^\dagger\Theta_\nu)\\
&+ 4|y_\nu|^2(\sigma^*\sigma)(\Theta_\nu^\dagger\Theta_\nu) + 8g^2(\sigma^*\sigma)^2.
\end{aligned} \tag{3.62}$$

After building the NCG SM+$\sigma+\nu_R$, we will consider the NCG 2HDM+$\sigma+\nu_R$. For making that we will have two options[5]: $\Theta_\nu = \widetilde{\Theta}_1$ or $\Theta_\nu = \widetilde{\Theta}_2$. We will focus on the case $\Theta_\nu = \widetilde{\Theta}_2$ to construct the Type II, as well as a Flipped-like, and Lepton-Specific-like models. The case $\Theta_\nu = \widetilde{\Theta}_1$ give rise to scalar fields with imaginary masses, which would correspond to tachyon fields [86] (see appendix C for the details). Only in that case, it will be possible to construct Type-I-like model called neutrinophilic Higgs doublet model [87].

3.6.1 The SM+$\sigma + \nu_R$

Here, we consider the NCG SM extended by a complex singlet scalar field. The extension to the NCG SM with a real scalar was first proposed in [15] to get the correct Higgs boson mass. Then, the complex case was also considered in [**?**].

The kinetic term in this case is given by

$$\frac{3(|y_d|^2+|y_u|^2)+|y_\nu|^2+|y_e|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma,$$

suggesting the following normalization

$$\Theta_2 \to \frac{2g}{\sqrt{3(|y_d|^2+|y_u|^2)+|y_\nu|^2+|y_e|^2}}\Phi_2. \tag{3.63}$$

[5]The tilde over the fields Θ_i is needed to be consistent with the charge assignment showed in table 3.11.

The quadratic scalar terms are given from Eq. (3.11) by

$$
\begin{aligned}
-\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &= \frac{1}{2g^2}\left(-\frac{2f_2\Lambda^2}{f_0}(|y_e|^2+3|y_d|^2+3|y_u|^2)(\Theta_2^\dagger\Theta_2)\right)\\
&= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_2^\dagger\Phi_2+\sigma^*\sigma\right). \qquad (3.64)
\end{aligned}
$$

The mixed quartic terms are

$$
\begin{aligned}
(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) &= \begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_2^+\\ \varphi_2^0\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^- & \varphi_2^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}\\
&= \left(\varphi_2^-\varphi_2^{0*}-\varphi_2^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_2^+-\varphi_2^+\varphi_2^0\right)\\
&= \varphi_2^-\varphi_2^{0*}\varphi_2^0\varphi_2^+-\varphi_2^-\varphi_2^{0*}\varphi_2^+\varphi_2^0-\varphi_2^{0*}\varphi_2^-\varphi_2^0\varphi_2^++\varphi_2^{0*}\varphi_2^-\varphi_2^+\varphi_2^0\\
&= 0.
\end{aligned}
$$

Hence, the quartic potential terms are

$$
\begin{aligned}
\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &\supseteq \frac{3(|y_d|^4+|y_u|^4)+|y_\nu|^4}{4g^2}(\Theta_2^\dagger\Theta_2)^2+4|y_\nu|^2(\sigma^*\sigma)(\Theta_2^\dagger\Theta_2)\\
&= \frac{3(|y_d|^4+|y_u|^4)+|y_\nu|^4+|y_e|^4}{(3(|y_d|^2+|y_u|^2)+|y_\nu|^2)^2}4g^2(\Phi_2^\dagger\Phi_2)^2\\
&+\frac{16|y_\nu|^2g^2}{3(|y_d|^2+|y_u|^2)+|y_\nu|^2+|y_e|^2}(\sigma^*\sigma)(\Phi_2^\dagger\Phi_2). \qquad (3.65)
\end{aligned}
$$

So by defining $R_u^\nu := \frac{|y_\nu|^2}{|y_u|^2}$, and taking into account again $|y_e|\sim|y_d|\ll|y_u|$ we make the following redefinition

$$
\begin{aligned}
&\mu_2^2 = -4\frac{f_2\Lambda^2}{f_0} = \mu_S^2, \qquad \lambda_2 \approx \frac{4((R_u^\nu)^2+3)}{(R_u^\nu+3)^2}g^2, \qquad \mu_1^2=\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=0\\
&\lambda_{S1}=0, \qquad 2\lambda_{S2}\approx\frac{16R_u^\nu}{R_u^\nu+3}g^2, \qquad \lambda_S=16g^2. \qquad (3.66)
\end{aligned}
$$

Hence, we get

$$
V=\mu_2^2(\Phi_2^\dagger\Phi_2)+\mu_S^2\sigma^*\sigma+\lambda_2(\Phi_2^\dagger\Phi_2)^2+\lambda_S(\sigma^*\sigma)^2+2\lambda_{S2}(\sigma^*\sigma)(\Phi_2^\dagger\Phi_2). \qquad (3.67)
$$

Now, the corresponding normalization for the top and neutrino Yukawa couplings are

$$
y_u := -i\sqrt{\frac{|y_e|^2+|y_\nu|^2+3|y_d|^2+3|y_u|^2}{2}}\frac{m_t}{gv_2}, \qquad (3.68)
$$

$$
y_\nu := -i\sqrt{\frac{|y_e|^2+|y_\nu|^2+3|y_d|^2+3|y_u|^2}{2}}\frac{m_\nu}{gv_2}, \qquad (3.69)
$$

so after multiplying by its complex conjugate and because $|y_e| \sim |y_d| \ll |y_u| \sim |y_\nu|$, we have

$$\frac{m_t}{v_2} \approx \frac{\sqrt{2}}{\sqrt{3+R_u^\nu}} g, \quad \text{and} \quad \frac{m_\nu}{v_2} \approx \sqrt{\frac{2R_u^\nu}{3+R_u^\nu}} g, \tag{3.70}$$

Hence, we get

$$y_u \approx \frac{2}{\sqrt{3+R_u^\nu}} g, \quad \text{and} \quad y_\nu \approx \sqrt{\frac{R_u^\nu}{3+R_u^\nu}} 2g, \tag{3.71}$$

Mass spectrum

First, we parametrized the scalar fields as follows

$$\Phi_2 = \begin{pmatrix} 0 \\ \frac{\rho_2 + i\eta_2 + v_2}{\sqrt{2}} \end{pmatrix}, \quad \text{and} \quad \sigma = \frac{S + i\chi + v_S}{\sqrt{2}} \tag{3.72}$$

Then, the minimum conditions for the potential in Eq. (3.67) are given by

$$\mu_2^2 = -\lambda_2 v_2^2 - \lambda_{S2} v_S^2, \tag{3.73a}$$
$$\mu_S^2 = -\lambda_S v_S^2 - \lambda_{S2} v_2^2, \tag{3.73b}$$

Then, after expanding that potential around the real components we get

$$\begin{aligned} V \supseteq & \left(-\lambda_2 v_2^2 - \lambda_{S2} v_S^2\right) \frac{\rho_2^2}{2} + \lambda_2 \frac{3}{2} \rho_2^2 v_2^2 \\ & + \left(-\lambda_S v_S^2 - \lambda_{S2} v_2^2\right) \frac{S^2}{2} + \lambda_S \frac{3}{2} S^2 v_S^2 \\ & + \lambda_{S2} \frac{S^2 v_2^2 + 4 S \rho_2 v_2 v_S + \rho_2^2 v_S^2}{2} \\ & = \lambda_2 \rho_2^2 v_2^2 + \lambda_S S^2 v_S^2 + 2\lambda_{S2} S \rho_2 v_2 v_S \\ & = \frac{1}{2} \begin{pmatrix} \rho_2, & S \end{pmatrix} \begin{pmatrix} 2\lambda_2 v_2^2 & 2\lambda_{S2} v_2 v_S \\ 2\lambda_{S2} v_2 v_S & 2\lambda_S v_S^2 \end{pmatrix} \begin{pmatrix} \rho_2 \\ S \end{pmatrix}, \end{aligned} \tag{3.74}$$

so the eigenvalues are

$$m_h^2 = \lambda_S v_S^2 + \lambda_2 v_2^2 - \sqrt{(\lambda_S v_S^2 - \lambda_2 v_2^2)^2 + 4\lambda_{S2}^2 v_S^2 v_2^2}, \tag{3.75a}$$
$$m_s^2 = \lambda_S v_S^2 + \lambda_2 v_2^2 + \sqrt{(\lambda_S v_S^2 - \lambda_2 v_2^2)^2 + 4\lambda_{S2}^2 v_S^2 v_2^2}. \tag{3.75b}$$

For the imaginary components we have that

$$
\begin{aligned}
V \supseteq & \left(-\lambda_2 v_2^2 - \lambda_{S2} v_S^2\right) \frac{\eta_2^2}{2} + \lambda_2 \frac{1}{2}\eta_2^2 v_2^2 \\
& + \left(-\lambda_S v_S^2 - \lambda_{S2} v_2^2\right) \frac{\chi^2}{2} + \lambda_S \frac{1}{2}\chi^2 v_S^2 \\
& + \lambda_{S2} \frac{\chi^2 v_2^2 + \eta_2^2 v_S^2}{2} \\
& = \lambda_2 \rho_2^2 v_2^2 + \lambda_S S^2 v_S^2 + 2\lambda_{S2} S \rho_2 v_2 v_S \\
& = 0,
\end{aligned}
\tag{3.76}
$$

which means that η_2 as well as χ are massless.

Next, with the following set of beta-functions for the corresponding RGEs

$$16\pi^2 \beta_{y_u} = y_u \left(-8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_1^2 + \frac{9}{2}y_u^2\right), \tag{3.77a}$$

$$16\pi^2 \beta_{y_\nu} = y_\nu \left(-\frac{3}{4}g_1^2 - \frac{9}{4}g_2^2 + \frac{5}{2}y_\nu^2 + 3y_u^2\right), \tag{3.77b}$$

$$16\pi^2 \beta_\lambda = 24\lambda^2 - 6y_u^4 + 12y_u^2\lambda - 3(g_1^2 + 3g_2^2)\lambda + \frac{9}{8}g_2^4 + \frac{3}{8}g_1^4 + \frac{3}{4}g_2^2 g_1^2, \tag{3.77c}$$

$$16\pi^2 \beta_{\lambda_S} = 18\lambda_S^2 + 8\lambda_{S2}^2, \tag{3.77d}$$

$$16\pi^2 \beta_{\lambda_{S2}} = \lambda_{S2} \left(12\lambda_2 + 6\lambda_S + 8\lambda_{S2} + 6y_u^2 + 2y_\nu^2 - \frac{3}{2}(3g_2^2 + g_1^2)\right), \tag{3.77e}$$

together the boundary conditions given in Eq. (3.66) for $R_u^\nu \approx 2$, $g = 0.5$ and an unification scale of $\sim 10^{16}$ GeV, we get the low energies values for λ_S and λ_2. In general, the higher is λ_S the higher is m_s. For $v_S \geq 1 \times 10^4$ GeV, the masses m_t and m_h get stable values. Hence, if we replace these values on Eqs. (3.75a) and (3.75a) we get the mass spectrum shown in table 3.12.

	Mass
m_t	166
m_h	125
m_s	6.4×10^5

Table 3.12: Mass spectrum for the NCG SM with right-handed neutrino and a singlet scalar field. Here we have fixed $v_S = 10^6$ GeV. The values are given in GeV.

3.6.2 Lepton-Specific'$+\sigma + \nu_R$

In this subsection as well as in the next two, we have highlighted in blue all the new terms regarding to the ones given in section 3.3, and by Eqs. (3.60), (3.61) and (3.62).

Regarding to our parity assignment $\Phi_1 \to -\Phi_1$, $\Phi_2 \to +\Phi_2$, and $u_R \to +u_R$, given in section 1.4, this model is set up by making $e_R \to -e_R$, $d_R \to +d_R$, and $\nu_R \to +\nu_R$. We use the apostrophe to distinguish this model from the one given in the appendix, which is the actual Lepton-Specific model (because both leptons couple to the same Higgs doublet). The kinetic term on Θ_1 remains just like the one in Eq. (3.24), and the same occurs for its corresponding normalization in Eq.(3.25b). For Θ_2, the kinetic term should be replaced as follows

$$\frac{|y_e|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{3(|y_d|^2+|y_u|^2)+|y_\nu|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma,$$

and the normalization in Eq. (3.25b) should be changed to get

$$\begin{aligned}\Theta_1 &\to \frac{2g}{\sqrt{|y_e|^2}}\Phi_1,\\ \Theta_2 &\to \frac{2g}{\sqrt{3(|y_d|^2+|y_u|^2)+|y_\nu|^2}}\Phi_2,\\ \Sigma &\to \frac{2\sqrt{2}g}{|y_R|}\sigma.\end{aligned} \tag{3.78}$$

Then, the quadratic term on Θ_2 is not altered, so we get

$$\begin{aligned}-\frac{f_2\Lambda^2}{f_0}\mathrm{Tr}(\Phi^2) &= -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2+3|y_d|^2+3|y_u|^2}{4g^2}\Theta_2^\dagger\Theta_2 + \Phi_1^\dagger\Phi_1 + \sigma^*\sigma\right)\\ &= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_2^\dagger\Phi_2 + \Phi_1^\dagger\Phi_1 + \sigma^*\sigma\right).\end{aligned} \tag{3.79}$$

Now, in contrast to Eq. (3.27), the mixed quartic terms are not zero

$$\begin{aligned}(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) &= \begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_1^+\\ \varphi_1^0\end{pmatrix}\cdot\begin{pmatrix}\varphi_1^- & \varphi_1^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}\\ &= \left(\varphi_1^-\varphi_2^{0*} - \varphi_1^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_1^+ - \varphi_2^+\varphi_1^0\right)\\ &= \varphi_1^-\varphi_2^{0*}\varphi_2^0\varphi_1^+ - \varphi_1^-\varphi_2^{0*}\varphi_2^+\varphi_1^0 - \varphi_1^{0*}\varphi_2^-\varphi_2^0\varphi_1^+ + \varphi_1^{0*}\varphi_2^-\varphi_2^+\varphi_1^0\\ &= (\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1).\end{aligned}$$

Hence, the quartic potential is given by

$$\begin{aligned}\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) \supseteq\ & 4g^2(\Phi_1^\dagger\Phi_1)^2 + \frac{3(|y_d|^4+|y_u|^4)+|y_\nu|^4}{4g^2}(\Theta_2^\dagger\Theta_2)^2 + 8g^2(\sigma^*\sigma)^2\\ &+ \frac{|y_e|^2|y_\nu|^2}{2g^2}\left((\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1)\right) + 4|y_\nu|^2(\sigma^*\sigma)(\Theta_2^\dagger\Theta_2)\\ &= 4g^2(\Phi_1^\dagger\Phi_1)^2 + \frac{3(|y_d|^4+|y_u|^4)+|y_\nu|^4}{(3(|y_d|^2+|y_u|^2)+|y_\nu|^2)^2}4g^2(\Phi_2^\dagger\Phi_2)^2\\ &+ 8g^2(\sigma^*\sigma)^2 + \frac{16|y_\nu|^2g^2}{3(|y_d|^2+|y_u|^2)+|y_\nu|^2}(\sigma^*\sigma)(\Phi_2^\dagger\Phi_2)\\ &+ \frac{8|y_\nu|^2g^2}{3(|y_d|^2+|y_u|^2)+|y_\nu|^2}((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) - (\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)).\end{aligned} \tag{3.80}$$

So we make the following redefinition

$$\mu_1^2 = \mu_2^2 = -4\frac{f_2\Lambda^2}{f_0} = \mu_S^2, \qquad \frac{\lambda_1}{2} = 4g^2, \qquad \frac{\lambda_2}{2} \approx \frac{4((R_u^\nu)^2+3)}{(R_u^\nu+3)^2}g^2,$$
$$\lambda_3 = -\lambda_4 \approx \frac{8R_u^\nu}{R_u^\nu+3}g^2, \quad \lambda_5 = 0, \quad \lambda_{S1} = 0, \quad \lambda_{S2} \approx \frac{16R_u^\nu}{R_u^\nu+3}g^2, \quad \lambda_S = 8g^2. \tag{3.81}$$

3.6.3 Flipped'$+\sigma + \nu_R$

As in the preceding subsection, we have used the apostrophe to differentiate this model from the actual Flipped model given in the appendix C.3. In this case, the parity assignment is reached by making $e_R \to +e_R$, $d_R \to -d_R$, and $\nu_R \to +\nu_R$. We keep both Eqs. (3.33), and (3.34a), which contain the kinetic and the respective normalization for the Θ_1 field. On his part, the kinetic term for the field Θ_2 should be changed as follows

$$\frac{3|y_d|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{|y_\nu|^2+|y_e|^2+3|y_u|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma, \tag{3.82}$$

and its normalization in Eq. (3.34b) is replaced to obtain

$$\Theta_1 \to \frac{2g}{\sqrt{3|y_d|^2}}\Phi_1,$$
$$\Theta_2 \to \frac{2g}{\sqrt{|y_\nu|^2+|y_e|^2+3|y_u|^2}}\Phi_2,$$
$$\Sigma \to \frac{2\sqrt{2}g}{|y_R|}\sigma. \tag{3.83}$$

Then, the quadratic potential remains the same

$$-\frac{f_2\Lambda^2}{f_0}\mathrm{Tr}(\Phi^2) = -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2+|y_e|^2+3|y_u|^2}{4g^2}\Theta_2^\dagger\Theta_2 + \Phi_1^\dagger\Phi_1 + \sigma^*\sigma\right)$$
$$= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_2^\dagger\Phi_2 + \Phi_1^\dagger\Phi_1 + \sigma^*\sigma\right). \tag{3.84}$$

Also, we do not have any extra mixed-quartic term apart from the one given in Eq. (3.36), since

$$(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) = 0.$$

Hence, the total quartic potential is given by

$$\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) = \frac{4}{3}g^2(\Phi_1^\dagger\Phi_1)^2 + \frac{|y_\nu|^4+|y_e|^4+3|y_u|^4}{4g^2}(\Theta_2^\dagger\Theta_2)^2 + 8g^2(\sigma^*\sigma)^2 + 4|y_\nu|^2(\sigma^*\sigma)(\Theta_2^\dagger\Theta_2)$$
$$+ 2|y_u|^2\left((\Phi_1^\dagger\Phi_1)(\Theta_2^\dagger\Theta_2) - (\Phi_1^\dagger\Theta_2)(\Theta_2^\dagger\Phi_1)\right)$$
$$= \frac{4}{3}g^2(\Phi_1^\dagger\Phi_1)^2 + \frac{4(|y_\nu|^4+|y_e|^4+3|y_u|^4)g^2}{(|y_\nu|^2+|y_e|^2+3|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2 + 8g^2(\sigma^*\sigma)^2$$
$$+\frac{16|y_\nu|^2g^2}{|y_\nu|^2+|y_e|^2+3|y_u|^2}(\sigma^*\sigma)(\Phi_2^\dagger\Phi_2) + \frac{8g^2|y_u|^2}{|y_\nu|^2+|y_e|^2+3|y_u|^2}\left((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) - (\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\right). \tag{3.85}$$

Then, the we redefine the coefficients by

$$\mu_1^2 = \mu_2^2 = -4\frac{f_2\Lambda^2}{f_0} = \mu_S^2, \qquad \frac{\lambda_1}{2} = \frac{4}{3}g^2, \qquad \frac{\lambda_2}{2} \approx \frac{4((R_u^\nu)^2+3)}{(R_u^\nu+3)^2}g^2,$$
$$\lambda_3 = -\lambda_4 = \frac{8}{R_u^\nu+3}g^2, \quad \lambda_5 = \lambda_{S1} = 0, \quad \lambda_{S2} \approx \frac{16R_u^\nu}{R_u^\nu+3}g^2, \quad \lambda_S = 16g^2. \tag{3.86}$$

3.6.4 Type II+$\sigma + \nu_R$

In the Type-II model [88], we have that $e_R \to -e_R$, $d_R \to -d_R$, and $\nu_R \to +\nu_R$. Then, the kinetic term on Θ_2 in Eq. (3.42) is modified as follows

$$\frac{|y_e|^2+3|y_d|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{3|y_u|^2+|y_\nu|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma,$$

suggesting a modification on the normalization in Eq. (3.43b) to get the following

$$\Theta_1 \to \frac{2g}{\sqrt{|y_e|^2+3|y_d|^2}}\Phi_1,$$
$$\Theta_2 \to \frac{2g}{\sqrt{3|y_u|^2+|y_\nu|^2}}\Phi_2,$$
$$\Sigma \to \frac{2\sqrt{2}g}{|y_R|}\sigma. \tag{3.87}$$

As in the preceding two subsections, we maintain the kinetic term for Θ_1 in Eq. (3.42) as well as its normalization in Eq. (3.43a). Hence, for the quadratic potential we have that

$$-\frac{f_2\Lambda^2}{f_0}\mathrm{Tr}(\Phi^2) = -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2+3|y_u|^2}{4g^2}\Theta_2^\dagger\Theta_2 + \Phi_1^\dagger\Phi_1 + \sigma^*\sigma\right)$$
$$= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_2^\dagger\Phi_2 + \Phi_1^\dagger\Phi_1 + \sigma^*\sigma\right). \tag{3.88}$$

