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Relation between SU(N) and SO(N) groups and the Pascal triangle

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ABSTRACT. In this article we describe an approach to obtain the dimensions and anomalies of SU(N) and SO(N) representations from the Pascal triangle.

RESUMEN. En este artículo se describe un procedimiento para obtener las dimensiones y las anomalías de las representaciones de los grupos SU(N) y SO(N) a partir del triángulo de Pascal.

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With the production of the t-quark recently at Fermilab, CDF and D∅, the Standard Model (SM) is the most attractive theory that unifies partially the electrodynamics and weak forces. For the last two decades, physicists have been trying to unify the non-gravitational forces with only one constant by using simple groups like SU(N), SO(N), E_6, etcetera. However, they face problems with the proton decay and the unification scale. Some new ideas are becoming like supersymmetry, which deals with the unification models and proton decay consistently.

On the other hand the SM can not explain the hierarchical mass problem but some new ideas have been worked out to find a solution, being horizontal gauge symmetry a clear example. Some models that include it are E_8, $SU(6)^3 \times Z_3$, $SU(6) \times U(1)$. To build these models it is necessary to know the dimensions and anomalies of the irreps.

In the Refs. 1–12 it is shown explicitly the dimensions of totally symmetric and antisymmetric representations by means of binomial coefficients, but they do not establish the relation with the Pascal triangle, maybe because it is straightforward. However this correspondence will become a very useful tool from a didactic point of view, because it should help the reader to calculate these dimensions immediately.

The aim of this article is to show how the reader can find in the Pascal triangle not only the dimensions of completely symmetric and antisymmetric representations of SU(N), but also its anomalies. Moreover we can find the dimension of the adjoint representation of SU(N) and the dimension of the spinorial representation of SO(N).

For reaching this purpose we shall write the Pascal triangle as a matrix. Then we shall show that with the elements of it we can get the dimensions and the anomalies of the representations mentioned above.

It is well known that the Pascal triangle is an arrangement of numbers, where the n-th row is conformed by the binomial coefficients of $(x+y)^n$. We can construct the Pascal triangle with the following rule for some three numbers:

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ .. & a & \rightarrow & b & .. \\ & \vdots & & \downarrow & \\ & & & a+b & .. \\ & & & \vdots & \end{array} \tag{1}$$

This rule allows us to write the Pascal triangle as the following matrix:

$$T = \begin{pmatrix} 1 & & \cdots & \cdots & 0 & \cdots \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & \vdots & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & & & & & \ddots \end{pmatrix}, \tag{2}$$

where the element in the n-th row and m-th column is given by

$$T_{nm} = \binom{n}{m} \equiv \frac{n!}{m!\,(n-m)!}. \tag{3}$$

We have studied two kinds of representations of SU(N). One of them is composed by one row with $m+1$ boxes and p ($\leq N-2$) rows with one box each one (see Fig. 1). This is the $(m+1, 1, \ldots, 1, 0, \ldots, 0)$ representation. The other one is composed by $p+1$ rows with $m+1$ boxes each one (see Fig. 2). This is the $(m+1, \ldots, m+1, 0, \ldots, 0)$ representation. We have dealt with these representations because we can get the totally symmetric and antisymmetric representations from them. If we take $m=0$ we obtain the totally antisymmetric representation. On the other hand, with $p=0$ the totally symmetric representation is obtained. Moreover, the adjoint representation of SU(N) is obtained from the Young tableau shown in Fig. 1, with $p=N-2$ and $m=1$.

For calculating the dimension of some representation of SU(N), we have gotten the dimension of the $(m+1, 1, \ldots, 1, 0, \ldots, 0)$ and $(m+1, \ldots, m+1, 0, \ldots, 0)$ SU(N) representations by means of binomial coefficients, *i.e.*, in function of the elements T_{nm} and obtained the same results as those in Refs. 1–12. The results are

$$\begin{aligned} \mathrm{Dim}_N(m+1, 1, \ldots, 1, 0, \ldots, 0) &\equiv \binom{N+m}{m+p+1}\binom{m+p}{m} \\ &= T_{N+m,m+p+1}\, T_{m+p,m} \end{aligned} \tag{4}$$

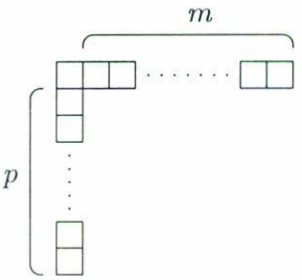

FIGURE 1. Young tableau for the $(m+1, 1, \ldots, 1, 0, \ldots, 0)$ representation of SU(N).