Next, the additional mixed-quartic term to the one given in Eq. (3.45), is

$$(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) = \begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_1^+\\ \varphi_1^0\end{pmatrix}\cdot\begin{pmatrix}\varphi_1^- & \varphi_1^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}$$
$$= \left(\varphi_1^-\varphi_2^{0*} - \varphi_1^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_1^+ - \varphi_2^+\varphi_1^0\right)$$
$$= \varphi_1^-\varphi_2^{0*}\varphi_2^0\varphi_1^+ - \varphi_1^-\varphi_2^{0*}\varphi_2^+\varphi_1^0 - \varphi_1^{0*}\varphi_2^-\varphi_2^0\varphi_1^+ + \varphi_1^{0*}\varphi_2^-\varphi_2^+\varphi_1^0$$
$$= (\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1).$$

Hence, the total quartic potential is as follows

$$\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) = \frac{4\cdot(|y_e|^4+3|y_d|^4)g^2}{(|y_e|^2+3|y_d|^2)^2}(\Phi_1^\dagger\Phi_1)^2 + \frac{3|y_u|^4+|y_\nu|^4}{4g^2}(\Theta_2^\dagger\Theta_2)^2 + 8g^2(\sigma^*\sigma)^2 + 4|y_\nu|^2(\sigma^*\sigma)(\Theta_2^\dagger\Theta_2)$$
$$+\frac{1}{2g^2}\left(|y_e|^2|y_\nu|^2+3|y_d|^2|y_u|^2\right)\Big((\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2)-(\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1)\Big)$$
$$=\frac{4\cdot(|y_e|^4+3|y_d|^4)g^2}{(|y_e|^2+3|y_d|^2)^2}(\Phi_1^\dagger\Phi_1)^2 + \frac{3|y_u|^4+|y_\nu|^4}{(3|y_u|^2+|y_\nu|^2)^2}4g^2(\Phi_2^\dagger\Phi_2)^2 + \frac{16|y_\nu|^2g^2}{3|y_u|^2+|y_\nu|^2}(\sigma^*\sigma)(\Phi_2^\dagger\Phi_2)$$
$$+8g^2(\sigma^*\sigma)^2 + \frac{8(|y_e|^2|y_\nu|^2+3|y_d|^2|y_u|^2)g^2}{(3|y_u|^2+|y_\nu|^2)(|y_e|^2+3|y_d|^2)}((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)-(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)). \tag{3.89}$$

Then, with $R_d^e \approx 1$, we can redefine the coefficients by

$$\mu_1^2=\mu_2^2=-4\frac{f_2\Lambda^2}{f_0}=\mu_S^2, \qquad \frac{\lambda_1}{2}\approx g^2, \qquad \frac{\lambda_2}{2}\approx\frac{4((R_u^\nu)^2+3)}{(R_u^\nu+3)^2}g^2,$$
$$\lambda_3=-\lambda_4\approx\frac{1}{2}g^2, \quad \lambda_5=0, \quad \lambda_{S1}=0, \quad \lambda_{S2}\approx\frac{16R_u^\nu}{R_u^\nu+3}g^2, \quad \lambda_S=16g^2. \tag{3.90}$$

3.7 Phenomenology of the NCG 2HDM $+\sigma+\nu_R$

The most general potential considered in the preceding section is obtained by adding to the potential in Eq. (1.40) the following terms [89, 90]

$$\mathcal{L}_S=\mu_S^2\sigma^*\sigma+\frac{\lambda_S}{2}(\sigma^*\sigma)^2+\lambda_{S1}(\sigma^*\sigma)(\Phi_1^\dagger\Phi_1)+\lambda_{S2}(\sigma^*\sigma)(\Phi_2^\dagger\Phi_2). \tag{3.91}$$

In terms of the VEVs in Eq. (1.44) and $\langle\sigma\rangle_0=\frac{v_S}{\sqrt{2}}$, the minimum conditions are given by

$$\mu_1^2=\frac{v_2}{v_1}\mu_{12}^2-\frac{\lambda_1}{2}v_1^2-\frac{\lambda_{345}}{2}v_2^2-\frac{\lambda_{S1}}{2}v_S^2,$$
$$\mu_2^2=\frac{v_1}{v_2}\mu_{12}^2-\frac{\lambda_2}{2}v_2^2-\frac{\lambda_{345}}{2}v_1^2-\frac{\lambda_{S2}}{2}v_S^2,$$
$$\mu_S^2=-\frac{\lambda_S}{2}v_S^2-\frac{\lambda_{S1}}{2}v_1^2-\frac{\lambda_{S2}}{2}v_2^2. \tag{3.92}$$

Now, expanding σ around the vacuum we have

$$\sigma\to\frac{S+i\chi+v_S}{\sqrt{2}}, \tag{3.93}$$

where S and χ are real fields.

So, we proceed to calculate the masses for the CP-even and the CP-odd scalar fields. In the charged sector, the mass for for $H^\pm$ will be given by Eq. (1.50). Let us start expanding the potential in Eq. (3.91) for the real components to get [91]

$$
\begin{aligned}
V &= \left(\frac{v_2}{v_1}\mu_{12}^2 - \frac{\lambda_1}{2}v_1^2 - \frac{\lambda_{345}}{2}v_2^2 - \frac{\lambda_{S1}}{2}v_S^2\right)\frac{\rho_1^2}{2} + \frac{\lambda_1}{2}\frac{3}{2}\rho_1^2 v_1^2 \\
&+ \left(\frac{v_1}{v_2}\mu_{12}^2 - \frac{\lambda_2}{2}v_2^2 - \frac{\lambda_{345}}{2}v_1^2 - \frac{\lambda_{S2}}{2}v_S^2\right)\frac{\rho_2^2}{2} + \frac{\lambda_2}{2}\frac{3}{2}\rho_1^2 v_1^2 \\
&- \mu_{12}^2\rho_1\rho_2 - \left(\frac{\lambda_S}{2}v_S^2 + \frac{\lambda_{S1}}{2}v_1^2 + \frac{\lambda_{S2}}{2}v_2^2\right)\frac{S^2}{2} + \frac{\lambda_S}{2}\frac{3}{2}S^2 v_S^2 \\
&+ \lambda_{345}\frac{\rho_1^2 v_2^2 + 4\rho_1\rho_2 v_1 v_2 + \rho_2^2 v_1^2}{4} \\
&+ \lambda_{S1}\frac{S^2 v_1^2 + 4S\rho_1 v_1 v_S + \rho_1^2 v_S^2}{4} \\
&+ \lambda_{S2}\frac{S^2 v_2^2 + 4S\rho_2 v_2 v_S + \rho_2^2 v_S^2}{4} \\
&= \frac{1}{2}\frac{v_2}{v_1}\mu_{12}^2\rho_1^2 + \frac{1}{2}\frac{v_1}{v_2}\mu_{12}^2\rho_2^2 + \frac{\lambda_1}{2}\rho_1^2 v_1^2 + \frac{\lambda_2}{2}\rho_2^2 v_2^2 + \frac{\lambda_S}{2}S^2 v_S^2 \\
&- \mu_{12}^2\rho_1\rho_2 + \lambda_{345}\rho_1\rho_2 v_1 v_2 + \lambda_{S1}S\rho_1 v_1 v_S + \lambda_{S2}S\rho_2 v_2 v_S,
\end{aligned}
\tag{3.94}
$$

which give us the mass-squared matrix for the real uncharged scalar fields

$$
\mathcal{L}_{\text{mass}}^{\rho_1,\rho_2,s} = \frac{1}{2}(\rho_1, \rho_2, S)\begin{pmatrix} \lambda_1 v_1^2 + \frac{v_2}{v_1}\mu_{12}^2 & \lambda_{345}v_1 v_2 - \mu_{12}^2 & \lambda_{S1}v_1 v_S \\ \lambda_{345}v_1 v_2 - \mu_{12}^2 & \lambda_2 v_2^2 + \frac{v_1}{v_2}\mu_{12}^2 & \lambda_{S2}v_2 v_S \\ \lambda_{S1}v_1 v_S & \lambda_{S2}v_2 v_S & \lambda_S v_S^2 \end{pmatrix}\begin{pmatrix}\rho_1 \\ \rho_2 \\ S\end{pmatrix}. \tag{3.95}
$$

So, diagonalization is now possible by defining the following orthogonal (3×3) matrix, in terms of the mixing angles α, γ and ϑ

$$
\mathcal{R} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{pmatrix}\begin{pmatrix} \cos\gamma & 0 & -\sin\gamma \\ 0 & 1 & 0 \\ \sin\gamma & 0 & \cos\gamma \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.96}
$$

$$
= \begin{pmatrix} \cos\alpha\cos\gamma & -\sin\alpha\cos\gamma & -\sin\gamma \\ \sin\alpha\cos\vartheta - \cos\alpha\sin\gamma\sin\vartheta & \cos\alpha\cos\vartheta + \sin\alpha\sin\gamma\sin\vartheta & -\cos\gamma\sin\vartheta \\ \sin\alpha\sin\vartheta + \cos\alpha\sin\gamma\cos\vartheta & \cos\alpha\sin\vartheta - \sin\alpha\sin\gamma\cos\vartheta & \cos\gamma\cos\vartheta \end{pmatrix}, \tag{3.97}
$$

so that the physical mass eigenstates H, h, and s in terms of the interaction basis (ρ_1, ρ_2, S), are given by

$$
\begin{pmatrix} H \\ h \\ s \end{pmatrix} = \mathcal{R}^T \begin{pmatrix} \rho_1 \\ \rho_2 \\ S \end{pmatrix}. \tag{3.98}
$$

Therefore, the square-mass eigenstates are

$$
\begin{pmatrix} m_H^2 & 0 & 0 \\ 0 & m_h^2 & 0 \\ 0 & 0 & m_s^2 \end{pmatrix} = \mathcal{R}^T \begin{pmatrix} \lambda_1 v_1^2 + \frac{v_2}{v_1}\mu_{12}^2 & \lambda_{345}v_1 v_2 - \mu_{12}^2 & \lambda_{S1}v_1 v_S \\ \lambda_{345}v_1 v_2 - \mu_{12}^2 & \lambda_2 v_2^2 + \frac{v_1}{v_2}\mu_{12}^2 & \lambda_{S2}v_2 v_S \\ \lambda_{S1}v_1 v_S & \lambda_{S2}v_2 v_S & \lambda_S v_S^2 \end{pmatrix}\mathcal{R} \tag{3.99}
$$

Now, expanding the potential for the CP-odd components we have that

$$
\begin{aligned}
V \supseteq & \left(\frac{v_2}{v_1}\mu_{12}^2 - \frac{\lambda_1}{2}v_1^2 - \frac{\lambda_{34}}{2}v_2^2 - \frac{\lambda_{S1}}{2}v_S^2\right)\frac{\eta_1^2}{2} + \frac{\lambda_1}{2}\frac{1}{2}\eta_1^2 v_1^2 \\
& + \left(\frac{v_1}{v_2}\mu_{12}^2 - \frac{\lambda_2}{2}v_2^2 - \frac{\lambda_{34}}{2}v_1^2 - \frac{\lambda_{S2}}{2}v_S^2\right)\frac{\eta_2^2}{2} + \frac{\lambda_2}{2}\frac{1}{2}\eta_2^2 v_1^2 \\
& - \mu_{12}^2\eta_1\eta_2 - \left(\frac{\lambda_S}{2}v_S^2 + \frac{\lambda_{S1}}{2}v_1^2 + \frac{\lambda_{S2}}{2}v_2^2\right)\frac{\chi^2}{2} + \frac{\lambda_S}{2}\frac{2\chi^2}{4}v_S^2 \\
& + \frac{\lambda_{34}}{4}\left(\eta_1^2 v_2^2 + \eta_2^2 v_1^2\right) + \frac{\lambda_5}{4}\left(4v_1v_2\eta_1\eta_2 - v_1^2\eta_2^2 - v_2^2\eta_1^2\right) \\
& + \lambda_{S1}\frac{\chi^2 v_1^2 + \eta_1^2 v_S^2}{4} + \lambda_{S2}\frac{\chi^2 v_2^2 + \eta_2^2 v_S^2}{4} \\
= & \frac{1}{2}\left(\mu_{12}^2 - \lambda_5 v_1 v_2\right)\begin{pmatrix}\eta_1 & \eta_2\end{pmatrix}\begin{pmatrix}\frac{v_2}{v_1} & -1 \\ -1 & \frac{v_1}{v_2}\end{pmatrix}\begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix}.
\end{aligned}
\tag{3.100}
$$

If we compare this result with the one obtained in equation (1.56), we can appreciate that both are exactly the same. Then, in case of μ_{12}^2 and λ_5 are both zero, the three pseudoscalar fields G^0, A, and χ become massless.

Now, for the parameters involving the potential in Eq. (3.91), the one-loop RGEs are given by [92]

$$16\pi^2\beta_{y_u} = y_u(-8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_1^2 + \frac{9}{2}y_u^2), \tag{3.101a}$$

$$
\begin{aligned}
16\pi^2\beta_{\lambda_1} = & \frac{3}{4}(g_1^4 + 3g_2^4 + 2g_1^2g_2^2) + 12\lambda_1^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2\lambda_5^2 \\
& - 3(3g_2^2 + g_1^2)\lambda_1 + 2\lambda_{S1}^2,
\end{aligned}
\tag{3.101b}
$$

$$
\begin{aligned}
16\pi^2\beta_{\lambda_2} = & \frac{3}{4}(g_1^4 + 3g_2^4 + 2g_1^2g_2^2) + 12\lambda_2^2 + 4\lambda_3^2 + 4\lambda_3\lambda_4 + 2\lambda_4^2 + 2\lambda_5^2 \\
& - 3(3g_2^2 + g_1^2 - 4y_1^2)\lambda_2 - 12y_1^4 + 2\lambda_{S2}^2,
\end{aligned}
\tag{3.101c}
$$

$$
\begin{aligned}
16\pi^2\beta_{\lambda_3} = & -3(3g_2^2 + g_1^2 - 2y_1^2)\lambda_3 + \frac{3}{4}(3g_1^4 + 9g_2^4 - 2g_1^2g_2^2) \\
& + 2(\lambda_1 + \lambda_2)(3\lambda_3 + \lambda_4) + 4\lambda_3^2 + 2\lambda_4^2 + 2\lambda_5^2 + 2\lambda_{S1}\lambda_{S2},
\end{aligned}
\tag{3.101d}
$$

$$16\pi^2\beta_{\lambda_4} = -3(3g_2^2 + g_1^2 - 2y_1^2)\lambda_4 + 3g_1^2g_2^2 + 2(\lambda_1 + \lambda_2)\lambda_4 + 8\lambda_3\lambda_4 + 4\lambda_4^2 + 8\lambda_5^2, \tag{3.101e}$$

$$16\pi^2\beta_{\lambda_5} = (2\lambda_1 + 2\lambda_2 + 8\lambda_3 + 12\lambda_4)\lambda_5 - 3(g_1^2 + 3g_2^2 - 2y_u^2)\lambda_5, \tag{3.101f}$$

$$16\pi^2\beta_{\lambda_S} = 10\lambda_S^2 + 4\lambda_{S1}^2 + 4\lambda_{S2}^2, \tag{3.101g}$$

$$16\pi^2\beta_{\lambda_{S1}} = (6\lambda_1 + 4\lambda_S + 4\lambda_{S1})\lambda_{S1} + (4\lambda_3 + 2\lambda_4)\lambda_{S2} - \frac{3}{2}(3g_2^2 + g_1^2)\lambda_{S1}, \tag{3.101h}$$

$$16\pi^2\beta_{\lambda_{S2}} = (6\lambda_2 + 4\lambda_S + 4\lambda_{S2})\lambda_{S2} + (4\lambda_3 + 2\lambda_4)\lambda_{S1} - \frac{3}{2}(3g_2^2 + g_1^2 - 4y_1^2)\lambda_{S2}, \tag{3.101i}$$

From Eqs (3.81), (3.86) and (3.90), we summarize the boundary conditions for the above RGE's in table 3.13.

	$\Theta_\nu = \widetilde{\Theta}_2$		
	Lepton-Specific'	Flipped'	Type-II
y_u	g	g	g
λ_1	$8g^2$	$\frac{8}{3}g^2$	$2g^2$
λ_2	$8\frac{(R^2+3)}{(R+3)^2}g^2$	$8\frac{(R^2+3)}{(R+3)^2}g^2$	$8\frac{(R^2+3)}{(R+3)^2}g^2$
λ_3	$8\frac{R}{R+3}g^2$	$\frac{8}{R+3}g^2$	$\frac{1}{2}g^2$
λ_4	$-8\frac{R}{R+3}g^2$	$-\frac{8}{R+3}g^2$	$-\frac{1}{2}g^2$
λ_S	$16g^2$	$16g^2$	$16g^2$
λ_{S1}	0	0	0
λ_{S2}	$16\frac{R}{R+3}g^2$	$16\frac{R}{R+3}g^2$	$16\frac{R}{R+3}g^2$

Table 3.13: Boundary conditions for the 2HDM+$\sigma + \nu_R$. For simplicity we have done $R := R_u^\nu$.

For the Flipped' and Type-II models we choose $R_u^\nu \approx 2.7$. For the Lepton Specific model we choose $R_u^\nu \approx 2$, because for $R_u^\nu \gtrsim 2.2$ it give rise to tachyon fields. For an unification scale of $\sim 10^{16}$ GeV, an unified gauge coupling $g = 0.5$ and $\tan\beta \approx 2.14$ together with the boundary conditions listed above, we may run down to low energies the RGEs. As for the SM+$\sigma + \nu_R$, the higher is λ_S the higher is m_s and for $v_S \geq 10^5$ GeV, the Higgs and top masses get stable values. The scalar mass spectrum for the three models is shown in table 3.14.

	$\Theta_\nu = \widetilde{\Theta}_2$		
	Lepton-Specific'	Flipped'	Type-II
R_u^ν	2.2	2.7	2.7
m_t	164	164	164
m_h	141	125.7	125.6
m_H	20.8	52.6	55.7
m_s	6.1×10^5	6×10^5	6×10^5
$m_{H^\pm}$	132	93.3	69.05
$m_{A,\chi}$	0	0	0

Table 3.14: Mass spectrum for an unification scale of $\sim 10^{16}$ GeV, $\tan\beta \approx 2.14$, $g = 0.5$, and $v_S = 10^6$ GeV. The mass values are given in GeV.

Notice that the top quark mass is $m_t \approx 164$ GeV, which is lower than its experimental value. However, this is compatible with the one obtained in [15]. This difficulty has been fixed up by a two-three-loops RGEs analysis [**?**]. In our case, we have the 'free' parameter $\tan\beta$ in order to make it higher. By fixing $v_S = 10^6$ GeV, we have the following model-independent rules:

1. When $\tan\beta \to 1$, then $m_t \to 130$ GeV.
2. When $\tan\beta \to 10$, then $m_t \to 180$ GeV.

3. Neither the mass for the charged scalar neither the mass for the singlet scalar depend on $\tan\beta$.
4. The change on the Higgs boson mass as a function of $\tan\beta$ is negligible.
5. The higher is $\tan\beta$ the lower is m_H.
6. The higher is R_u^ν the lower is m_H.

For $\tan\beta \approx 3.27$ we found the mass spectrum shown in the tables 3.15, 3.16, and 3.17.

	Lepton-Specific'		
R_u^ν	0.8	1.0	2.1
m_t	173.5	173.5	173.5
m_h	160	156	137
m_H	44.6	42.2	6.36
m_s	6.5×10^5	6.4×10^5	6.1×10^5
$m_{H^\pm}$	100	107	134
$m_{A,\chi}$	0	0	0

Table 3.15: Lepton-Specific' mass spectrum. The Higgs boson mass is always higher than 125 GeV. For $R_u^\nu \gtrsim 2.2$, the mass value m_H becomes imaginary. The mass values are given in GeV.

	Flipped'		
R_u^ν	1.0	2.0	3.15
m_t	173.5	173.5	173.5
m_h	155	139	125
m_H	33.3	36.1	37.7
m_s	6.4×10^5	6.3×10^5	6.1×10^5
$m_{H^\pm}$	113	104	95.4
$m_{A,\chi}$	0	0	0

Table 3.16: Flipped' mass spectrum. For $R_u^\nu \approx 3.15$ we get $m_h \approx 125$ GeV. The mass values are given in GeV.

	Type-II		
R_u^ν	1.0	2.0	3.3
m_t	173.5	173.5	173.5
m_h	157	140	125
m_H	39.4	39.1	38.6
m_s	6.4×10^5	6.1×10^5	5.9×10^5
$m_{H^\pm}$	69	69.1	68.9
$m_{A,\chi}$	0	0	0

Table 3.17: Type-II model mass spectrum. For $R_u^\nu \approx 3.3$ we get $m_h \approx 125$ GeV. The mass values are given in GeV.

In general, we have shown that for certain values of $\tan\beta$, and R_u^ν (after fixing $g = 0.5$ and an unification scale of $\sim 10^{16}$ GeV) the NCG 2HDM+$\sigma + \nu_R$ is compatible with the top quark and Higgs boson experimental mass values, except for the Lepton-Specific' developed in section 3.6.2. Despite that, as in section 3.4, the potential in Eq. (3.91) does not contain terms of the form $\Phi_1^\dagger\Phi_2$ neither $\sigma\Phi_1^\dagger\Phi_2$. Then, the pseudoscalar A becomes a massless Nambu-Goldestone field to tree level. Also, the new potential terms due to the presence of σ do not break the accidental $U(1)$ symmetry. Then, we still have a domain wall problem.

As an alternative solution, it has been proposed the existence of a period of exponential inflation during the earliest times of the Universe. This is since if a domain wall forming phase transition occurred before the period of inflation, then the domain walls would have been inflated beyond the current cosmological horizon. This removes the domain wall problem from the current horizon as the energy per unit volume of the current observable Universe would gain no contribution from the domain wall energy per unit volume [93].

Beyond this possible solution, in this work, as it was done in section 3.5, we adopt the insertion of terms into the potential which explicitly break the $U(1)$ symmetry, so that we can get a phenomenologically viable NCG 2HDM.

3.8 A viable NCG 2HDM+$\sigma + \nu_R$ Type-II

We now consider the insertion of the term $\mu_{12}^2\Phi_1^\dagger\Phi_2$ into the potential (3.91) in order to calculate the scalar mass spectrum for the NCG 2HDM+$\sigma + \nu_R$ Type-II. In contrast to section 3.5, we do not consider here terms proportional to λ_5 since it will not be necessary to get a fully phenomenologically viable model. In such case, from Eq. (3.90) we set (for $g = 0.5$) the following boundary conditions

$$\lambda_1 = \frac{1}{2},\ \ \lambda_2 = 2\frac{(R_u^\nu)^2 + 3}{(R_u^\nu + 3)^2},\ \ \lambda_3 = -\lambda_4 = \frac{1}{8},\ \ \lambda_5 = 0,\ \ \lambda_{S1} = 0,\ \ \lambda_{S2} = \frac{4R_u^\nu}{R_u^\nu + 3},\ \ \lambda_S = 4. \tag{3.102}$$

As before, we choose $\tan\beta = 2.14$ to guarantee a top quark mass value of 173 GeV. Again, the main restriction in this section is to have a Higgs boson mass of approximately 125 GeV, satisfying the alignment limit constrain $C_V^h \gtrsim 0.99$, as required by the alignment limit [42, 78].

Now, we search for configuration settings for v_S and μ_{12}^2, which are in agreement with the above constraints we have established. The favorable values for μ_{12}^2 and v_S are marked by colors and correspond to the region delimited by the dashed lines (localized upward on the right) in figure 3.4. Their corresponding scalar mass spectrum is exhibited in tables 3.18 and 3.19.

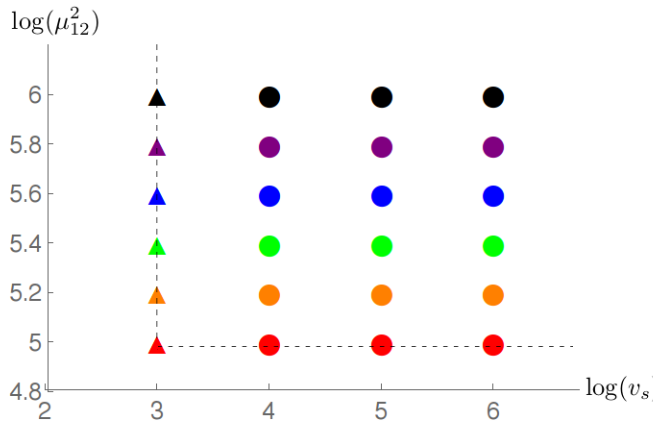


Figure 3.4: The colors stand for values of $v_S = 10^{\log v_S^2}$ and $\mu_{12}^2 = 10^{\log \mu_{12}^2}$ which are in agreement with a Higgs boson mass of approximately 125 GeV. The circles are for $R_u^\nu = 2.43$ whereas the triangles are for $R_u^\nu = 2.32$, indicating an inflection point since for $v_S \lesssim 10^3$ GeV it is not always possible to find a value for R_u^ν so that $m_h \approx 125$ GeV.

	m_H[GeV]	m_A[GeV]	$m_{H^\pm}$[GeV]
▲	626	598	602
▲	757	753	756
▲	951	948	950
▲	1195	1193	1195
▲	1504	1502	1504
▲	1893	1891	1892

Table 3.18: Scalar mass spectrum for $v_S = 10^3$ GeV ($m_s = 6.1 \times 10^2$ GeV) and $R_u^\nu = 2.32$. From red to black the scalar masses increases and have nearly the same values when $\mu_{12}^2 \gtrsim 10^6$ GeV^2. The Higgs boson mass is $m_h \approx 125$ GeV

	m_H[GeV]	m_A[GeV]	$m_{H^\pm}$[GeV]
●	601	598	602
●	756	753	556
●	950	948	950
●	1195	1193	1195
●	1504	1502	1504
●	1892	1891	1892

Table 3.19: Scalar mass spectrum for $v_S \gtrsim 10^4$ GeV ($m_s \gtrsim 6.1 \times 10^3$ GeV) and a stable value of $R_u^\nu = 2.43$. For $\mu_{12}^2 \gtrsim 10^6$ GeV^2 we obtain the decoupling limit. We have fixed $m_h \approx 125$ GeV.

Therefore, for a zero value of λ_5 the parameter space to get a Higgs boson mass of approximately 125 GeV is defined by v_S and μ_{12}^2. The area upward on the right and bounded by the dashed lines, as shown in figure 3.4, correspond to the allowed one since it is in accordance with the theoretical and experimental constraints, as the suppression of domain walls, and the ones coming from the searches of heavy multi-Higgs signals at LHC. We can visualise this latter because the intensity of the interaction between h and the vector bosons is so that $C_V^h = \sin(\beta - \alpha) \approx 1$, whereas the one between h and the quark sector (when μ_{12}^2 goes from 10^5 to 10^6 GeV^2) are in the following intervals

$$0.995 \leq C_u^h \leq 0.999, \;\; \text{and} \;\; 1.056 \leq C_d^h \leq 1.005, \tag{3.103}$$

as required by the alignment limit.

So, we have encountered that by adding a non-null μ_{12}^2 parameter into the the noncommutative geometry formulation of the two Higgs doublte model (with right-handed neutrino ν_R and a singlet scalar field σ), it is possible to get a model which is in agreement with the phenomenological requirements.

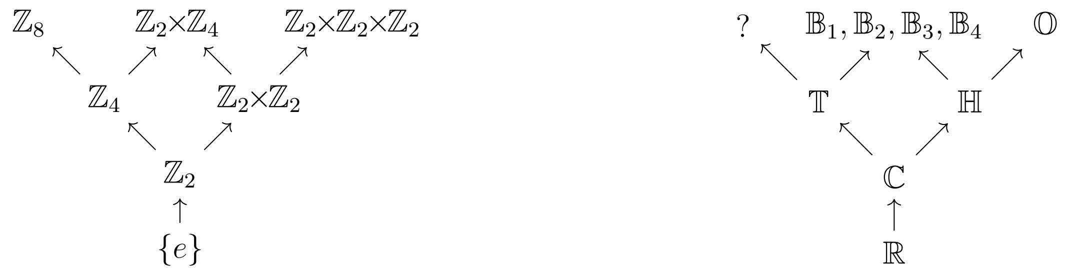
Chapter 4

A chiral nonassociative symmetry for the SM leptons

With the prospect of finding better options to describe the NCG Higgs sector to low energies, and inspired by the work done by S. Farnsworth and L. Boyle in [17, 18, 19, 16, 20, 21], and M. Dubois-Violette in [22, 23, 24], we so introduce a 'chiral' nonassociative algebra in the context of NCG. For doing it, we will follow the classification of not necessarily associative division algebras over $\mathbb{R}$ developed by Wills Toro [25]. In figure 4.1, we show the main finding of his work.

After we introduce the essential points of how to obtain this new class of non-associative algebras, we will center our attention in the Bison algebra $\mathbb{B}_2$ since it not only has the symmetry (automorphism) group $SU(2) \times U(1)$ as observed in the Glashow-Weinberg-Salam model, but it also naturally assign the correct hypercharge values for the SM leptons. Subsequently, we will address the construction of the electroweak theory for the SM leptons by fitting it together on the framework of NCG. First, we define a bi-representation for this algebra with the correct number of leptons and anti-leptons states on a inner product space. When constructing the corresponding almost-associative geometry, we will face up with an increase of the number of the fermion degrees of freedom as well as with the need to handle with a 'twisted commutator' in order to have well defined bosons in the theory.

Once we trace a road map, we ask for the construction of a chirality operator to finally build the Dirac operator, which is the object containing the Higgs sector of the theory.



(a) Finite grading groups

(b) Graded division $\mathbb{R}$-algebras

Figure 4.1: Not necessarily associative algebras (b) with their corresponding grading groups (a).

4.1 New non-associative division $\mathbb{R}$-algebras

We start this section by defining the associator for an algebra $\mathcal{A}$ as follows

$$(x, y, z) := (xy)z - x(yz), \qquad x, y, z \in \mathcal{A}. \tag{4.1}$$

The algebra $\mathcal{A}$ is said to be associative if $\forall x, y, z \in \mathcal{A}$, the equation (4.1) is null.

An unital (with identity element $\mathbf{1}$) $\mathbb{R}$-algebra $\mathcal{A}$ is called a 'division algebra' if for every $x, y \in \mathcal{A}$ the condition $x \cdot y = 0$ implies that either x or y or both are zero. In addition, given a finite group G with identity element e, the function

$$C : G \times G \to \{1, -1\} \subset \mathbb{R}, \tag{4.2}$$

such that $C(e, g) = C(g, e) = 1$ for any $g \in G$, is called a 'unital structure constant of G in $\mathcal{A}$'. Furthermore, $\mathcal{A}$ is called a 'twisted group $\mathbb{R}$-algebra' [94], if as a vector space it satisfies

$$A = \bigoplus_{g \in G} W_g, \text{ where } \dim_{\mathbb{R}} W_g = 1, \text{ and } W_g \cdot W_h \subset W_{g \cdot h}, \tag{4.3}$$

and for any choice of base elements $v_g \in W_g$, $g \in G$, we have $v_e = 1$, and there exists a unital structure constant C for G in $\mathbb{R}$, so that

$$v_g \cdot v_h = C(g, h) v_{gh}, \qquad \forall g, h \in G. \tag{4.4}$$

Next, we want to list two very important results (see propositions 1 and 2 of [25]. Let $\mathcal{A}$ be a twisted group $\mathbb{R}$-algebra with grading group G, then we have that:

1. If the order[1] of G (denoted by $|G|$) is not a power of 2 then $\mathcal{A}$ has zero divisors.

2. If $\{v_g | g \in G\}$ is the basis associated to the structure constant C and $e \neq g \in G$, then $L(v_g)^{|g|} = -\mathbb{I}$, and $R(v_g)^{|g|} = -\mathbb{I}$, where $|g|$, $\mathbb{I}$, $L(v_g)$, and $R(v_g)$ are the order[2] of g, the identity matrix, the left and the right action of v_g, respectively.

We will describe some of the the twisted group algebras with grading groups of order 2, 4, and 8.

[1] The order of a group coincides with its cardinal number [95, pag 24].

[2] The order of a group element g is the order of the (cyclic) subgroup generated by g [95, pag 32].

4.1.1 Grading group of order 2

There is only one option: $\mathbb{Z}_2$. In this case, the unital structure constant can be represented as shown in table 4.1. In particular, we have that $C(1,1) := \alpha \in \{1, -1\}$.

By fixing the basis $[v_0, v_1]$, and from the second result before mentioned, we get that $L(v_1)^2 = -\mathbb{I}$ (the order of 1 is the same as the one of the whole $\mathbb{Z}_2$ since it is the group generator), and so $\alpha = -1$ to avoid zero divisors. So we get the complex numbers by identifying $v_0 := 1$, and $v_1 := i$ because we have that

$$v_1 \cdot v_1 = C(1,1)v_0 \quad \Leftrightarrow \quad i \cdot i = -1. \tag{4.5}$$

The multiplication rules are given in table 4.2.

C	0	1
0	1	1
1	1	α

Table 4.1: Structure constant for the grading group $\mathbb{Z}_2$, with $\alpha \in \{1, -1\}$.

$\mathbb{C}$	v_0	v_1
v_0	v_0	v_1
v_1	v_1	$-v_0$

Table 4.2: Multiplication table for the $\mathbb{Z}_2$ graded algebra isomorphic to $\mathbb{C}$.

4.1.2 Grading groups of order 4

In this case, we have two options: the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and the cyclic group $\mathbb{Z}_4$. For $\mathbb{Z}_2 \times \mathbb{Z}_2$ we only get one division algebra isomorphic to the quaternion algebra. For the Cyclic group $\mathbb{Z}_4$ we get a twisted division algebra, that as a vector space can be written in the basis $[v_0 := \mathbf{1},\ v_1 := \omega,\ v_2 := \omega^2,\ v_3 := \omega^3]$. Next, the relations $L(v_g)^{|g|} = -\mathbb{I}$ and $R(v_g)^{|g|} = -\mathbb{I}$ imply two sets of values for the constants u, v, w, x, y, z, but only one of them lead us to a division (nonassociative) algebra. This is called the Tesseranion algebra and is denoted by $\mathbb{T}$. Its multiplication table is shown in table 4.4.

C	0	1	2	3
0	1	1	1	1
1	1	1	1	u
2	1	v	-1	w
3	1	x	y	z

Table 4.3: Structure constant for $\mathbb{Z}_4$, with $u, v, w, x, y, z \in \{1, -1\}$.

$\mathbb{T}$	$\mathbf{1}$	ω	ω^2	ω^3
$\mathbf{1}$	$\mathbf{1}$	ω	ω^2	ω^3
ω	ω	ω^2	ω^3	$-\mathbf{1}$
ω^2	ω^2	$-\omega^3$	$-\mathbf{1}$	ω
ω^3	ω^3	$\mathbf{1}$	$-\omega$	ω^2

Table 4.4: Multiplication table for the Tesseranion algebra $\mathbb{T}$.

For this $\mathbb{R}$-algebra with basis $\{\mathbf{1}, \omega, \omega^2, \omega^3\}$, we can write its elements $x \in \mathbb{T}$ as follows

$$x \equiv x_0\mathbf{1} + x_1\omega + x_2\omega^2 + x_3\omega^3, \qquad \text{with} \qquad x_0, x_1, x_2, x_3 \in \mathbb{R}.$$

Since $\omega^2\omega^2 = -\mathbf{1}$, then ω^2 generates a subalgebra isomorphic to $\mathbb{C}$ (identifying ω^2 whit imaginary unit i). Hence, $\mathbb{T}$ can be written as

$$\mathbb{T} \equiv \underbrace{\mathbb{C}}_{\text{even}} \oplus \underbrace{\omega\mathbb{C}}_{\text{odd}},$$

with elements of the form $x = (x_0 + x_2\omega^2) + \omega(x_1 + x_3\omega^2) = x_{\text{even}} + x_{\text{odd}} \equiv (X_{\text{even}}, X_{\text{odd}})$ and product

$$xy = (X_{\text{even}}, X_{\text{odd}})(Y_{\text{even}}, Y_{\text{odd}}) = (X_{\text{even}}Y_{\text{even}} + i\overline{X_{\text{odd}}}Y_{\text{odd}}, \overline{X_{\text{even}}}Y_{\text{odd}} + X_{\text{odd}}Y_{\text{even}}).$$

The main associative relation for $x, y, z \in \mathbb{T}$ is

$$(x, y, z) = 0, \quad \Leftrightarrow \quad x \text{ or } y \text{ or } z \text{ is even.} \tag{4.6}$$

In order to introduce a involution such that $(xy)^* = y^*x^*$ for any $x, y \in \mathbb{T}$, we introduce the following conditions

$$\mathbf{1}^* = \alpha_0\mathbf{1}, \qquad (\omega)^* = \alpha_1\omega, \qquad (\omega^2)^* = \alpha_2\omega^2, \qquad (\omega^3)^* = \alpha_3\omega^3, \tag{4.7}$$

where $\alpha_i \in \{1, -1\}$. We know the following:

$$\begin{aligned}
\alpha_0\mathbf{1} = \mathbf{1}^* = \mathbf{1}^*\mathbf{1}^* &= \alpha_0^2\mathbf{1}, && \rightarrow \alpha_0 = \alpha_0^2,\\
\alpha_0\mathbf{1} = \mathbf{1}^* = (\omega)^*(\omega^3)^* &= -\alpha_1\alpha_3\mathbf{1}, && \rightarrow \alpha_0 = -\alpha_1\alpha_3,\\
\alpha_1\omega = \omega^* = (\omega^3)^*(\omega^2)^* &= -\alpha_2\alpha_3\omega, && \rightarrow \alpha_1 = -\alpha_2\alpha_3,\\
\alpha_3\omega^3 = (\omega^3)^* = (\omega^2)^*(\omega)^* &= -\alpha_2\alpha_1\omega^3, && \rightarrow \alpha_3 = -\alpha_2\alpha_1.
\end{aligned}$$

This implies that $\alpha_0 = \alpha_2 = 1$, and $\alpha_1 = -\alpha_3$.

4.1.3 Grading group of order 8

In this case, there are five options [95, pag 99]: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_8$, Q_8, and D_4. The last two are known as the quaternion and the dihedral groups, respectively and we will not consider them. As suggested by figure 4.1, the case for $\mathbb{Z}_8$ remains unanswered in [25]. For the grading group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the only (twisted) algebra is isomorphic to the (nonassociative) octonion algebra. For the case $\mathbb{Z}_2 \times \mathbb{Z}_4$ one gets four not isomorphic nonassociative division $\mathbb{R}$-algebras: $\mathbb{B}_1$, $\mathbb{B}_2$, $\mathbb{B}_3$, and $\mathbb{B}_4$, for which the author has suggested the name of 'Bison algebras'. For all of them, any element can be decomposed as a sum of two Tesseranion elements as well as a sum of two quaternions. Only $\mathbb{B}_2$ and $\mathbb{B}_4$ are isomorphic to its opposite algebra which means that only these two are involutive algebras. The physical interest underlies in that the automorphism group of all the $\mathbb{B}_i$ is $SU(2) \times U(1)$. Then, in the setting of NCG, an almost commutative manifold with a finite space given by $\mathbb{B}_2$ or $\mathbb{B}_4$ would give rise to the electroweak symmetry (gauge) group. In particular, $\mathbb{B}_2$ will allow us to assign the correct hypercharge values for the SM leptons. In that follows, we just call the Bison algebra to refer us to $\mathbb{B}_2$.

4.2 The $\mathbb{B}_2$ algebra

The Bison algebra $\mathbb{B}_2$ is an 8-dimensional division non-associative $\mathbb{R}$-algebra with grading group $\mathbb{Z}_2 \times \mathbb{Z}_4$. The multiplication table for this algebra is shown in table 4.5.

	$\mathbf{1}$	ω	ω^2	ω^3	θ	$\theta\omega$	$\theta\omega^2$	$\theta\omega^3$
$\mathbf{1}$	$\mathbf{1}$	ω	ω^2	ω^3	θ	$\theta\omega$	$\theta\omega^2$	$\theta\omega^3$
ω	ω	ω^2	ω^3	$-\mathbf{1}$	$\theta\omega$	$-\theta\omega^2$	$\theta\omega^3$	θ
ω^2	ω^2	$-\omega^3$	$-\mathbf{1}$	ω	$-\theta\omega^2$	$\theta\omega^3$	θ	$-\theta\omega$
ω^3	ω^3	$\mathbf{1}$	$-\omega$	ω^2	$\theta\omega^3$	$-\theta$	$-\theta\omega$	$-\theta\omega^2$
θ	θ	$\theta\omega$	$\theta\omega^2$	$\theta\omega^3$	$-\mathbf{1}$	$-\omega$	$-\omega^2$	$-\omega^3$
$\theta\omega$	$\theta\omega$	$\theta\omega^2$	$-\theta\omega^3$	θ	$-\omega$	ω^2	ω^3	$\mathbf{1}$
$\theta\omega^2$	$\theta\omega^2$	$\theta\omega^3$	$-\theta$	$-\theta\omega$	ω^2	ω^3	$-\mathbf{1}$	$-\omega$
$\theta\omega^3$	$\theta\omega^3$	$-\theta$	$\theta\omega$	$\theta\omega^2$	$-\omega^3$	$-\mathbf{1}$	$-\omega$	ω^2

Table 4.5: Multiplication table for the Bison algebra $\mathbb{B}_2$

The elements belonging to $\mathbb{B}_2$ are defined by:

$$x := x_0\mathbf{1} + x_1\omega + x_2\omega^2 + x_3\omega^3 + x_4\theta + x_5\theta\omega + x_6\theta\omega^2 + x_7\theta\omega^3, \tag{4.8}$$

where $x_i \in \mathbb{R}$, for $i = 0, 1...7$. As we have already mentioned, this algebra admits a double decomposition:

$$\begin{aligned} \mathbb{B}_2 &= \mathbb{T} \oplus \theta\,\mathbb{T}, \\ \mathbb{B}_2 &= \mathbb{H} \oplus \omega\mathbb{H}, \end{aligned} \tag{4.9}$$

where the first one (in terms of Tesseranions) is evident by factoring θ to the left for the last four components in Eq. (4.8). The decomposition in terms of quaternions, is reached by making the following identification [95, pag 117]

$$[\mathbf{1}, i, j, k] \quad \Leftrightarrow \quad [\mathbf{1}, \omega^2, \theta, \theta\omega^2]. \tag{4.10}$$

Both decomposition suggest a simpler representation of each $x \in \mathbb{B}_2$ as four copies of the complex numbers

$$x := \underbrace{x_0\mathbf{1} + x_2\omega^2}_{z_{02}} + \omega\,\underbrace{(x_1\mathbf{1} + x_3\omega^2)}_{z_{13}} + \theta\,\underbrace{(x_4\mathbf{1} + x_6\omega^2)}_{z_{46}} + \theta\omega\,\underbrace{(x_5\mathbf{1} + x_7\omega^2)}_{z_{57}}, \tag{4.11}$$

or simply as

$$x := \{\underbrace{z_{02}, z_{46}}_{\text{even}}, \underbrace{z_{13}, z_{57}}_{\text{odd}}\} \tag{4.12}$$