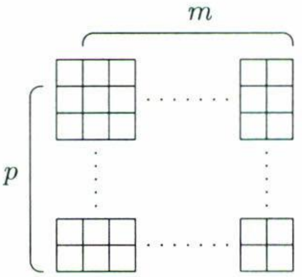

FIGURE 2. Young tableau for the $(m+1, \ldots, m+1, 0, \ldots, 0)$ representation of SU(N).

and

$$\mathrm{Dim}_N(m+1, \ldots, m+1, 0, \ldots, 0) = \frac{\prod_{j=0}^{p} \binom{N+m-j}{m+1}}{\prod_{j=0}^{p} \binom{m+j+1}{m+1}} = \frac{\prod_{j=0}^{p} T_{N+m-j,m+1}}{\prod_{j=0}^{p} T_{m+j+1,m+1}}. \tag{5}$$

From Eqs. (4) or (5) we obtain:

1. $m = 0$ leads to the totally antisymmetric SU(N) representations. They are easily obtained from Eq. (4) instead of Eq. (5), which implies a rotation by 45 degrees. Their dimensions are $T_{N,p+1}$, given by the elements in the N-th row of the matrix T [see Eq. (2)], except the elements $T_{N,0}$ and $T_{N,N}$.
2. $p = 0$ yields to the totally symmetric SU(N) representations. Their dimensions are $T_{N+m,m+1}$, which corresponds to the diagonal elements $T_{N,1}$, $T_{N+1,2}$, ... Note that $T_{n,m} = T_{n,n-m}$; in consecuence, $T_{N+m,m+1} = T_{N+m,N-1}$. In other words, the dimension is given by the elements in the $(N-1)$-th column of the T-matrix.

On the other hand, from Eq. (4), the dimension of the adjoint SU(N) representation (with $m = 1$ and $p = N-2$) is $T_{N+1,N} \times T_{N-1,1} = N^2-1$. Or equivalently, $T_{N+1,1} \times T_{N-1,1}$. Thus the dimension is calculated only with the help of the second column of the T.

By the way, from Eqs. (14) and (16) in Ref. 3, we have re-written those expressions for anomalies of completely symmetric and antisymmetric representations of SU(N) groups, respectively, by means of the binomial coefficients. The results are

$$A_s = \binom{N+m+1}{N+2} + \binom{N+m}{N+2} = T_{N+m+1,N+2} + T_{N+m,N+2} \tag{6}$$

and

$$A_a = \binom{N-3}{p-1} + \binom{N-3}{p-2} = T_{N-3,p-1} + T_{N-3,p-2}, \tag{7}$$

where A_s and A_a are the anomalies for the totally symmetric and antisymmetric SU(N) representations, respectively. Here m is the completely symmetric product of m fundamental N-dimensional representations and p the completely antisymmetric combination of p fundamentals. Thus we only need the $(N+2)$-th column of the T-matrix if we want to compute the anomalies for the totally symmetric representations of SU(N), and only the $(N-3)$-th row of the T-matrix for the completely antisymmetric representations.

In Ref. 12 the authors point out that the anomalies have a similar behavior as the Pascal triangle. We have proved this with Eqs. (6) and (7).

On the other hand we are going to mention some issues about SO(N) groups. It is well known that the minimum dimension of the hermitian Γ_i matrices, $i = 1, 2, \ldots, N$, which satisfy the Clifford algebra of rank N,

$$\{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \tag{8}$$

is $2^n \times 2^n$ with $N = 2n(2n+1)$ for N even(odd). The number of independent matrices that arises from of the product of m different matrices Γ_i is given by $T_{2n,m}$ [5]. Moreover, $\sum_m T_{2n,m} = 2^{2n}$ gives the total number of independent hermitian matrices of dimension $2^n \times 2^n$ whose squares are the identity matrix.

The matrices $\Gamma_{i_1 \ldots i_m}$, where

$$\Gamma_{i_1 \ldots i_m} = i^{\frac{1}{2}m(m-1)} \prod_{j=1}^{m} \Gamma_{i_j} \tag{9}$$

form a basis for the representation of dimension $T_{2n,m}$ of SO($2n$) group.

Finally, we would like to summarize our results. From the T-matrix (or Pascal triangle) given by Eq. (2), we can obtain the following: (i) the N-th row and the $(N-1)$-th column give the dimensions of the totally antisymmetric and symmetric representations of SU(N), respectively; (ii) by means of the second column the reader can compute the dimension of the adjoint representation of SU(N); and (iii) from the $(N+2)$-th column and $(N-3)$-th row the reader can obtain the anomalies for the totally symmetric and antisymmetric representations of SU(N), respectively.

We would like to stress that the tool presented in this work is very useful for N *small* because the T-matrix is constructed very quickly and easily.

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REFERENCES

1. A. Pais, *Rev. Mod. Phys.* **38** (1966) 215.
2. C. Itzykson, *Rev. Mod. Phys.* **38** (1966) 95.
3. J. Banks and H. Georgi, *Phys. Rev. D* **14** (1976) 1159.
4. R. Slansky, *Phys. Rep.* **79** (1981) 1.
5. H. Georgi, *Lie Algebras in Particle Physics*, (The Benjamin Company, Reading, Mass., 1982).
6. D.B. Lichtenberg, *Unitary Symmetry and Elementary Particles*, (Academic Press, New York, 1978).
7. J.Q. Chen, *Group Representation Theory for Physicists*, (World Scientific, Singapore, 1989).
8. W. Greiner and B. Müller, *Quantum Mechanics*, (Springer-Verlag, New York, 1989).
9. M. Hamermesh, *Group Theory and its Application to Physical Problems*, (Addison-Wesley, Reading, Mass., 1962).
10. W.K. Tung, *Group Theory in Physics*, (World Scientific, Singapore, 1984).
11. J.J. Sakurai, *Modern Quantum Mechanics*, (The Benjamin Company, Reading, Mass., 1985).
12. A. Fernández and R. Martínez, *Rev. Mex. Fís.* **35** (1989) 379.