Next, to introduce an involution so that $(xy)^* = y^* x^*$ for all $x, y \in \mathbb{B}_2$, we introduce the relations in Eq. (4.7) together with the following:

$$\theta^* = \beta_0 \theta, \qquad (\theta\omega)^* = \beta_1 \theta\omega, \qquad (\theta\omega^2)^* = \beta_2 \theta\omega, \qquad (\theta\omega^3)^* = \beta_3 \theta\omega^3,$$

where the $\alpha, \beta = \pm 1$. We known the following:

$$\begin{aligned}
\alpha_0 \mathbf{1} = \mathbf{1}^* = (\theta\omega^3)^*(\theta\omega)^* = -\beta_3\beta_1 \mathbf{1}, && \Rightarrow \alpha_0 = -\beta_3\beta_1,\\
\alpha_1 \omega = \omega^* = -(\theta)^*(\theta\omega)^* = \beta_0\beta_1 \omega, && \Rightarrow \alpha_1 = \beta_0\beta_1,\\
\alpha_1 \omega = \omega^* = -(\theta\omega^3)^*(\theta\omega^2)^* = \beta_3\beta_2 \omega, && \Rightarrow \alpha_1 = \beta_3\beta_2,\\
\alpha_2 \omega^2 = (\omega^2)^* = -(\theta\omega^2)^*(\theta)^* = -\beta_2\beta_0 \omega^2, && \Rightarrow \alpha_2 = -\beta_2\beta_0,\\
\alpha_3 \omega^3 = (\omega^3)^* = -(\theta\omega^3)^*(\theta)^* = \beta_3\beta_0 \omega^3, && \Rightarrow \alpha_3 = \beta_3\beta_0,\\
\alpha_3 \omega^3 = (\omega^3)^* = (\theta\omega^2)^*(\theta\omega)^* = \beta_2\beta_1 \omega^3, && \Rightarrow \alpha_3 = \beta_2\beta_1,\\
\beta_0 \theta = (\theta)^* = (\theta\omega^3)^*(\omega)^* = -\beta_3\alpha_1 \theta, && \Rightarrow \beta_0 = -\beta_3\alpha_1,\\
\beta_0 \theta = (\theta)^* = -(\theta\omega)^*(\omega^3)^* = -\beta_1\alpha_3 \theta, && \Rightarrow \beta_0 = -\beta_1\alpha_3,\\
\beta_1(\theta\omega) = (\theta\omega)^* = (\omega)^*(\theta)^* = \beta_0\alpha_1 \theta\omega, && \Rightarrow \beta_1 = \beta_0\alpha_1,\\
\beta_1(\theta\omega) = (\theta\omega)^* = -(\theta\omega^3)^*(\omega^2)^* = -\beta_3\alpha_2 \theta\omega, && \Rightarrow \beta_1 = -\beta_3\alpha_2,\\
\beta_1(\theta\omega) = (\theta\omega)^* = -(\theta\omega^2)^*(\omega^3)^* = \beta_2\alpha_3 \theta\omega, && \Rightarrow \beta_1 = \beta_2\alpha_3,\\
\beta_2(\theta\omega^2) = (\theta\omega^2)^* = -(\theta\omega)^*(\omega)^* = -\beta_1\alpha_1 \theta\omega^2, && \Rightarrow \beta_2 = -\beta_1\alpha_1,\\
\beta_2(\theta\omega^2) = (\theta\omega^2)^* = -(\theta)^*(\omega^2)^* = -\beta_0\alpha_2 \theta\omega^2, && \Rightarrow \beta_2 = -\beta_0\alpha_2,\\
\beta_2(\theta\omega^2) = (\theta\omega^2)^* = -(\theta\omega^3)^*(\omega^3)^* = -\beta_3\alpha_3 \theta\omega^2, && \Rightarrow \beta_2 = -\beta_3\alpha_3,\\
\beta_3(\theta\omega^3) = (\theta\omega^3)^* = (\theta)^*(\omega^3)^* = \beta_0\alpha_3 \theta\omega^3, && \Rightarrow \beta_3 = \beta_0\alpha_3,\\
\beta_3(\theta\omega^3) = (\theta\omega^3)^* = (\theta\omega)^*(\omega^2)^* = -\beta_1\alpha_2 \theta\omega^3, && \Rightarrow \beta_3 = -\beta_1\alpha_2,\\
\beta_3(\theta\omega^3) = (\theta\omega^3)^* = (\theta\omega^2)^*(\omega)^* = \beta_2\alpha_1 \theta\omega^3, && \Rightarrow \beta_3 = \beta_2\alpha_1.
\end{aligned}$$

This implies that $\beta_1 = -\beta_3$, $\beta_2 = -\beta_0$, $\alpha_1 = \beta_0\beta_1$, besides the encountered before $\alpha_0 = 1 = \alpha_2$, and $\alpha_1 = -\alpha_3$, . So the involution becomes:

$$\begin{aligned}
\mathbf{1}^* = \alpha_0 \mathbf{1}, \qquad (\omega)^* = \alpha_1 \omega, \qquad (\omega^2)^* = \alpha_2 \omega^2, \qquad (\omega^3)^* = \alpha_3 \omega^3,\\
\theta^* = \beta_0 \theta, \qquad (\theta\omega)^* = \beta_1 \theta\omega, \qquad (\theta\omega^2)^* = -\beta_0 \theta\omega, \qquad (\theta\omega^3)^* = -\beta_1 \theta\omega^3.
\end{aligned} \tag{4.13}$$

Then, we have four different choises for the involution on $\mathbb{B}_2$:

1. $x^* = \{x_0, x_2, x_1, -x_3, x_4, -x_6, x_5, -x_7\} := \{z_{02}, \overline{z_{13}}, \overline{z_{46}}, \overline{z_{57}}\}$,
2. $x^* = \{x_0, x_2, x_1, -x_3, x_6, -x_4, x_7, -x_5\} := \{z_{02}, \overline{z_{13}}, \overline{z_{64}}, \overline{z_{75}}\}$
3. $x^* = \{x_0, x_2, x_3, -x_1, x_4, -x_6, x_7, -x_5\} := \{z_{02}, \overline{z_{31}}, \overline{z_{46}}, \overline{z_{75}}\}$,
4. $x^* = \{x_0, x_2, x_3, -x_1, x_6, -x_4, x_5, -x_7\} := \{z_{02}, \overline{z_{31}}, \overline{z_{64}}, \overline{z_{57}}\}$.

One can fix one of the four involutions arbitrarily without changing the underlying physical interpretation. More specifically, we choose the fourth option

$$
\begin{aligned}
x^* &= \{x_0, -x_1, x_2, x_3, -x_4, x_5, x_6, -x_7\} \\
&:= \{z_{02}, \overline{z_{31}}, \overline{z_{64}}, \overline{z_{57}}\},
\end{aligned} \tag{4.14}
$$

wich is reached by taking $\beta_0 = -1$ and $\beta_1 = +1$, so that

$$
\begin{array}{llll}
\mathbf{1}^* = \mathbf{1}, & (\omega)^* = -\omega, & (\omega^2)^* = \omega^2, & (\omega^3)^* = \omega^3, \\
\theta^* = -\theta, & (\theta\omega)^* = \theta\omega, & (\theta\omega^2)^* = \theta\omega, & (\theta\omega^3)^* = -\theta\omega^3.
\end{array}
$$

Note that this involution choice is equivalent to make $\{x_1, x_4, x_7\}$ to be the imaginary components of the complex numbers $z_{31} = x_3 + ix_1$, $z_{64} = x_6 + ix_4$, and $z_{57} = x_5 + ix_7$. The number z_{02} can be interpreted (for any involution) merely as a couple of real numbers.

We simplify our notation by relabeling as follows

$$
z_{02} \to (x_0, x_2), \quad z_{31} \to z_1, \quad z_{64} \to z_2, \text{ and } \quad z_{57} \to z_3. \tag{4.15}
$$

In that case, we can rearrange the basis by spliting it in its even and odd parts

$$
x = \{\underbrace{(x_0, x_2), z_2}_{\text{even}}, \underbrace{z_1, z_3}_{\text{odd}}\} \tag{4.16}
$$

4.2.1 Representation of the $\mathbb{B}_2$-product

In NCG, the algebra multiplication is represented by composition between operators. Because composition is an associative operation, when working with a nonassociative algebra it will be necessary to go further as is done in [19]. There, a bimodule H over $\mathcal{A}$ is defined as the new algebra $\mathcal{A} \oplus H$ with the product given by[3] $(x + h) \cdot (x' + h') := xx' + xh' + hx'$, for $x, y \in \mathcal{A}$, and $h, h' \in H$.

Moreover, here we will consider only the natural birepresentation of an algebra on itself, or in copies of itself. Let us fix an element $y \in \mathbb{B}_2$ and then select any other $x \in \mathbb{B}_2$. The action of x to the left and to the right are given explicitly by

$$
L_x(y) := xy = \begin{pmatrix}
x_0 & x_3 & -x_2 & -x_1 & -x_4 & -x_7 & -x_6 & x_5 \\
x_1 & x_0 & -x_3 & x_2 & -x_5 & -x_4 & -x_7 & -x_6 \\
x_2 & x_1 & x_0 & x_3 & x_6 & x_5 & -x_4 & x_7 \\
x_3 & -x_2 & x_1 & x_0 & -x_7 & x_6 & x_5 & -x_4 \\
x_4 & -x_7 & -x_6 & x_5 & x_0 & -x_3 & x_2 & x_1 \\
x_5 & x_4 & x_7 & -x_6 & x_1 & x_0 & -x_3 & -x_2 \\
x_6 & x_5 & x_4 & x_7 & -x_2 & -x_1 & x_0 & -x_3 \\
x_7 & x_6 & -x_5 & x_4 & x_3 & x_2 & x_1 & x_0
\end{pmatrix}
\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix}, \tag{4.17}
$$

[3]The product xx' is the product inherited from $\mathcal{A}$, while xh' and hx' are the left- and right-actions defined in Eq. (2.13). As it is explained in [19], the order zero condition is contained in the case when $\mathcal{A} \oplus H$ is an associative algebra.

$$R_x(y) := yx = \begin{pmatrix} x_0 & -x_3 & -x_2 & x_1 & -x_4 & x_7 & -x_6 & -x_5 \\ x_1 & x_0 & x_3 & -x_2 & -x_5 & -x_4 & -x_7 & -x_6 \\ x_2 & x_1 & x_0 & x_3 & -x_6 & x_5 & x_4 & x_7 \\ x_3 & x_2 & -x_1 & x_0 & -x_7 & x_6 & x_5 & -x_4 \\ x_4 & x_7 & x_6 & -x_5 & x_0 & x_3 & -x_2 & -x_1 \\ x_5 & x_4 & -x_7 & -x_6 & x_1 & x_0 & -x_3 & x_2 \\ x_6 & -x_5 & -x_4 & -x_7 & x_2 & x_1 & x_0 & x_3 \\ x_7 & x_6 & x_5 & x_4 & x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \\ y_7 \end{pmatrix}. \tag{4.18}$$

4.2.2 $\mathbb{B}_2$-derivations

For the Bison algebra, the set of derivations is given by:

$$\begin{aligned} \delta_0 =& \frac{2}{3}[R_{(\omega^3)}R_{(\omega^3)} - L_{(\omega^3)}L_{(\omega^3)}] + \frac{1}{6}[R_{(\theta\omega^3)}L_{(\theta\omega^3)} - L_{(\theta\omega)}R_{(\theta\omega)} \\ &+ R_{(\omega)}L_{(\omega)} - L_{(\omega^3)}R_{(\omega^3)}], \\ \delta_1 =& \frac{1}{2}[R_{(\omega)}L_{(\omega)} - R_{(\theta\omega^3)}L_{(\theta\omega^3)} + R_{(\omega^3)}L_{(\omega^3)} - R_{(\theta\omega)}L_{(\theta\omega)}], \\ \delta_2 =& \frac{1}{2}[L_{(\omega)}L_{(\theta\omega^3)} + L_{(\omega^3)}L_{(\theta\omega)}] - L_{(\theta\omega)}R_{(\omega^3)} - L_{(\theta\omega^3)}R_{(\omega)}, \\ \delta_3 =& \frac{1}{2}[L_{(\omega)}L_{(\theta\omega)} - L_{(\omega^3)}L_{(\theta\omega^3)}] + L_{(\theta\omega^3)}R_{(\omega^3)} - L_{(\theta\omega)}R_{(\omega)}, \end{aligned} \tag{4.19}$$

which satisfy that $[\delta_i, \delta_j] = 2\epsilon_{ijk}\delta_k$, where ϵ is totally anti-symmetric. More specifically, they satisfy $[\delta_0, \delta_i] = 0$, $[\delta_i, \delta_j] = 2\delta_k$ for $[i, j, k] = \{[1, 2, 3], [3, 1, 2], [2, 3, 1]\}$. Thus, they form a basis for the Lie algebra $\mathrm{Der}(\mathbb{B}_2) \approxeq \mathfrak{u}(1) \oplus \mathfrak{su}(2)$.

In a more compact form, the derivations can be written as

$$\begin{aligned} \delta_0 &= e_{24} - e_{42} - 2(e_{57} - e_{75}) - (e_{68} - e_{86}), \\ \delta_1 &= e_{24} - e_{42} + (e_{68} - e_{86}), \\ \delta_2 &= e_{26} - e_{62} - (e_{48} - e_{84}), \\ \delta_3 &= -(e_{28} - e_{82}) - (e_{46} - e_{64}), \end{aligned}$$

with e_{ij} a matrix where the only nonzero entry is a 1 in the $i-$th row and $j-$th column. Also, taking into account the 2 dimensional matrix representation of the imaginary unit given by $i := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, we get

$$\delta_0 = \begin{pmatrix} 0 & & & \\ & -2i & & \\ & & i & \\ & & & i \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & i & \\ & & & -i \end{pmatrix},$$
$$\delta_2 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & i \\ & & i & 0 \end{pmatrix}, \quad \delta_3 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}, \tag{4.20}$$

so, the values on the $\mathfrak{u}(1)$ generator δ_0, should be compared with the hypercharges for the SM leptons. This suggest the following identification

$$\begin{pmatrix} x_0 \\ x_2 \\ x_6 \\ x_4 \\ x_3 \\ x_1 \\ x_5 \\ x_7 \end{pmatrix} \to \begin{pmatrix} x_0, x_2 \\ z_2 \\ z_1 \\ z_3 \end{pmatrix} := \begin{pmatrix} \xi \\ e_R \\ (\nu_L)^\dagger \\ (e_L)^\dagger \end{pmatrix}, \tag{4.21}$$

where ξ is a new hypercharge zero (singlet) fermion. Therefore, we see that the $SU(2)$ generators act only on the odd (left) whereas its Abelian generator act on both even (right) and odd (left) components of the algebra representation. But not only that, the diagonal elements of δ_0 corresponds exactly with the SM leptons hypercharges. So, this is why we say that $\mathbb{B}_2$ is chiral by nature.

Finally, from Eqs. (4.14) and (4.20), we have that

$$\delta_i x^* = -(\delta_i x)^*, \quad \text{for} \quad i = 0, 1, 2, 3, \quad \text{and} \quad \forall x \in \mathbb{B}_2. \tag{4.22}$$

4.3 The Bison Algebra Model

In this section, we reconstruct the Glashow-Weinberg-Salam model discussed in section 2.3.4 by replacing the algebra $\mathbb{C} \oplus \mathbb{H}$ with the Bison algebra $\mathbb{B}_2$. This shift is done since the Bison algebra is naturally chiral in the sense that (from Eq. (4.20)) the $\mathfrak{su}(2)$ generators δ_1, δ_2, and δ_3 act only on the odd elements, whereas the $\mathfrak{u}(1)$ generator δ_0 acts on both the even and odd elements.

4.3.1 The SM leptons representation

Following the NCG philosophy, one has a continuous manifold where chiralities and particle and anti-particle states are already defined. Then, we also have to include such states in the finite space in order to achieve the correct representations of the gauge group. Furthermore, as it was explained in Chapter 2, to get rid of the extra fermion degrees of freedom, we should impose the conditions in Eqs. (2.30) and (2.32).

So, in order to impose first the 'Majorana' condition $J\Psi = \Psi$ (Eq. (2.32)), we will define a $\mathbb{B}_2$ representation and a charge conjugation operator J_F so that they commute. In this way, we choose the following 16 dimensional birepresentation of an arbitrary element $a \in \mathbb{B}_2$ on the space $H = \mathbb{B}_2 \oplus \mathbb{B}_2$.

$$\rho_L(a) = \begin{pmatrix} L_a & \\ & R_{a^*} \end{pmatrix}, \qquad \rho_R(x) = \begin{pmatrix} R_a & \\ & L_{a^*} \end{pmatrix}. \tag{4.23}$$

Its action on H_F is defined by

$$\begin{aligned} \rho_L(a)h &= \begin{pmatrix} L_a & \\ & R_{a^*} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax \\ ya^* \end{pmatrix}, \\ \rho_R(a)h &= \begin{pmatrix} R_a & \\ & L_{a^*} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xa \\ a^*y \end{pmatrix}. \end{aligned} \tag{4.24}$$

where $h = (x, y) \in \mathbb{B}_2 \oplus \mathbb{B}_2$.

Note that the second copy of $\mathbb{B}_2$ is necessary to get the anti-particle states corresponding to the particles in Eq. (4.21) and

$$y = \begin{pmatrix} y_0 \\ y_2 \\ y_4 \\ y_6 \\ y_1 \\ y_3 \\ y_5 \\ y_7 \end{pmatrix} \to \begin{pmatrix} (y_0, y_2) \\ y_6 + iy_4 \\ y_3 + iy_1 \\ y_5 + iy_7 \end{pmatrix} := \begin{pmatrix} y_0, y_2 \\ u_2 \\ u_1 \\ u_3 \end{pmatrix} := \begin{pmatrix} \xi^\dagger \\ (e_R)^\dagger \\ \nu_L \\ e_L \end{pmatrix}, \tag{4.25}$$

where we have introduced the complex numbers u_i for $i = 1, 2, 3$.

Next, we make use of our chosen involution in Eq. (4.14) to define the charge conjugation operator on H_F as follows[4]

$$J_F := \begin{pmatrix} & 1_8 \\ 1_8 & \end{pmatrix} \circ * = \left(\begin{array}{cccccccc|cccccccc}
&&&&&&&& 1 &&&&&&& \\
&&&&&&&&& -1 &&&&&& \\
&&&&&&&&&& 1 &&&&& \\
&&&&&&&&&&& 1 &&&& \\
&&&&&&&&&&&& -1 &&& \\
&&&&&&&&&&&&& 1 && \\
&&&&&&&&&&&&&& 1 & \\
&&&&&&&&&&&&&&& -1 \\
\hline
1 &&&&&&&&&&&&&&& \\
& -1 &&&&&&&&&&&&&& \\
&& 1 &&&&&&&&&&&&& \\
&&& 1 &&&&&&&&&&&& \\
&&&& -1 &&&&&&&&&&& \\
&&&&& 1 &&&&&&&&&& \\
&&&&&& 1 &&&&&&&&& \\
&&&&&&& -1 &&&&&&&&
\end{array}\right). \tag{4.26}$$

The action of such J_F on H_F is given by

$$J_F h = \begin{pmatrix} & 1_8 \\ 1_8 & \end{pmatrix} \circ * \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^* \\ x^* \end{pmatrix}. \tag{4.27}$$

So we have that

$$\rho_L(a)(Jh) = \begin{pmatrix} ay^* \\ x^* a^* \end{pmatrix}, \tag{4.28}$$

and by Eq. (4.23)

$$J\left(\rho_L(a)h\right) = J \begin{pmatrix} ax \\ ya^* \end{pmatrix} = \begin{pmatrix} ay^* \\ x^* a^* \end{pmatrix}. \tag{4.29}$$

Therefore, J_F commutes with the $\mathbb{B}_2$ birepresentation

$$[J_F, \rho_L] = [J_F, \rho_R] = 0. \tag{4.30}$$

Note that this property is unaltered whatever choice for an involution we make.

In order to have a well defined anti-unitary real structure, it is necessary to define inner product on $\mathbb{B}_2 \oplus \mathbb{B}_2$.

[4] The matrix form of the charge conjugation operator is given in the standard basis $\{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7\}$, in order to make it more evident its relation with the involution in Eq. (4.14), where only the 1^{th}, 2^{th}, and 3^{th} components are afected by a minus $(-)$ sign.

Inner product

Having a look to Eqs (4.21) and (4.25), it follows that there are 4 real and 6 complex physical degrees of freedom. Then, we make the following assumption

$$H_F = \mathbb{B}_2 \oplus \mathbb{B}_2 \cong \mathbb{R}^4 \oplus \mathbb{C}^6. \tag{4.31}$$

Hence, by using Eq. (4.15), any $h \in H_F$ can be labeled by setting

$$h = \{x_0, x_2, \underbrace{x_6, x_4}_{z_2}, \underbrace{x_3, x_1}_{z_1}, \underbrace{x_5, x_7}_{z_3}, y_0, y_2, \underbrace{y_6, y_4}_{u_2}, \underbrace{y_3, y_1}_{u_1}, \underbrace{y_5, y_7}_{u_3}\}. \tag{4.32}$$

where $z_i, u_i \in \mathbb{C}$ and x_0, x_2, y_0, y_2 still considered as real elements.

Now we define the (natural) inner product for any pair of elements $h, h' \in H_F = \mathbb{B}_2 \oplus \mathbb{B}_2$ as given by

$$\begin{aligned} h \cdot h' &= x_0 x_0' + x_2 x_2' + y_0 y_0' + y_0 y_0' \\ &+ \sum_{i=1}^{i=3} \overline{z_i} z_i' + \sum_{i=1}^{i=3} \overline{u_i} u_i'. \end{aligned} \tag{4.33}$$

Let us now calculate

$$\begin{aligned} Jh &= \{y_0, y_2, \underbrace{y_6, -y_4}_{\overline{u_2}}, \underbrace{y_3, -y_1}_{\overline{u_1}}, \underbrace{y_5, -y_7}_{\overline{u_3}}, x_0, x_2, \underbrace{x_6, -x_4}_{\overline{z_2}}, \underbrace{x_3, -x_1}_{\overline{z_1}}, \underbrace{x_5, -x_7}_{\overline{z_3}}\}, \\ Jh' &= \{y_0', y_2', \underbrace{y_6', -y_4'}_{\overline{u_2'}}, \underbrace{y_3', -y_1'}_{\overline{u_1'}}, \underbrace{y_5', -y_7'}_{\overline{u_3'}}, x_0', x_2', \underbrace{x_6', -x_4'}_{\overline{z_2'}}, \underbrace{x_3', -x_1'}_{\overline{z_1'}}, \underbrace{x_5', -x_7'}_{\overline{z_3'}}\}, \end{aligned}$$

so we have that

$$\begin{aligned} Jh' \cdot Jh &= y_0' y_0 + y_2' y_2 + x_0' x_0 + x_2' x_2 \\ &+ \sum_{i=1}^{i=3} z_i \overline{z_i'} + \sum_{i=1}^{i=3} u_i \overline{u_i'}. \end{aligned} \tag{4.34}$$

Then, taking into account that $\overline{\overline{z_i}} = z_i$, and $\overline{\overline{u_i}} = u_i$, one gets that $Jh' \cdot Jh = h \cdot h'$, which means that J is antiunitary.

Now we want to summarize what we have so far: an algebra $A_F = \mathbb{B}_2$ (real-nonassociative) together to a (real) birepresentation on the (inner product) space $H_F = \mathbb{B}_2 \oplus \mathbb{B}_2$ and an (anti-unitary) charge conjugation operator J. Therefore, it remains to us to build Dirac and chirality operators. Before constructing these two operators, let us have a look to what happen when we take the tensor product with the elements of the continuous space.

4.3.2 Almost-commutative $\mathbb{B}_2$-manifold

Following the NCG paradigm, we now consider the product space $M \times \mathbb{B}_2$, where the factor M is the continuous (curved) manifold encoding the space-time structure. In this way, the total space will be defined by the tensor product between the canonical spectral triple and the finite space corresponding to the nonassociative $\mathbb{B}_2$ algebra. This construction will be reached in five steps:

1. For the total algebra we take $A := C^\infty(M,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{B}_2$. Note that we can not take $C^\infty(M,\mathbb{C})$ since $\mathbb{B}_2$ is a real algebra and their tensor product would double the number fermion states[5]. Note that for $C^\infty(M,\mathbb{C})$ the real tensor product $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}\cong\mathbb{R}^2$, which give us a total of $2\times 16 = 32$ fermion degrees of freedom. Lets compare the last relation with $\mathbb{C}\otimes_{\mathbb{C}}\mathbb{C}\cong\mathbb{C}$ and $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}\cong\mathbb{R}^4$.

2. For the total Hilbert space, we consider again the real tensor product $L^2(M,S)\otimes(\mathbb{B}_2\oplus\mathbb{B}_2)$, since $H_F = \mathbb{B}_2\oplus\mathbb{B}_2$ is a real vector space. Even though, this is an $16\times 4 = 64$ dimensional complex vector space. So, if we first impose the Majorana condition $J_M\xi = \xi$ for any $\xi \in L^2(M,S)$, and call this space $\widetilde{L^2(M,S)}$, then the tensor product over the reals $H := \widetilde{L^2(M,S)}\otimes(\mathbb{B}_2\oplus\mathbb{B}_2)$, lead us to a 64 dimensional real vector space. Then, it could be thought as a 32 dimensional complex Hilbert space.

3. The (total) chirality operator Γ should be Hermitian. It would be also desirable to have a Γ commuting with the algebra representation in order to impose the 'Weyl' condition $\Gamma\Psi = \Psi$ to break up the fermion degrees of freedom to one half.

   We have two options:

   - In a continuous space with Lorentzian signature, we can select all of the γ^μ matrices to be real:

$$\gamma^0 = \begin{pmatrix} & & & -1 \\ & & -1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & -1 & & \\ 1 & & & \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}. \tag{4.35}$$

     Despite that, it is not possible to have a real γ^5. To see this, lets suppose that all of the gamma matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are real and that

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3, \tag{4.36}$$

[5] One would be tempted to consider the space $H_F = \mathbb{B}_2\oplus\mathbb{B}_2$ as a complexification [96] of $\mathbb{B}_2$ (since $\mathbb{B}_2\oplus\mathbb{B}_2 \cong \mathbb{C}\otimes_{\mathbb{R}}\mathbb{B}_2$) by defining $(a+ib)(h_1,h_2) = (ah_1 - bh_2, ah_2 + bh_1)$, with $a+ib\in\mathbb{C}$ and $(h_1,h_2)\in H_F$. In that case, we would take the tensor product $L^2(M,S)\otimes_{\mathbb{C}} H_F$ obtaining the correct degrees of freedom. But even so, the representation in Eq. (4.23) still not to be a complex representation.

is satisfied. Then, it implies that

$$\begin{aligned}(\gamma^5)^\dagger &= (\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger \\ &= \gamma^3\gamma^2\gamma^1(-\gamma^0),\end{aligned} \tag{4.37}$$

because one of the 4 gamma matrices should be anti-Hermitian. So, unless we multiply the right hand side of Eq. (4.36) by i, γ_5 will be anti-Hermitian. Let us now to consider the real matrix

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}. \tag{4.38}$$

To have an Hermitian Γ, instead of γ^5 we can take the tensor product of $\gamma^0\gamma^1\gamma^2\gamma^3$ with some representation of the imaginary unit I (attached to γ_F) acting on the finite space as follows

$$\Gamma = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes I\gamma_F. \tag{4.39}$$

- Conversely, if we choose γ^5 to be a real matrix, then one of the γ^μ's should be imaginary. In this case, it is evident that the total chirality operator can be defined as $\Gamma = \gamma^5 \otimes \gamma_F$, without any complication. However, as we will see, this option causes some difficulties when defining the bosons of the theory.

4. The next step is to construct the Dirac operator. As before, we have two options:

   - Let us consider γ^5 real. In this case, we can construct the Dirac operator in the usual way as

$$\not{D} \otimes 1_{16} + \gamma^5 \otimes D_F. \tag{4.40}$$

But in this case, the commutator $[\not{D} \otimes 1_{16} + \gamma^5 \otimes D_F, \hat{f} \otimes \rho(a)]$ is not bounded, which would spoil the well definition of the boson fields. To show that, let us suppose that γ^0 is the imaginary one. Instead of taking it as imaginary, we consider it as real but tensored with some anti-Hermitian representation of the imaginary unit I acting on H_F.

$$[\not{D} \otimes 1_{16} + \gamma^5 \otimes D_F, \hat{f} \otimes \rho(a)] = [\gamma^0\nabla_0 \otimes I, \hat{f} \otimes \rho(a)] + \underbrace{[\gamma^i\nabla_i \otimes 1_{16}, \hat{f} \otimes \rho(a)]}_{\text{bounded}} + \underbrace{[\gamma^5 \otimes D_F, \hat{f} \otimes \rho(a)]}_{\text{bounded}}, \tag{4.41}$$

then, for the remaining part we have

$$\begin{aligned}[\gamma^0\nabla_0 \otimes I, \hat{f} \otimes \rho(a)](\psi \otimes h) &= \gamma^0\nabla_0\hat{f}\psi \otimes I\rho(a)h - \hat{f}\gamma^0\nabla_0\psi \otimes \rho(a)Ih \\ &= (\gamma^0\hat{f}\nabla_0^S + \gamma^0\partial_0\hat{f})\psi \otimes I\rho(a)h - \hat{f}\gamma^0\nabla_0^S\psi \otimes \rho(a)Ih \\ &= \gamma^0\hat{f}\nabla_0^S\psi \otimes (I\rho(a) - \rho(a)I)h + \gamma^0\partial_0\hat{f}\psi \otimes I\rho(a)h,\end{aligned}$$

thus, we get

$$[\gamma^0\nabla_0 \otimes I, \hat{f}\otimes\rho(a)] = \gamma^0\hat{f}\nabla_0^S \otimes (I\rho(a)-\rho(a)I) + \gamma^0\partial_0\hat{f}\otimes I\rho(a)$$
$$= \gamma^0\hat{f}\nabla_0^S \otimes [I,\rho(a)] + \underbrace{\gamma^0\partial_0\hat{f}\otimes I\rho(a)}_{\text{bounded}}.$$

When $[I,\rho(a)]=0$, for a 16×16 matrix I acting on H_F, we have a bounded commutator $[\gamma^\mu\nabla_\mu^S\otimes 1_{16}+\gamma^5\otimes D_F, \hat{f}\otimes\rho(a)]$. But it can be shown that there are not anti-Hermitian matrices commuting with $\rho(a)$. If this commutator is not zero it would be problematic because the connection 1-form containing the bosons of the theory would be unbounded. Rather than require a null commutator, one could ask whether there exist an automorphism $\widetilde{\rho}$ of A such that the twisted commutator [97, 98, 99, 100] defined by

$$[\not{\mathcal{D}}, \rho(a)]_{\widetilde{\rho}} := \not{\mathcal{D}}\rho(a) - \widetilde{\rho}(a)\not{\mathcal{D}}, \tag{4.42}$$

is bounded for any $a\in A$?

- Let us take the four gamma matrices to be real as in Eq. (4.35). To avoid the using of γ^5 (which is not real), we can alternatively to define the total Dirac operator as given by [58, 101]

$$\not{\mathcal{D}}\otimes\gamma_F + 1_4\otimes D_F. \tag{4.43}$$

Usually, the Dirac operator is required to be 'krein-hermitian', which means that $\gamma_0\not{D}^\dagger\gamma_0 = \not{D}$ (where $\not{D} = i\gamma^\mu\nabla_\mu^S$). But if we do not compose the gamma matrices with i (i.e. $\not{D} := \gamma^\mu\nabla_\mu^S$), then we end up with a Dirac operator that is krein anti-Hermitian: $\gamma_0\not{D}^\dagger\gamma_0 = -\not{D}$. Because the Dirac operator should be Hermitian, one option is to take the product between the continuous Dirac operator $\gamma^\mu\nabla_\mu^S$ with some representation I of the imaginary unit on the finite space. In this case, we will need to find a twist for the commutator $[\mathcal{D}, \hat{f}\otimes\rho(a)]$. Another option, is by defining an special kind of Hermiticity. More specifically, we can say that $\not{D}$ is a Hermitian if it satisfies that:

$$\gamma_1\gamma_2\gamma_3\not{D}^\dagger\gamma_1\gamma_2\gamma_3 = \not{D}. \tag{4.44}$$

In such a case, we can replace the usual krein-inner product $\Psi^\dagger\gamma_0\Psi$, with the inner product given by

$$\int \Psi^T\gamma^1\gamma^2\gamma^3\Psi, \tag{4.45}$$

this makes sure the Dirac operator is Hermitian with respect to this inner product.

5. The last step is to build an anti-unitary charge conjugation operator satisfying Eqs. (2.11). Our before definition of Hermiticity would be equivalent to changing the signature of the continuous space from $2 =_8 -6$ to $-2 =_8 6$. Therefore we have to change the signature of the internal space from $6 =_8 -2$ to $-6 =_8 2$ to ensure the total space remains 0. The internal space is then either KO-dimension 2 or 6. The difference will just be whether D_F commutes or anti-commutes with J. For a continuous space with Lorentz signature, the total space should be KO dimension 0.

4.3.3 Chirality and Dirac operators

Following the NCG principles, one should build chirality and Dirac operators acting on $H_F = \mathbb{B}_2 \oplus \mathbb{B}_2$. We will do it in two different ways:

- By fixing an (natural) Hermitian chirality operator γ_F guided by the 'chiral' form of the representations (symmetries) as manifested in Eqs.(4.21) and (4.25).
- By building a Dirac operator containing the minimal Yukawa interaction for the leptons of the SM.

From γ_F to D_F

We start by calculating the most general matrix commuting with the birepresntation given in Eq.(4.24). Let us consider a 16×16 matrix M so that $M\rho(a) = \rho(a)M$

$$M = \left(\begin{array}{cccccccc|cccccccc}
a &&&&&&&& -b &&&&&&& \\
& a &&&&&&&& b &&&&&& \\
&& a &&&&&&&& -b &&&&& \\
&&& a &&&&&&&& -b &&&& \\
&&&& a &&&&&&&& b &&& \\
&&&&& a &&&&&&&& -b && \\
&&&&&& a &&&&&&&& -b & \\
&&&&&&& a &&&&&&&& b \\
\hline
-c &&&&&&&& d &&&&&&& \\
& c &&&&&&&& d &&&&&& \\
&& -c &&&&&&&& d &&&&& \\
&&& -c &&&&&&&& d &&&& \\
&&&& c &&&&&&&& d &&& \\
&&&&& -c &&&&&&&& d && \\
&&&&&& -c &&&&&&&& d & \\
&&&&&&& c &&&&&&&& d
\end{array}\right), \tag{4.46}$$

where $a, b, c, d \in \mathbb{R}$. The off-diagonal elements with $b = c = -1$ give rise to J, which (as we have already shown) it commutes with the birepresentation.

Now, we will proceed to construct a chirality operator so that it commutes with the bi-representation. Hence, by focusing on the diagonal elements of M, we may define γ_F by

$$\gamma_F = \begin{pmatrix} -1_4 &&& \\ & -1_4 && \\ && 1_4 & \\ &&& 1_4 \end{pmatrix},$$

which correspond to the chirality presumed by the sub-index on Eqs. (4.21) and (4.25). It that case, it would be natural to take ξ as a right-handed particle. Then, we will consider this as just as the right-handed neutrino. Let us show that γ_F is Hermitian, first

$$\gamma_F h = \{-x_0, -x_2, \underbrace{-x_6, -x_4}_{-z_2}, \underbrace{-x_3, -x_1}_{-z_1}, \underbrace{-x_5, -x_7}_{-z_3}, y_0, y_2, \underbrace{y_6, y_4}_{u_2}, \underbrace{y_3, y_1}_{u_1}, \underbrace{y_5, y_7}_{u_3}\},$$
$$\gamma_F h' = \{-x'_0, -x'_2, \underbrace{-x'_6, -x'_4}_{-z'_2}, \underbrace{-x'_3, -x'_1}_{-z'_1}, \underbrace{-x'_5, -x'_7}_{-z'_3}, y'_0, y'_2, \underbrace{y'_6, y'_4}_{u'_2}, \underbrace{y'_3, y'_1}_{u'_1}, \underbrace{y'_5, y'_7}_{u'_3}\}.$$

So, by comparing with Eq. (4.32), we conclude that

$$\gamma_F h \cdot h' = h \cdot \gamma_F h', \tag{4.47}$$

which means that γ_F is Hermitian.

Now, the total Γ will be sought such that it squares to the identity and that either commutes or anti-commutes with J.

Once we have such a chirality operator, we can impose $\Gamma\Psi = \Psi$, for any $\Psi \in L^2\widetilde{(M, S)} \otimes (\mathbb{B}_2 \oplus \mathbb{B}_2)$, to avoid to doubling the fermion degrees of freedom [69]. Then, the total chirality operator will be given by

$$\Gamma = \underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}_{\text{a-H}} \otimes \underbrace{I}_{\text{a-H}} \underbrace{\gamma_F}_{\text{H}}, \tag{4.48}$$

where a-H, and H mean for anti-Hermitian (Hermitian).

We will go further by imposing that $I\gamma_F$ should not to commute with the algebra but just with its inner derivations

$$[I\gamma_F, \delta_i] = 0, \qquad i \in \{0, 1, 2, 3\}. \tag{4.49}$$

Let us suppose that $I = \rho(\omega^2)$, which is given explicitly by

$$\rho(\omega^2) = \begin{pmatrix} i &&&&&&& \\ & i &&&&&& \\ && i &&&&& \\ &&& i &&&& \\ &&&& i &&& \\ &&&&& -i && \\ &&&&&& -i & \\ &&&&&&& -i \end{pmatrix} \tag{4.50}$$

where we have used $i := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$. It is straightforward to show that $[\rho(\omega^2), \delta_1] = 0$, for $i = 0, 1, 2, 3$.

Before to build a charge conjugation operator, first note that

$$\begin{aligned}\gamma^5 \not{D} = -\not{D}\gamma^5 \quad &\Leftrightarrow \quad i\gamma^0\gamma^1\gamma^2\gamma^3\not{D} = -i\not{D}\gamma^0\gamma^1\gamma^2\gamma^3 \\ &\Leftrightarrow \quad \gamma^0\gamma^1\gamma^2\gamma^3\not{D} = -\not{D}\gamma^0\gamma^1\gamma^2\gamma^3 \\ &\Leftrightarrow \quad \{\not{D}, \gamma^0\gamma^1\gamma^2\gamma^3\} = 0. \end{aligned} \tag{4.51}$$

From the last result and by imposing the rule $\{D, \Gamma\} = 0$, we have

$$\begin{aligned}\{D, \Gamma\} = 0 &\Leftrightarrow \{\not{D} \otimes \gamma_F, \gamma^0\gamma^1\gamma^2\gamma^3 \otimes I\gamma_F\} + \{1_4 \otimes D_F, \gamma^0\gamma^1\gamma^2\gamma^3 \otimes I\gamma_F\} = 0 \\ &\Leftrightarrow \underbrace{\{\not{D}, \gamma^0\gamma^1\gamma^2\gamma^3\}}_{0} \otimes (I\gamma_F)^2 + \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \{D_F, I\gamma_F\} = 0 \\ &\Leftrightarrow \{D_F, I\gamma_F\} = 0. \end{aligned} \tag{4.52}$$

We have also the following property

$$\gamma_F J_F = -J_F\gamma_F \quad \Leftrightarrow \quad \gamma_F(\gamma_F J_F) = -(\gamma_F J_F)\gamma_F. \tag{4.53}$$

The charge conjugation operator will be either $1 \otimes J_F$ or $\gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F$ or $1 \otimes \gamma_F J_F$ or $\gamma^0\gamma^1\gamma^2\gamma^3 \otimes \gamma_F J_F$.

For the total Dirac operator $\mathcal{D} = \not{D} \otimes \gamma_F + 1_4 \otimes D_F$, for an arbitrary D_F, we impose $[\mathcal{D}, J] = 0$, so we have four options:

1. For $J = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F$, we get (by using $\{\gamma_F, J_F\} = 0$)

$$\begin{aligned}[\mathcal{D}, J] = 0 &\Leftrightarrow [\not{D} \otimes \gamma_F, \gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F] + [1_4 \otimes D_F, \gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F] = 0 \\ &\Leftrightarrow \underbrace{\{\not{D}, \gamma^0\gamma^1\gamma^2\gamma^3\}}_{0} \otimes \gamma_F J_F + \gamma^0\gamma^1\gamma^2\gamma^3 \otimes [D_F, J_F] = 0 \\ &\Leftrightarrow [D_F, J_F] = 0. \end{aligned} \tag{4.54}$$

2. For $J = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \gamma_F J_F$ we may use Eq. (4.53) to get

$$\begin{aligned}[\mathcal{D}, J] = 0 &\Leftrightarrow [\not{D} \otimes \gamma_F, \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \gamma_F J_F] + [1_4 \otimes D_F, \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \gamma_F J_F] = 0 \\ &\Leftrightarrow \underbrace{\{\not{D}, \gamma^0\gamma^1\gamma^2\gamma^3\}}_{0} \otimes \gamma_F J_F + \gamma^0\gamma^1\gamma^2\gamma^3 \otimes [D_F, \gamma_F J_F] = 0 \\ &\Leftrightarrow [D_F, \gamma_F J_F] = 0. \end{aligned} \tag{4.55}$$

3. For $J = 1_4 \otimes J_F$ we obtain

$$\begin{aligned}[\mathcal{D}, J] = 0 &\Leftrightarrow [\not{D} \otimes \gamma_F, 1_4 \otimes J_F] + [1_4 \otimes D_F, 1_4 \otimes J_F] = 0 \\ &\Leftrightarrow \not{D} \otimes [\gamma_F, J_F] + 1_4 \otimes [D_F, J_F] = 0 \\ &\Leftrightarrow [\gamma_F, J_F] = 0 \wedge [D_F, J_F] = 0. \end{aligned} \tag{4.56}$$

4. In this case, by the property $\{\gamma_F, J_F\} = 0$ we know that $[\gamma_F, J_F] = 0$ is not true.

5. For $J = 1_4 \otimes \gamma_F J_F$ and using Eq. (4.53) we obtain

$$\begin{aligned}
[\mathcal{D}, J] = 0 &\Leftrightarrow [\not{\partial} \otimes \gamma_F, 1_4 \otimes \gamma_F J_F] + [1_4 \otimes D_F, 1_4 \otimes \gamma_F J_F] = 0 \\
&\Leftrightarrow \not{\partial} \otimes [\gamma_F, \gamma_F J_F] + 1_4 \otimes [D_F, \gamma_F J_F] = 0 \\
&\Leftrightarrow [\gamma_F, \gamma_F J_F] = 0 \wedge [D_F, \gamma_F J_F] = 0.
\end{aligned} \tag{4.57}$$

By Eq. (4.53) we see that $[\gamma_F, \gamma_F J_F] = 0$ is not true.

Then, there are just two viable options: $J = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F$ and $J = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \gamma_F J_F$.

Finally, to construct a finite Dirac operator lets start with a general 16×16 real matrix which anti-commutes with $I\gamma_F$ (for some I acting on H_F), as indicated by Eq. (4.43). Depending of which of the cases we are handle, we will require that this (general) matrix also commutes with J_F (Eq. (4.54),) or $\gamma_F J_F$ (Eq.(4.55)). Next, from all of these matrices we selected the Hermitian ones. Unfortunately, when one compares the obtained matrices with the SM Dirac operator (like the one given by Eqs. (2.21) and (2.23)) one realizes that there is not a matrix (candidate for D_F) satisfying the before commutation properties that simultaneously accommodate the lepton Yukawa interactions of the SM.

From D_F to γ_F

Here we fix D_F such that it contains the minimal Yukawa interaction for the leptonic sector of the SM

$$D_F = \left(\begin{array}{cccccccc|cccccccc}
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & -a_1 & a_2 & a_3 & -a_4 \\
& & & & & & & & & & & & -a_2 & -a_1 & a_4 & a_3 \\
& & & & & & & & & & a_1 & -a_2 & & & & \\
& & & & & & & & & & a_2 & a_1 & & & & \\
& & & & & & & & & & -a_3 & a_4 & & & & \\
& & & & & & & & & & -a_4 & -a_3 & & & & \\
\hline
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & & \\
& & & & a_1 & a_2 & -a_3 & -a_4 & & & & & & & & \\
& & & & -a_2 & a_1 & a_4 & -a_3 & & & & & & & & \\
-a_1 & -a_2 & & & & & & & & & & & & & & \\
a_2 & -a_1 & & & & & & & & & & & & & & \\
a_3 & a_4 & & & & & & & & & & & & & & \\
-a_4 & a_3 & & & & & & & & & & & & & &
\end{array}\right). \tag{4.58}$$

To construct the chirality operator Γ, will take

$$\Gamma = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \gamma_F, \tag{4.59}$$

where the finite chirality operator γ_F is required to be anti-Hermitian (because $\gamma^0\gamma^1\gamma^2\gamma^3$ is anti-Hermitian) and such that $\gamma_F^2 = -1_{16}$ (because $(\gamma^0\gamma^1\gamma^2\gamma^3)^2 = -1_4$). We also demand it commutes with the set of derivations on $\mathbb{B}_2$

$$[\gamma_F, \delta_0] = [\gamma_F, \delta_1] = [\gamma_F, \delta_2] = [\gamma_F, \delta_3] = 0. \tag{4.60}$$

To have a more general model we do not impose any kind of chirality to the new fermion. We call this ξ

$$h = \{\underbrace{x_0, x_2}_{\xi}, \underbrace{x_6, x_4}_{e_R}, \underbrace{x_3, x_1}_{(\nu_L)^\dagger}, \underbrace{x_5, x_7}_{(e_L)^\dagger}, \underbrace{y_0, y_2}_{\xi^\dagger}, \underbrace{y_6, y_4}_{(e_R)^\dagger}, \underbrace{y_3, y_1}_{\nu_L}, \underbrace{y_5, y_7}_{e_L}\}. \tag{4.61}$$

The charge conjugation operator J can be either $1 \otimes J_F$ or $\gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F$. Whatever we select it should either commutes or anti-commutes with Γ:

1. Lets consider the case when Γ and J anti-commutes. Having in mind the two possibilities $J = \gamma^0\gamma^1\gamma^2\gamma^3 \otimes J_F$ or $J = 1_4 \otimes J_F$, we have that

$$\begin{aligned}\{\Gamma, J\} = 0 &\Leftrightarrow \gamma^0\gamma^1\gamma^2\gamma^3 \otimes \{\gamma_F, J_F\} = 0 \ \vee (\gamma^0\gamma^1\gamma^2\gamma^3)^2 \otimes \{\gamma_F, J_F\} = 0 \\ &\Leftrightarrow \{\gamma_F, J_F\} = 0.\end{aligned} \tag{4.62}$$

   Again, this relation does not depend on the kind of J we have selected. This condition an the one demanded in Eq. (4.60), give us five options. Let us show the explicit form of two of them

$$\begin{pmatrix} 0 & & & & \alpha 1_2 & & & \\ & -ip & & & & & & \\ & & ir & & & & & \\ & & & iq & & & & \\ -\alpha 1_2 & & & & ip & & & \\ & & & & & ir & & \\ & & & & & & iq & \\ & & & & & & & -iq \end{pmatrix}, \begin{pmatrix} 0 & & & & -\Xi & & & \\ & ir & & & & & & \\ & & iq & & & & & \\ & & & iq & & & & \\ \Xi & & & & 0 & & & \\ & & & & & rt & & \\ & & & & & & iq & \\ & & & & & & & iq \end{pmatrix}, \tag{4.63}$$

   where $p, r, q \in \mathbb{R}$, $\alpha := \sqrt{1-p^2}$ and $\Xi = \begin{pmatrix} u & \\ & v \end{pmatrix}$ with $u, v \in \mathbb{R}$. For each of the five options the relation $\{D_F, \gamma_F\} = 0$ implies $r = -q$. Then, it is not possible to consider any of these 5 options like a truly chirality operator since particles and antiparticles would have the same chirality (which is not possible).

2. For the commuting case we have

$$\begin{aligned}[\Gamma, J] = 0 &\Leftrightarrow \gamma^0\gamma^1\gamma^2\gamma^3 \otimes [\gamma_F, J_F] = 0 \ \vee (\gamma^0\gamma^1\gamma^2\gamma^3)^2 \otimes [\gamma_F, J_F] = 0 \\ &\Leftrightarrow [\gamma_F, J_F] = 0.\end{aligned} \tag{4.64}$$

One may note here that the last relation is independent of the which option we have taken for J. The last condition together with the one imposed in Eq. (4.60), give rise to the following two options. The first one given by

$$\gamma_F = \left(\begin{array}{cccccccc|cccccccc}
 & s & & & & & & & & & & & & & & \\
-s & & & & & & & & & & & & & & & \\
 & & & t & & & & & & & & & & & & \\
 & & -t & & & & & & & & & & & & & \\
 & & & & & -u & & & & & & & & & & \\
 & & & & u & & & & & & & & & & & \\
 & & & & & & & -u & & & & & & & & \\
 & & & & & & u & & & & & & & & & \\
\hline
 & & & & & & & & & s & & & & & & \\
 & & & & & & & & -s & & & & & & & \\
 & & & & & & & & & & & -t & & & & \\
 & & & & & & & & & & t & & & & & \\
 & & & & & & & & & & & & & u & & \\
 & & & & & & & & & & & & -u & & & \\
 & & & & & & & & & & & & & & & u \\
 & & & & & & & & & & & & & & -u &
\end{array}\right),$$

with $s, u, t \in \mathbb{R}$. This can be written in a more compact form like

$$\gamma_F = \begin{pmatrix}
-is & & & & & & & \\
 & -it & & & & & & \\
 & & iu & & & & & \\
 & & & iu & & & & \\
 & & & & -is & & & \\
 & & & & & it & & \\
 & & & & & & -iu & \\
 & & & & & & & -iu
\end{pmatrix}. \tag{4.65}$$

The second one (in compact form) is given by

$$\gamma_F = \begin{pmatrix}
0 & & & & -ir & & & \\
 & -it & & & & & & \\
 & & iu & & & & & \\
 & & & iu & & & & \\
-ir & & & & 0 & & & \\
 & & & & & it & & \\
 & & & & & & -iu & \\
 & & & & & & & -iu
\end{pmatrix}, \tag{4.66}$$

with $r \in \mathbb{R}$. Now, we apply the condition $\{D_F, \gamma_F\} = 0$, which implies that $u = -t$. Any one of the last two candidates for γ_F lead us to the correct lepton chiralities. For instance, if we choose $u = -t = -1$, for both options we have

$$\begin{aligned} \gamma_F\ e_R &= -\ e_R, & \gamma_F e_L &= +e_L, \\ \gamma_F(\nu_L)^\dagger &= -(\nu_L)^\dagger, & \gamma_F(e_R)^\dagger &= +(e_R)^\dagger, \\ \gamma_F(e_L)^\dagger &= -(e_L)^\dagger, & \gamma_F\nu_L &= +\nu_L. \end{aligned} \tag{4.67}$$

But the action on ξ is different. From Eq. (4.65) we have $\gamma_F\xi = -s\xi$ and $\gamma_F\xi^\dagger = -s\xi^\dagger$. Whereas from Eq. (4.66) we have $\gamma_F\xi = -r\xi^\dagger$ and $\gamma_F\xi^\dagger = -r\xi$. So, we may to interpret ξ as a new fermion having not a defined chirality.

The twisting of $[\mathcal{D}, \hat{f} \otimes \rho(a)]$

Finally we fix γ_F to be the one in Eq. (4.65). Then, we want to show that despite the commutator $[\not{D} \otimes \gamma_F, \hat{f} \otimes \rho(a)]$ is not bounded the twisted commutator in Eq. (4.42) is bounded for

$$\widetilde{\rho}(a) := L_{\omega^2}^{-1}\rho(a)L_{\omega^2}. \tag{4.68}$$

Therefore, since $\mathcal{D} = \not{D} \otimes \gamma_F + 1_4 \otimes D_F$, the commutator $[\mathcal{D}, \hat{f} \otimes \rho(a)]$ is bounded if and only if $[\not{D} \otimes \gamma_F, \hat{f} \otimes \rho(a)]$ is bounded. So, by using Eq. (2.6) we have

$$\begin{aligned} [\not{D} \otimes \gamma_F, \hat{f} \otimes \rho(a)] &= (\not{D} \circ \hat{f}) \otimes \gamma_F\rho(a) - (\hat{f} \circ \not{D}) \otimes \rho(a)\gamma_F \\ &= \left(\hat{f} \circ \not{D} - i\gamma^\mu\partial_\mu(\hat{f})\right) \otimes \gamma_F\rho(a) - (\hat{f} \circ \not{D}) \otimes \rho(a)\gamma_F \\ &= (\hat{f} \circ \not{D}) \otimes [\gamma_F, \rho(a)] - \underbrace{i\gamma^\mu\partial_\mu(\hat{f}) \otimes \gamma_F\rho(a)}_{\text{bounded}}, \end{aligned} \tag{4.69}$$

Then, by making the twist we get

$$\begin{aligned} [\gamma_F, \rho(a)] \to [\gamma_F, \rho(a)]_{\widetilde{\rho}(a)} &= \gamma_F\rho(a) - \widetilde{\rho}(a)\gamma_F \\ &= \gamma_F\rho(a) - ({L_{\omega^2}}^{-1}\rho(a)L_{\omega^2})\gamma_F \\ &= 0, \end{aligned} \tag{4.70}$$

which is what we wanted. Now, we might to get well defined boson sector.

Conclusions and outlooks

In this work, we have demonstrated that the NCG SM admits one Higgs doublet per Yukawa coupling into the fluctuated Dirac operator. Based on that, we built the minimal Lepton-Specific, Flipped, and Type II 2HDMs, which were achieved by matching the doublet fields $\Theta_u = \Theta_d$, $\Theta_u = \Theta_e$, and $\Theta_e = \Theta_d$, respectively. Then, by using the spectral action, we encountered the form of the scalar potential for each one of the models. At this stage, λ_i (for $i = 1, ..., 4$) become non-free parameters since they are determined to unification scale as functions of g, R_u^ν and therefore of unification scale itself. Further, in contrast to the canonical 2HDM formulation, in NCG the mixing angle β must be adjusted in order to get the correct value for the top quark mass. Although for $\tan\beta \approx 2.14$ one recovers $m_t \approx 173$ GeV, we found that none of the minimal 2HDMs provide any remarkable change on the originally NCG Higgs boson mass prediction of ≈ 170 GeV unless we introduce non-null terms μ_{12}^2 and λ_5 into the scalar potential. For the special case that $\mu_{12}^2 \gtrsim 10^5$ GeV^2 and $\lambda_5 \approx -0.62$ or $\lambda_5 \approx 0.36$, we have a 2HDM type II so that we get a 125 GeV Higgs boson in accordance with the alignment limit which is the main current phenomenological constrain for the 2HDM.

For the 2HDM with right-handed neutrino and a singlet scalar field, we got two different classes of models: the Neutrinophilic models discussed in the appendix and those where ν_R couples the SM Higgs boson. In the Neutrinophilic models we got only tachyon masses for the CP-even non-SM Higgs field H. For the other case, with $\tan\beta \approx 3.27$ the Type-II as well as the Flipped' (but not the Lepton-Specifc') can fix the Higgs boson mass to its experimental value of approximately 125 GeV while keeping at the same time a top quark mass of ≈ 173 GeV. Even though, this model predicts a charged scalar field with a mass close to 95 GeV, which is below the current experimental bounds.

As a general issue, the potential for the scalar sector of the NCG 2HDM either with or without an extra singlet scalar field possesses an exact $U(1)$ symmetry, which implies a null mass for the CP-odd (pseudoscalar) field A. This accidental symmetry also leads us to the presence of unwanted stable domain walls. In order have a completely phenomenological viable model, for both cases we have introduced $\mu_{12}^2 \gtrsim 10^5$ GeV^2, and as opposed to minimal case, in presence of right-handed neutrino (and a singlet scalar field) we have maintained $\lambda_5 = 0$. In this context, we have used the type II boundary conditions to run down the RGEs. So, for both cases, we get a 125 GeV Higgs boson whose coupling with the gauge bosons deviates a maximum of 1% from the unity, while the respective coupling with the quark sector deviates by a maximum of 3.2%, as requested by current experimental limits.

There are at least three paths to explore for:

1. One option is to explore the possibility to handle an inert doublet. In such a case, we should

make $\mu_1 > 0$ so that the doublet Φ_1 does not acquire vacuum expectation value. In which case, it would be necessary to work based on the Neutinophilic model since otherwise, after electroweak symmetry breaking, there would always be some SM fermion that would not acquire mass. Here, the the insertion of $\lambda_5 \neq 0$ could be key to generate one-loop radiative seesaw mechanism to get masses to the observed left-handed neutrinos [4]

2. Another option, is to investigate for a rigorous modification of the spectral action in Eq. (2.42), where the finite part of the fluctuated Dirac operator is replaced by $\mathfrak{M}(\Phi)$, as done in [14, 102] for some Hermitian operator $\mathfrak{M}$ so that it commutes with the algebra symmetries (derivations). The task of this operator would be to 'mix' terms on Φ such that the resulting scalar potential now includes terms proportional to λ_5 and μ_{12}^2.

3. It also would be interesting to explore the phenomenology of the 2HDM on the context of the $U(1)_{B-L}$ symmetry obtained from the 'fused' algebra reformulation of the NCG SM developed in [17].

In our second approach, we have extended NCG to include the 8 dimensional real nonassociative algebra $\mathbb{B}_2$. The automorphisms group of this algebra is $SU(2) \times U(1)$ and naturally offers an explanation for the chiral nature of the electroweak theory. This is because it behaves in such a way that the $SU(2)$ generators act only on the odd components of the algebra while the generator of $U(1)$ acts on both the odd and even components. Furthermore, the most notably aspect is that the Abelian generator encodes exactly the SM lepton charges, suggesting so a natural interpretation of the even (odd) parts of the algebra with the right (left) handedness of fermions. Based on the algebra involution, we have captured the SM leptons (and anti-leptons) degrees of freedom by representing this algebra on two copies of itself. Then, we equipped such fermion space with a inner product to set an anti-unitary charge conjugation operator J_F. We observed that the right way to build both Dirac and chirality (finite) operators is by fixing first an Hermitian D_F so that it contains the correct SM leptons Yukawa interaction. Then, we asked for an anti-Hermitian γ_F either commuting or anti-commuting with J_F. From these two options, we found that $\{D_F, J_F\} = 0$ is necessary to have SM particles an anti-particles with opposite chiralities. We found that this model also predicts a new kind of singlet fermion per SM family, which does not have a definite chirality. Finally, we proceeded to built an almost commutative geometry for this model as follows:

1. The total algebra was formed by taking the real tensor product between the set of real-valued coordinate functions $\mathcal{C}^\infty(M, \mathbb{R})$ and the Bison algebra

2. For the Hilbert space, we took the real tensor product between the space of spinors (viewed as a real vector space) and the 'inner product' space $\mathbb{B}_2 \oplus \mathbb{B}_2$.

3. We constructed a Hermitian chirality operator by taking the (real) tensor product between the anti-Hermitian operators $\gamma^0\gamma^1\gamma^3\gamma^3$ and γ_F. The four continuous gamma matrices were chosen to be real.

4. We defined the Dirac operator $\mathcal{D} := \gamma^\mu \nabla_\mu^S \otimes \gamma_F + 1_4 \otimes D_F$, whose Hermiticity is guaranteed since its first term is given by the product between anti-hermitian operators.

Once developed this non-associative almost-commutative manifold, we found a twisted commutator leading the way towards the possibility to have a complete picture for the scalar sector of this theory. We must work on this route in order to find a rigorous definition for scalar bosons as differential forms over the non-associative algebra $\mathbb{B}_2$, like the work done for the exceptional Jordan algebra in [24]. Concerning to the extra fermion state we found into this formalism, it lead us to ask if this formalism has anything to say about the nature of neutrino masses. As an additional point, it would be also desirable to enlarge this model to include quarks together with its quantum numbers as well as the color symmetry. One possibility could be by fitting it together with the octonion algebra, following the study developed in [22].

Appendices

Appendix A

Renormalization group equations for the gauge couplings

The Green function for scalar (Klein-Gordon) and fermion (Dirac) fields is equal to the sum of all the probability amplitudes for the corresponding particle to go from one space-time point to another. Thus, it is known as the field's *propagator*. We can get the propagator by representing the Dirac delta function as the Fourier transform of the identity function.

Next, we review the Feynman rules[1] ([103, pag 135]) for the propagators (times the imaginary unity) as shown in figure A.1.

Fermion propagator $$S_F(P) = \frac{i}{\not{p} - m}$$

Boson propagator $$D_{\mu\nu}(k) = \frac{-ig^{\mu\nu}}{k^2}$$

Ghost propagator $$S\Delta_{\mu\nu}(p) = \frac{-i\delta^{ab}}{p^2}$$

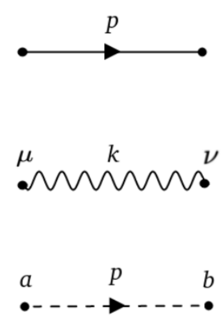
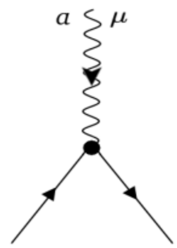


Figure A.1: fermion, boson and ghost propagators.

Here, the indices $a, b, c, ..$ mean for isospin components whereas $\mu, \nu, ...$ mean for space time coordinates.

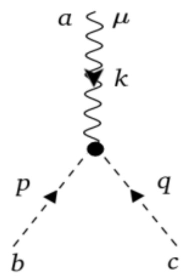
Secondly, we present the Feynman rules (figure A.2) for the Lagrangian's interaction terms between three fields, which are known as *vertices*. So, we are going to give the Feynman rules for the vertex.

[1]In the path integral formulation, it is necessary to fix the gauge in order to avoid integrating over unphysical degrees of freedom. In that case, the gauge invariance is lost and to re-establishe it one should introduce (nilpotent) BRST symmetry (which stands for Becchi, Rouet, Stora and Tyutin [104, pag 517]), which is still present even after one has fixed the gauge. This is reached by including the 'ghost fields' and their conjugate anti-fields.



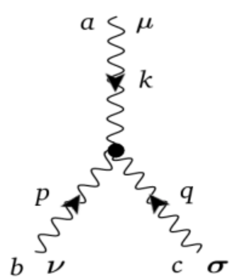
Fermion vertex

$$(\Gamma^\mu)^a = ig\gamma^\mu T^a$$



Ghost vertex

$$\Gamma^\mu_{abc}(q) = gf^{abc}q_\mu$$



Triple vertex

$$\Gamma^{\mu\nu\sigma}_{abc} = igf^{abc}[g_{\mu\nu}(k-p)_\sigma + g_{\nu\sigma}(p-q)_\mu + g_{\sigma\mu}(q-k)_\nu]$$

Figure A.2: Feynman rules for the vertices.

In order to calculate the 1-loop renormalization group equations for the SM gauge couplings, we will consider the (1-loop) corrections to the vertices and propagators.

A.0.1 Correction to the fermionic propagator

The only contribution to the fermionic propagator to 1-loop in perturbation theory is given by the fermion self energy as shown in the diagram in figure A.3

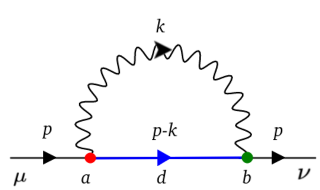


Figure A.3: Fermion self-energy diagram

By taking into account the specific color corresponding to each vertex and propagator (into the loop) we see that get the contribution to the probability amplitude is given by [105]

$$
\begin{aligned}
-i\Sigma^{ab}(p) &= \mu^{\epsilon}\int \frac{dk^4}{4\pi^2} ig\gamma_\mu (T^c)_{ad} \frac{i}{\not p - \not k - m} ig\gamma_\mu (T^c)_{db} \frac{-ig^{\mu\nu}}{k^2} \\
&= \frac{g^2}{8\pi^2\epsilon}(-\not p + 4m)(T^c T^c)_{ab} + \text{finite} \\
&= \frac{g^2}{8\pi^2\epsilon}(-\not p + 4m) C_2(F)\delta^{ab} + \text{finite},
\end{aligned}
\tag{A.1}
$$

where we have defined $\epsilon := 4 - d$, where d means for the space time dimension. We are not going through the details of the dimensional regularization procedure to calculate this divergent integral.

Before we go ahead, note that

$$
\begin{aligned}
S'_F(p)^{-1} &:= S_F(p)^{-1} - \Sigma^{ab}(p) \\
&= \not p - m - \frac{g^2}{8\pi^2\epsilon}(-\not p + 4m) \\
&= \not p \left(1 + \frac{g^2}{8\pi^2\epsilon}\right) - m\left(1 + \frac{g^2}{2\pi^2\epsilon}\right).
\end{aligned}
\tag{A.2}
$$

Hence, two additional diagrams (counter-terms) proportional to $\not p$ and m should be added to the (inverse) fermionic propagator to have a finite quantity. By ignoring finite terms, the 1-loop contributions to the (inverse) fermion propagator with its counter-terms are given by

$$
S'_F(p)^{-1} := S_F(p)^{-1} \quad - \quad \Sigma^{ab}(p) \quad - \quad B^{ab}\not p \quad - \quad A^{ab}, \tag{A.3}
$$

and are represented in figure A.4.

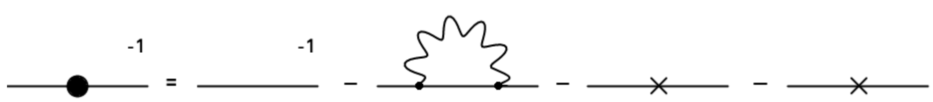


Figure A.4: 1-loop fermion propagator correction and counter-terms.

In figure A.4, the last two diagrams represented by $B^{ab}\not p$ and A^{ab} are the corresponding counter terms, which are given by

$$
\begin{aligned}
B^{ab} &= -\frac{g^2}{8\pi^2\epsilon} C_2(F)\delta^{ab}, \\
A^{ab} &= -\frac{g^2}{2\pi^2\epsilon} m C_2(F)\delta^{ab}.
\end{aligned}
\tag{A.4}
$$

So, for the Lagrangian

$$\mathcal{L} = i\overline{\psi}\partial\!\!\!/\psi - m\overline{\psi}\psi,$$

we should add the following counter-terms

$$\mathcal{L}_{\text{ct}} = iB\overline{\psi}\partial\!\!\!/\psi - A\overline{\psi}\psi.$$

Then, the total bare Lagrangian is given by

$$\mathcal{L} + \mathcal{L}_{\text{ct}} = i(1+B)\overline{\psi}\partial\!\!\!/\psi - (m+A)\overline{\psi}\psi. \tag{A.5}$$

In this way, the bare wave function will be given by $\psi_B = \sqrt{1+B}\psi := \sqrt{Z_2}\psi$, where we have defined

$$Z_2 := 1 + B = 1 - \frac{g^2}{8\pi^2\epsilon}C_2(F). \tag{A.6}$$

A.0.2 Correction to the boson propagator

The contribution to the bosonic propagator are depicted in figure A.5.

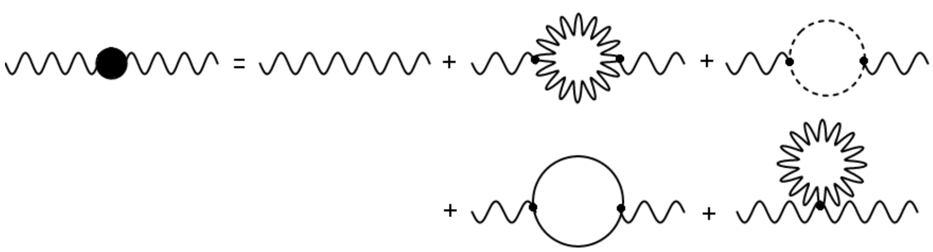


Figure A.5: 1-loop contributions to the boson propagator.

1. We first consider the Boson self-interaction represented in figure A.6.

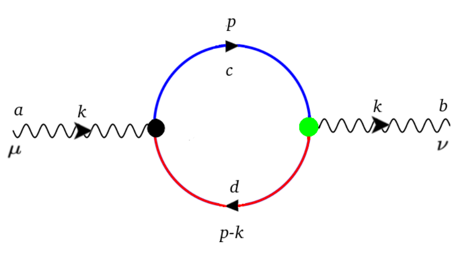


Figure A.6: Boson self-energy diagram.

The contribution to the probability amplitude is given by

$$\begin{aligned}
-i\Pi^{ab}_{\mu\nu}(k)\mid_1 &= \mu^{\epsilon}\mathrm{Tr}\int \frac{dk^4}{4\pi^2}\underbrace{ig\gamma_\mu(T^a)_{dc}}_{\text{vertex}}\frac{i}{\not p - m}\underbrace{ig\gamma_\mu(T^b)_{cd}}_{\text{vertex}}\frac{i}{\not p - \not k - m}\\
&= \frac{g^2}{6\pi^2\epsilon}\left(p_\mu p_\nu - g_{\mu\nu}p^2\right)\mathrm{Tr}\left(T^aT^b\right) + \text{finite}\\
&= \frac{g^2}{12\pi^2\epsilon}\left(p_\mu p_\nu - g_{\mu\nu}p^2\right)n_f\delta^{ab} + \text{finite} \qquad (A.7)
\end{aligned}$$

where n_f is the number of contributing fermions.

2. The next 1-loop correction to the boson propagator we consider is the ghost contribution (figure A.7)

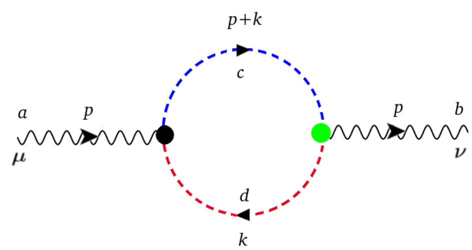


Figure A.7: Ghost contribution

The ghost contribution to the probability amplitude is given by

$$\begin{aligned}
-i\Pi^{ab}_{\mu\nu}(k)\mid_2 &= \mu^{\epsilon}\mathrm{Tr}\int \frac{dk^4}{4\pi^2}\underbrace{gf^{adc}(-p_\mu - k_\mu)}_{\text{vertex}}\frac{-i\delta^{ab}}{k^2}\underbrace{gf^{bcd}(-k_\nu)}_{\text{vertex}}\frac{-i\delta^{ab}}{(p+k)^2}\\
&= \frac{g^2}{16\pi^2\epsilon}\left(\frac{1}{3}p_\mu p_\nu + \frac{1}{6}g_{\mu\nu}p^2\right)f^{acd}f^{bcd} + \text{finite}\\
&= \frac{g^2}{16\pi^2\epsilon}\left(\frac{1}{3}p_\mu p_\nu + \frac{1}{6}g_{\mu\nu}p^2\right)C_2(G)\delta^{ab} + \text{finite} \qquad (A.8)
\end{aligned}$$

where we have used the relation $f^{acd}f^{bcd} = C_2(G)\delta^{ab}$. Here $G_2(G)$ is the eigenvalue of the Quadratic Casimir operator on the regular (adjoint) representation of the group G.

3. The last contribution to the boson propagator is depicted in figure A.8 and it contains two triple vertex

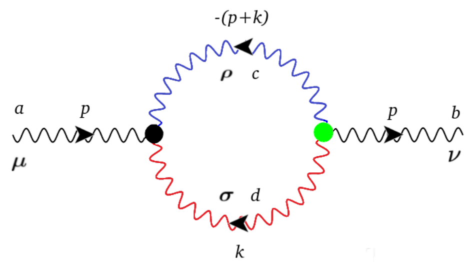


Figure A.8: Double triple vertex correction

Then, the contribution to the amplitude is

$$
\begin{aligned}
-i\Pi^{ab}_{\mu\nu}(k)\mid_3 &= \int \frac{dk^4}{4\pi^2} \underbrace{\Gamma^1}_{\text{vertex}} \frac{-ig^{\mu\nu}}{(p+k)^2} \underbrace{\Gamma^2}_{\text{vertex}} \frac{-ig^{\mu\nu}}{k^2} \\
&= -\frac{g^2}{16\pi^2\epsilon}\left(\frac{11}{3}p_\mu p_\nu - \frac{19}{6}g_{\mu\nu}p^2\right) f^{acd}f^{bcd} + \text{finite} \\
&= -\frac{g^2}{16\pi^2\epsilon}\left(\frac{11}{3}p_\mu p_\nu - \frac{19}{6}g_{\mu\nu}p^2\right) C_2(G)\delta^{ab} + \text{finite}. \qquad \text{(A.9)}
\end{aligned}
$$

Now by adding the three contributions we get

$$
\begin{aligned}
\Pi^{ab}_{\mu\nu}(k) &= \Pi^{ab}_{\mu\nu}(k)\mid_1 + \Pi^{ab}_{\mu\nu}(k)\mid_2 + \Pi^{ab}_{\mu\nu}(k)\mid_3 \\
&= \frac{g^2}{8\pi^2\epsilon}\left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left(\frac{5}{3}C_2(G) - \frac{2}{3}n_f\right)\delta^{ab}. \qquad \text{(A.10)}
\end{aligned}
$$

Hence, to the kinetic gauge Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu},$$

we should add the counter-terms

$$\mathcal{L}_{ct} = -\frac{C}{4}F^{\mu\nu}F_{\mu\nu}.$$

Then, the bare Lagrangian will be given by

$$\mathcal{L}_B = \mathcal{L} + \mathcal{L}_{ct} - \frac{1+C}{4}F^{\mu\nu}F_{\mu\nu}. \qquad \text{(A.11)}$$

In consequence, the bare gauge field is defined like $A^\mu_B = \sqrt{Z_3}A^\mu$, by means of

$$Z_3 := 1 + C = 1 + \frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(G) - \frac{2}{3}n_f\right). \qquad \text{(A.12)}$$

A.0.3 Vertex correction

The 1-loop correction to the vertex is pictured in the diagram (A.9).

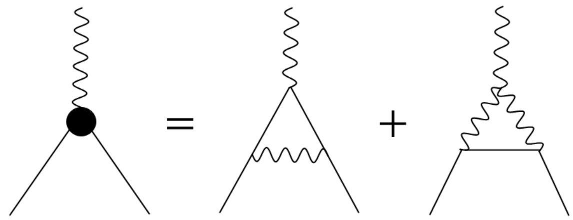


Figure A.9: 1-loop contribution o the vertex

1. The first vertex correction we consider is shown in figure A.10. Here we use the colours (blue, green, red) to describe vertices only.

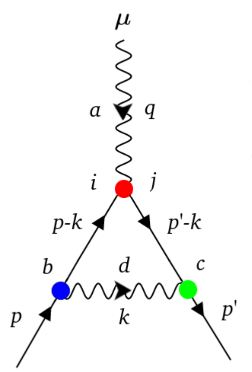


Figure A.10: vertex correction with two fermions and one boson in the 1-loop diagram

The contribution to the amplitude is given by

$$\Lambda^a_\mu(p,q,p')\mid_1 = \mu^{\frac{\epsilon^2}{4}} \int \frac{dk^4}{4\pi^2} \underbrace{-g\gamma_\nu (T^d)_{cj}}_{\text{vertex}} \frac{i}{\not{p}' - \not{k} - m} \underbrace{ig\gamma_\mu (T^a)_{ji}}_{\text{vertex}}$$
$$\times \frac{i}{\not{p} - \not{k} - m} \underbrace{ig\gamma_\rho (T^d)_{ib}}_{\text{vertex}} \frac{-ig^{\rho\nu}}{k^2}$$
$$= \frac{g^2}{8\pi^2\epsilon}\gamma_\mu T^d T^a T^d + \text{finite}$$
$$= \frac{g^2}{8\pi^2\epsilon}\left(-\frac{1}{2}C_2(G) + C_2(F)\right)\gamma_\mu T^a + \text{finite}, \qquad \text{(A.13)}$$

where

$$T^dT^aT^d = \frac{i}{2}f^{adc}[T^c, T^d] + C_2(F)T^a$$
$$= \left(-\frac{1}{2}C_2(G) + C_2(F)\right)T^a.$$

2. Now, we consider the vertex correction shown in figure A.11.

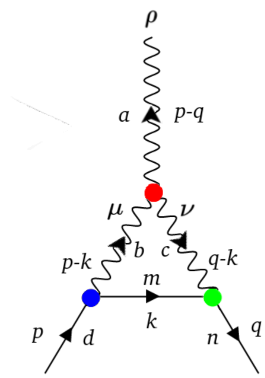


Figure A.11: 1-loop diagram with two boson and one fermion propagators in the

For this case, the contribution to the vertex amplitude is given by

$$\Lambda^a_\rho(k, p, q) \mid_2 = -i\frac{3g^2}{8\pi^2\epsilon}\gamma_\rho f^{abc}T^bT^c + \text{finite}$$
$$= \frac{3g^2}{16\pi^2\epsilon}\gamma_\rho C_2(G)T^a + \text{finite}. \tag{A.14}$$

Next, the two 1-loop contributions to the vertex are

$$\Lambda^a_\rho(k, p, q) = \Lambda^a_\mu(p, p', q) \mid_1 + \Lambda^a_\rho(k, p, q) \mid_2$$
$$= \frac{g^2}{8\pi^2\epsilon}\left(C_2(G) + C_2(F)\right)\gamma_\mu T^a. \tag{A.15}$$

The bare Lagrangian is

$$\mathcal{L} = \mathcal{L} + \mathcal{L}_{ct} = -(1 + D)g\mu^{\frac{\epsilon}{2}}\overline{\psi}\, \not{A}^\mu\psi$$
$$= -Z_1 g\mu^{\frac{\epsilon}{2}}\overline{\psi}\, \not{A}^\mu\psi,$$

where D is the corresponding counter-term, and we have defined

$$Z_1 := 1 + D = 1 - \frac{g^2}{8\pi^2\epsilon}\left(C_2(G) + C_2(F)\right). \tag{A.16}$$

So, in terms of the bare fields we have that

$$\begin{aligned}\mathcal{L}_B &= -Z_1 g \mu^{\frac{\epsilon}{2}} \frac{\overline{\psi}_B}{\sqrt{Z_2}} \frac{\not{A}^\mu_B}{\sqrt{Z_3}} \frac{\psi_B}{\sqrt{Z_2}} \\ &= -\frac{Z_1}{Z_2\sqrt{Z_3}} g\mu^{\frac{\epsilon}{2}} \overline{\psi} \not{A}^\mu \psi,\end{aligned}$$

and so the bare gauge coupling is given by

$$\begin{aligned}g_B &= \mu^{\frac{\epsilon}{2}} Z_1 Z_2^{-1} Z_3^{-\frac{1}{2}} g \\ &\approx \mu^{\frac{\epsilon}{2}} g \left(1 - \frac{g^2}{8\pi^2\epsilon}(C_2(G) + C_2(F))\right)\left(1 + \frac{g^2}{8\pi^2\epsilon}C_2(F)\right) \\ &\quad \times \left(1 - \frac{1}{2}\frac{g^2}{8\pi^2\epsilon}\left(\frac{5}{3}C_2(G) - \frac{2}{3}n_f\right)\right) \\ &= g\mu^{\frac{\epsilon}{2}} \left(1 + \frac{g^2}{8\pi^2\epsilon}\left(-\frac{11}{3}C_2(G) + \frac{2}{3}n_f\right)\right).\end{aligned}$$

Finally, we define the beta functions or the RGEs for the gauge coupling g like the limit when $d \to 4$, i.e, when $\epsilon = 4 - d \to 0$, we have

$$\begin{aligned}\beta_g &:= \lim_{\epsilon \to 0} \mu \frac{\partial g_B}{\partial \mu} \\ &= \lim_{\epsilon \to 0} \left[g \frac{\epsilon}{2} \mu^{\frac{\epsilon}{2}-1} \mu \left(1 + \frac{g^2}{8\pi^2\epsilon}\left(-\frac{11}{3}C_2(G) + \frac{2}{3}n_f\right)\right)\right] \\ &= \frac{g^3}{16\pi^2}\left(-\frac{11}{3}C_2(G) + \frac{2}{3}n_f\right). \qquad \text{(A.17)}\end{aligned}$$

Appendix B

The heat kernel expansion

If $E \to M$ is a vector bundle, then the Laplacian Δ^E of the connection ∇^E on E is a second order differential operator. A generalized Laplacian is a second order differential operator D^2 such that $D^2 = \Delta^E + Q$, for some $Q \in \Gamma(\text{End}(E))$. If D^2 is a generalized Laplacian then we have the following expansion in t, which is known as the heat expansion

$$\text{Tr}(e^{-tD^2}) \sim \sum_{k\geq 0} t^{\frac{k-d}{2}} a_k(D^2), \tag{B.1}$$

where d is the dimension of the manifold, the trace is taken over the Hilbert space $L^2(M, E)$ and the coefficients are in turn given by the Seely-De Witt coefficients $a_k(x, D^2)$

$$a_k(D^2) = \int_M a_k(x, D^2)\sqrt{|g|}d^4x. \tag{B.2}$$

Since the fluctuated Dirac operator of an almost commutative manifold is a generalized Laplacian, so we can compute the spectral action for an almost commutative space time by means of the heat kernel expansion

$$\text{Tr}(e^{-tD_\omega^2}) \sim \sum_{k\geq 0} t^{\frac{k-4}{2}} a_k(D_\omega^2), \tag{B.3}$$

We start by considering a function $g(tD_\omega^2)$ together its Laplace transform

$$g(tD_\omega^2) = \int_0^\infty e^{-stD_\omega^2} h(s)ds. \tag{B.4}$$

So we take the trace and use the heat kernel expansion of D_ω^2 to get

$$\begin{aligned}\text{Tr}(g(tD_\omega^2)) &= \int_0^\infty \text{Tr}(e^{-stD_\omega^2})h(s)ds \approx \int_0^\infty \sum_{k\geq 0}(st)^{\frac{k-4}{2}} a_k(D_\omega^2)h(s)ds \\ &= \sum_{k\geq 0} t^{\frac{k-4}{2}} a_k(D_\omega^2)\int_0^\infty s^{\frac{k-4}{2}} h(s)ds. \end{aligned}\tag{B.5}$$

The parameter t is considered to be very small and so we can drop terms for $k \geq 4$. The Seeley-De Witt coefficients vanish for odd values of k, so for $k = 4$ we have

$$a_4(D_\omega^2)\int_0^\infty s^0 h(s)ds = a_4(D_\omega^2)g(0). \tag{B.6}$$

For $k = 0$ and $k = 2$ we use the definition of the Γ function as

$$\Gamma(z) = \int_0^\infty r^{z-1}e^{-r}dr, \tag{B.7}$$

for $z \in \mathbb{C}$, and by inserting $r = sv$, we see that

$$\Gamma\left(\frac{4-k}{2}\right) = \int_0^\infty (sv)^{\frac{4-k}{2}-1}e^{-sv}d(sv) = s^{\frac{4-k}{2}}\int_0^\infty v^{\frac{4-k}{2}-1}e^{-sv}dv, \tag{B.8}$$

for $k < 4$. Then we have

$$s^{\frac{k-4}{2}} = \frac{1}{\Gamma\left(\frac{4-k}{2}\right)}\int_0^\infty v^{\frac{4-k}{2}-1}e^{-sv}dv. \tag{B.9}$$

So, we get

$$\mathrm{Tr}(g(tD_\omega^2)) \sim a_4(D_\omega^2)f(0) + \sum_{k=0,2,4} t^{\frac{k-4}{2}}a_k(D_\omega^2)\frac{1}{\Gamma\left(\frac{4-k}{2}\right)}\int_0^\infty v^{\frac{4-k}{2}-1}g(v)dv + O(\Lambda^{-1}). \tag{B.10}$$

Next, if we choose $g(u^2) = f(u)$, and rewrite the integration over v by substituting $v = u^2$, then we get

$$\int_0^\infty v^{\frac{4-k}{2}-1}g(v)dv = \int_0^\infty u^{\frac{4-k}{2}-1}g(u^2)d(u^2) = 2\int_0^\infty u^{4-k-1}f(u)du := 2f_{4-k}. \tag{B.11}$$

Let us now to write $t = \Lambda^{-2}$, so we obtain

$$\begin{aligned}\mathrm{Tr}\left(f\left(\frac{D_\omega}{\Lambda}\right)\right) &= \mathrm{Tr}(g(\Lambda^{-2}D_\omega^2)) \\ &\sim a_4(D_\omega^2)f(0) + 2\sum_{0\leq k<4} f_{4-k}\Lambda^{4-k}a_k(D_\omega^2)\frac{1}{\Gamma(\frac{4-k}{2})} \\ &= a_4(D_\omega^2)f(0) + 2f_4\Lambda^4 a_0(D_\omega^2) + 2f_2\Lambda^2 a_2(D_\omega^2)\end{aligned} \tag{B.12}$$

Appendix C

The Neutrinophilic model

Here, we present the 2HDMs with right-handed neutrinos (discussed in [106]) that suppress FCNC at tree-level. With we are able to get a kind of Type-I model called neutrinophilic Higgs doublet model [87]. We remember our parity choice for the two Higgs doublets and the type up quark: $\Phi_1 \to -\Phi_1$, $\Phi_2 \to +\Phi_2$, and $u_R \to +u_R$.

C.1 Neutrinophilic Higgs doublet model

The basis for this model is the Type-I 2HDM where the not-SM Higgs doublet Φ_1 only couples to neutrinos[1]. The parity assignment for the remaining fermions is $e_R \to +e_R$, $d_R \to +d_R$, and $\nu_R \to -\nu_R$. Then, the kinetic terms are given by

$$\frac{|y_\nu|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{|y_e|^2+3|y_u|^2+3|y_d|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma. \tag{C.1}$$

Then, we should take the following normalization

$$\Theta_1 \to \frac{2g}{\sqrt{|y_\nu|^2}}\Phi_1, \quad \Theta_2 \to \frac{2g}{\sqrt{|y_e|^2+3|y_u|^2+3|y_d|^2}}\Phi_2, \quad \text{and} \quad \Sigma \to \frac{2\sqrt{2}g}{|y_R|}\sigma. \tag{C.2}$$

Now, the quadratic terms are given by

$$\begin{aligned} -\frac{f_2\Lambda^2}{2\pi^2}\mathrm{Tr}(\Phi^2) &= -\frac{2f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2}{2g^2}(\Theta_1^\dagger\Theta_1) + \frac{|y_e|^2+3|y_d|^2+3|y_u|^2}{2g^2}(\Theta_2^\dagger\Theta_2) + \sigma^*\sigma\right) \\ &= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_1^\dagger\Phi_1 + \Phi_2^\dagger\Phi_2 + \sigma^*\sigma\right). \end{aligned} \tag{C.3}$$

[1]Note that the model contemplated here is different from the one studied in [107], benchmark scenario 2.

Furthermore, the mixed-quartic terms are

$$
\begin{aligned}
(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d) &= \begin{pmatrix}\varphi_2^- & \varphi_2^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^0 & -\varphi_2^+\end{pmatrix}\begin{pmatrix}\varphi_2^+\\ \varphi_2^0\end{pmatrix}\\
&= \left(\varphi_2^-\varphi_2^{0*} - \varphi_2^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_2^+ - \varphi_2^+\varphi_2^0\right)\\
&= \varphi_2^{0*}\varphi_2^-\varphi_2^+\varphi_2^0 + \varphi_2^-\varphi_2^{0*}\varphi_2^0\varphi_2^+ - \varphi_2^-\varphi_2^{0*}\varphi_2^+\varphi_2^0 - \varphi_2^{0*}\varphi_2^-\varphi_2^0\varphi_2^+\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) &= \begin{pmatrix}\varphi_1^0 & -\varphi_1^+\end{pmatrix}\begin{pmatrix}\varphi_2^+\\ \varphi_2^0\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^- & \varphi_2^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}\\
&= \left(\varphi_1^-\varphi_2^{0*} - \varphi_1^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_1^+ - \varphi_2^+\varphi_1^0\right)\\
&= \varphi_1^-\varphi_2^{0*}\varphi_2^0\varphi_1^+ - \varphi_1^-\varphi_2^{0*}\varphi_2^+\varphi_1^0 - \varphi_1^{0*}\varphi_2^-\varphi_2^0\varphi_1^+ + \varphi_1^{0*}\varphi_2^-\varphi_2^+\varphi_1^0\\
&= (\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1).
\end{aligned}
$$

Then, the quartic terms in the potential are given by

$$
\begin{aligned}
\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) \supseteq{}& \frac{|y_\nu|^4}{4g^2}(\Theta_1^\dagger\Theta_1)^2 + \frac{(|y_e|^4+3|y_d|^4+3|y_u|^4)}{4g^2}(\Theta_2^\dagger\Theta_2)^2 + 8g^2(\sigma^*\sigma)^2\\
&+ 4|y_\nu|^2(\sigma^*\sigma)(\Theta_1^\dagger\Theta_1) + \frac{|y_e|^2|y_\nu|^2}{2g^2}\Big((\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1)\Big)\\
={}& 4g^2(\Phi_1^\dagger\Phi_1)^2 + \frac{4(|y_e|^4+3|y_d|^4+3|y_u|^4)g^2}{(y_e|^2+3|y_d|^2+3|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2 + 16g^2(\sigma^*\sigma)(\Phi_1^\dagger\Phi_1)\\
&+ \frac{8|y_e|^2g^2}{|y_e|^2+3|y_u|^2+3|y_d|^2}\Big((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2) - (\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\Big). \qquad \text{(C.4)}
\end{aligned}
$$

Then, we redefine the coefficients as follows

$$
\begin{aligned}
&\mu_1^2 = \mu_2^2 = \mu_S^2 = -4\frac{f_2\Lambda^2}{f_0}, \qquad \frac{\lambda_1}{2} = 4g^2, \qquad \frac{\lambda_2}{2} \approx \frac{4}{3}g^2,\\
&\lambda_3 = -\lambda_4 \approx 0, \quad \lambda_5 = \lambda_{S2} = 0, \quad \lambda_{S1} \approx 16g^2, \quad \frac{\lambda_S}{2} = 8g^2. \qquad \text{(C.5)}
\end{aligned}
$$

Finally, the Yukawa interaction is given by

$$
\begin{aligned}
-\frac{1}{2}\langle J\Psi, (\gamma_5\otimes\Phi)\Psi\rangle ={}& \langle J_M\overline{\boldsymbol{e}_R}, y_e((\phi_e)_2^*\boldsymbol{e}_L + (\phi_e)_1^*\boldsymbol{\nu}_L)\rangle + \langle J_M\overline{\boldsymbol{\nu}_R}, y_\nu((\phi_\nu)_1^*\boldsymbol{\nu}_L + (\phi_\nu)_2^*\boldsymbol{e}_L)\rangle\\
&+ \langle J_M\overline{\boldsymbol{u}_R}, y_u((\phi_u)_1^*\boldsymbol{u}_L + (\phi_u)_2^*\boldsymbol{d}_L)\rangle + \langle J_M\overline{\boldsymbol{d}_R}, y_d((\phi_d)_1^*\boldsymbol{u}_L + (\phi_d)_2^*\boldsymbol{d}_L)\rangle\\
&+ \frac{1}{2}\langle J_M\overline{\boldsymbol{\nu}_R}, y_R^*\Sigma\overline{\boldsymbol{\nu}_R}\rangle + \text{h.c.} \qquad \text{(C.6)}
\end{aligned}
$$

Hence, by redefining the yukawa couplings as follows

$$
\begin{aligned}
y_e &:= -i\sqrt{\frac{|y_e|^2+3|y_d|^2+3|y_u|^2}{2}}\frac{m_e}{gv_2},\\
y_u &:= -i\sqrt{\frac{|y_e|^2+3|y_d|^2+3|y_u|^2}{2}}\frac{m_t}{gv_2},\\
y_d &:= -i\sqrt{\frac{|y_e|^2+3|y_d|^2+3|y_u|^2}{2}}\frac{m_d}{gv_2},\\
y_\nu &:= -i\sqrt{\frac{|y_\nu|^2}{2}}\frac{m_\nu}{gv_1},\\
y_R &:= -i\frac{|y_R|}{2}\frac{m_R}{gv_S},
\end{aligned}
\tag{C.7}
$$

we get

$$
\begin{aligned}
-\frac{1}{2}\langle J\Psi,(\gamma_5\otimes\Phi)\Psi\rangle &= \langle J_M\overline{\boldsymbol{e}_R}, y_e(\phi_2^0\boldsymbol{e}_L+\phi_2^-\boldsymbol{\nu}_L)\rangle + \langle J_M\overline{\boldsymbol{\nu}_R}, y_\nu(\phi_1^0\boldsymbol{\nu}_L+\phi_1^+\boldsymbol{e}_L)\rangle\\
&+ \langle J_M\overline{\boldsymbol{u}_R}, y_u(\phi_2^0\boldsymbol{u}_L+\phi_2^+\boldsymbol{d}_L)\rangle + \langle J_M\overline{\boldsymbol{d}_R}, y_d(\phi_2^-\boldsymbol{u}_L+\phi_2^0\boldsymbol{d}_L)\rangle\\
&+ \frac{1}{2}\langle J_M\overline{\boldsymbol{\nu}_R}, y_R^*\sigma\overline{\boldsymbol{\nu}_R}\rangle + \text{h.c.}
\end{aligned}
\tag{C.8}
$$

C.2 Lepton-Specific+$\sigma + \nu_R$

In this case, $e_R \to -e_R$, $d_R \to +d_R$, and $\nu_R \to -\nu_R$. Hence, we can keep the terms on Θ_2, whereas the terms for Θ_1 in Eq. Θ_1 (3.24) should be replaced as follows

$$
\frac{|y_\nu|^2+|y_e|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{3(|y_d|^2+|y_u|^2)}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma,
\tag{C.9}
$$

so, we should change our normalization in Eq. (3.25) for Θ_1 to get

$$
\Theta_1 \to \frac{2g}{\sqrt{|y_\nu|^2+|y_e|^2}}\Phi_1,\quad \Theta_2 \to \frac{2g}{\sqrt{3(|y_d|^2+|y_u|^2)}}\Phi_2,\quad \text{and}\quad \Sigma \to \frac{2\sqrt{2}g}{|y_R|}\sigma.
\tag{C.10}
$$

Now, the quadratic terms are given by

$$
\begin{aligned}
-\frac{f_2\Lambda^2}{f_0}\mathrm{Tr}(\Phi^2) &= -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2+|y_e|^2}{4g^2}\Theta_1^\dagger\Theta_1 + \Phi_2^\dagger\Phi_2 + \sigma^*\sigma\right)\\
&= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_1^\dagger\Phi_1 + \Phi_2^\dagger\Phi_2 + \sigma^*\sigma\right).
\end{aligned}
\tag{C.11}
$$

The mixed-quartic terms are zero in this case

$$
(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) = 0,
$$

and so, the quartic potential is given by

$$\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) = \frac{|y_\nu|^4+|y_e|^4}{4g^2}(\Theta_1^\dagger\Theta_1)^2 + \frac{4\cdot 3(|y_d|^4+|y_u|^4)g^2}{9(|y_d|^2+|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2$$
$$+ 4|y_\nu|^2(\sigma^*\sigma)(\Theta_\nu^\dagger\Theta_\nu) + 8g^2(\sigma^*\sigma)^2$$
$$= \frac{4(|y_\nu|^4+|y_e|^4)g^2}{(|y_\nu|^2+|y_e|^2)^2}(\Phi_1^\dagger\Phi_1)^2 + \frac{4(|y_d|^4+|y_u|^4)g^2}{3(|y_d|^2+|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2$$
$$+ \frac{16|y_\nu|^2g^2}{|y_\nu|^2+|y_e|^2}(\sigma^*\sigma)(\Phi_1^\dagger\Phi_1) + 8g^2(\sigma^*\sigma)^2. \tag{C.12}$$

Then, we make the following redefinition

$$\mu_1^2=\mu_2^2=\mu_S^2=-4\frac{f_2\Lambda^2}{f_0}, \qquad \frac{\lambda_1}{2}\approx 4g^2, \qquad \frac{\lambda_2}{2}\approx\frac{4}{3}g^2,$$
$$\lambda_3=\lambda_4=\lambda_5=0 \quad \lambda_{S1}\approx 16g^2, \quad \lambda_{S2}=0, \quad \frac{\lambda_S}{2}=8g^2. \tag{C.13}$$

C.3 Flipped

This model is characterized for the following parity assignment: $e_R \to +e_R$, $d_R \to -d_R$, and $\nu_R \to -\nu_R$. Hence, we can keep the terms on Θ_2 whereas the terms for Θ_1 in Eq. Θ_1 (3.31) should be replaced as follows

$$\frac{|y_\nu|^2+3|y_d|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{|y_e|^2+3|y_u|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma, \tag{C.14}$$

so, we should change our normalization in Eq. (3.25) for Θ_1 to obtain

$$\Theta_1 \to \frac{2g}{\sqrt{|y_\nu|^2+3|y_d|^2}}\Phi_1, \quad \Theta_2 \to \frac{2g}{\sqrt{|y_e|^2+3|y_u|^2}}\Phi_2, \quad \text{and} \quad \Sigma \to \frac{2\sqrt{2}g}{|y_R|}\sigma. \tag{C.15}$$

Hence, the quadratic potential terms are given by

$$-\frac{f_2\Lambda^2}{f_0}\mathrm{Tr}(\Phi^2) = -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2+3|y_d|^2}{4g^2}\Theta_1^\dagger\Theta_1 + \Phi_2^\dagger\Phi_2 + \sigma^*\sigma\right)$$
$$= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_1^\dagger\Phi_1 + \Phi_2^\dagger\Phi_2 + \sigma^*\sigma\right). \tag{C.16}$$

The mixed-quartic terms are

$$(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu) = \begin{pmatrix}\varphi_1^0 & -\varphi_1^+\end{pmatrix}\begin{pmatrix}\varphi_2^+\\ \varphi_2^0\end{pmatrix}\cdot\begin{pmatrix}\varphi_2^- & \varphi_2^{0*}\end{pmatrix}\begin{pmatrix}\varphi_2^{0*}\\ -\varphi_2^-\end{pmatrix}$$
$$= \left(\varphi_1^-\varphi_2^{0*} - \varphi_1^{0*}\varphi_2^-\right)\left(\varphi_2^0\varphi_1^+ - \varphi_2^+\varphi_1^0\right)$$
$$= \varphi_1^-\varphi_2^{0*}\varphi_2^0\varphi_1^+ - \varphi_1^-\varphi_2^{0*}\varphi_2^+\varphi_1^0 - \varphi_1^{0*}\varphi_2^-\varphi_2^0\varphi_1^+ + \varphi_1^{0*}\varphi_2^-\varphi_2^+\varphi_1^0$$
$$= (\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2) - (\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1),$$

so the quartic potential is given by

$$\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) = \frac{|y_\nu|^4+3|y_d|^4}{4g^2}(\Theta_1^\dagger\Theta_1)^2 + \frac{4\cdot(|y_e|^4+3|y_u|^4)g^2}{(|y_e|^2+3|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2$$
$$+ 4|y_\nu|^2(\sigma^*\sigma)(\Theta_1^\dagger\Theta_1) + \frac{1}{2g^2}\left(|y_e|^2|y_\nu|^2+3|y_d|^2|y_u|^2\right)\Big((\Theta_1^\dagger\Theta_1)(\Theta_2^\dagger\Theta_2)-(\Theta_1^\dagger\Theta_2)(\Theta_2^\dagger\Theta_1)\Big)$$
$$= \frac{4(|y_\nu|^4+3|y_d|^4)g^2}{(|y_\nu|^2+3|y_d|^2)^2}(\Phi_1^\dagger\Phi_1)^2 + \frac{4(|y_e|^4+3|y_u|^4)g^2}{(|y_e|^2+3|y_u|^2)^2}(\Phi_2^\dagger\Phi_2)^2 + \frac{16|y_\nu|^2g^2}{|y_\nu|^2+3|y_d|^2}(\sigma^*\sigma)(\Phi_1^\dagger\Phi_1)$$
$$+ \frac{8(|y_e|^2|y_\nu|^2+3|y_d|^2|y_u|^2)g^2}{(3|y_d|^2+|y_\nu|^2)(|y_e|^2+3|y_u|^2)}\Big((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)-(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\Big). \qquad \text{(C.17)}$$

Thus, we make the following coefficients redefinition

$$\mu_1^2=\mu_2^2=\mu_S^2=-4\frac{f_2\Lambda^2}{f_0}, \quad \frac{\lambda_1}{2}\approx 4g^2, \quad \frac{\lambda_2}{2}\approx\frac{4}{3}g^2, \quad \lambda_3=-\lambda_4\approx\lambda_{S2}=\lambda_5=0, \quad \lambda_{S1}\approx 16g^2, \quad \frac{\lambda_S}{2}=8g^2. \qquad \text{(C.18)}$$

C.4 Type-II’

In this case we have $\Theta_e=\Theta_d=\widetilde{\Theta_\nu}=\Theta_1$, which is equivalent to take $e_R\to -e_R$, $d_R\to -d_R$, and $\nu_R\to -\nu_R$. So we will keep the terms on Θ_2 whereas the terms for Θ_1 in Eq. Θ_1 (3.31) should be replaced as follows

$$\frac{|y_\nu|^2+|y_e|^2+3|y_d|^2}{4g^2}(D_\mu\Theta_1)^\dagger(D_\mu\Theta_1) + \frac{3|y_u|^2}{4g^2}(D_\mu\Theta_2)^\dagger(D_\mu\Theta_2) + \frac{|y_R|^2}{8g^2}\partial_\mu\Sigma^*\partial_\mu\Sigma, \qquad \text{(C.19)}$$

so, we should change our normalization in Eq. (3.25) for Θ_1 as follows

$$\Theta_1\to\frac{2g}{\sqrt{|y_\nu|^2+|y_e|^2+3|y_d|^2}}\Phi_1, \quad \Theta_2\to\frac{2g}{\sqrt{3}|y_u|}\Phi_2, \quad \text{and} \quad \Sigma\to\frac{2\sqrt{2}g}{|y_R|}\sigma. \qquad \text{(C.20)}$$

Now, the quadratic terms on Θ_1 are given by

$$-\frac{f_2\Lambda^2}{f_0}\mathrm{Tr}(\Phi^2) = -\frac{4f_2\Lambda^2}{f_0}\left(\frac{|y_\nu|^2+|y_e|^2+3|y_d|^2}{4g^2}\Theta_1^\dagger\Theta_1+\Phi_2^\dagger\Phi_2+\sigma^*\sigma\right)$$
$$= -\frac{4f_2\Lambda^2}{f_0}\left(\Phi_1^\dagger\Phi_1+\Phi_2^\dagger\Phi_2+\sigma^*\sigma\right). \qquad \text{(C.21)}$$

The mixed-quartic term is

$$(\Theta_\nu^\dagger\Theta_e)(\Theta_e{}^\dagger\Theta_\nu)=0,$$

so the quartic potential terms are given by

$$\begin{aligned}\frac{f_0}{8\pi^2}\mathrm{Tr}(\Phi^4) &= \frac{|y_\nu|^4+|y_e|^4+3|y_d|^4}{4g^2}(\Theta_1^\dagger\Theta_1)^2+\frac{4g^2}{3}(\Phi_2^\dagger\Phi_2)^2+8g^2(\sigma^*\sigma)^2\\ &+4|y_\nu|^2(\sigma^*\sigma)(\Theta_1^\dagger\Theta_1)+\frac{3}{2g^2}|y_u|^2|y_d|^2(\Theta_d^\dagger\Theta_u)(\Theta_u^\dagger\Theta_d)\\ &=\frac{4(|y_\nu|^4+|y_e|^4+3|y_d|^4)g^2}{(|y_\nu|^2+|y_e|^2+3|y_d|^2)^2}(\Phi_1^\dagger\Phi_1)^2+\frac{16|y_\nu|^2g^2}{|y_\nu|^2+|y_e|^2+3|y_d|^2}(\sigma^*\sigma)(\Phi_1^\dagger\Phi_1)\\ &+\frac{8g^2|y_d|^2}{|y_\nu|^2+|y_e|^2+3|y_d|^2}\Big((\Phi_1^\dagger\Phi_1)(\Phi_2^\dagger\Phi_2)-(\Phi_1^\dagger\Phi_2)(\Phi_2^\dagger\Phi_1)\Big)+\frac{4g^2}{3}(\Phi_2^\dagger\Phi_2)^2+8g^2(\sigma^*\sigma)^2.\end{aligned} \tag{C.22}$$

Hence, we redefine as follows

$$\begin{aligned}&\mu_1^2=\mu_2^2=\mu_S^2=-4\frac{f_2\Lambda^2}{f_0}, \qquad \frac{\lambda_1}{2}\approx 4g^2, \qquad \frac{\lambda_2}{2}=\frac{4}{3}g^2,\\ &\lambda_3=-\lambda_4\approx\lambda_5=\lambda_{S2}=0 \quad \lambda_{S1}\approx 16g^2, \quad \frac{\lambda_S}{2}=8g^2.\end{aligned} \tag{C.23}$$

C.5 Mass spectrum

Let us consider the 2HDM potential in Eq.(1.40) together with the singlet scalar potential in Eq. (3.91). Then, for the minimum conditions given by Eq. (3.92), we get the charged, CP-even, and CP-odd masses as given by Eqs. (1.50), (3.95), and (3.100), respectively.

Next, by comparing the boundary conditions in Eqs. (C.5), (C.13), (C.18), and (C.23), we realize that all of them are exactly the same:

$$\mu_1^2=\mu_2^2=\mu_S^2=-4\frac{f_2\Lambda^2}{f_0}, \quad \frac{\lambda_1}{2}=4g^2, \quad \frac{\lambda_2}{2}=\frac{4}{3}g^2, \quad \lambda_{S1}=16g^2, \quad \frac{\lambda_S}{2}=8g^2, \quad \lambda_3=\lambda_4=\lambda_5=\lambda_{S2}=0. \tag{C.24}$$

Note that no one of these conditions depends on R_u^ν. So, after running to low energies the set of RGEs in Eqs. (3.101), we get the mass spectrum depicted in table C.1

	Neutrinophilic
m_t	173.5
m_h	174
m_H	$129i$
m_s	4.2×10^5
$m_{H^\pm}$	37.1
m_A	0

Table C.1: The mass spectrum for the four models considered in this appendix are identical and do not depend on R_u^ν. We have called this simply as the Neutrinophilic model. We have selected an unification scale of $\sim 10^{16}$ GeV, $\tan\beta\approx 2.14$, $g=0.5$, and $v_S=10^6$ GeV. The singlet scalar field possess an imaginary mass, which corresponds to a tachyon field. The values are given in GeV

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