The lorentz group, a galilean approach

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We present a pedagogical approach to the Lorentz group. We start by introducing a compact notation to express the elements of the fundamental representation of the rotations group. Lorentz coordinate transformations are derived in a novel and compact form. We show how to make a Lorentz transformation on the electromagnetic fields as well. A covariant time-derivative is introduced in order to deal with non-inertial systems. Examples of the usefulness of these results such as the rotating system and the Thomas precession, are also presented.

Keywords: Special relativity; Lorentz transformations.

En este trabajo se presenta una aproximación pedagógica al grupo de Lorentz. Se comienza introduciendo una notación compacta para expresar los elementos de la representación fundamental del grupo de rotaciones. Las transformaciones de Lorentz de las coordenadas se derivan de una manera compacta. Se muestra también cómo realizar las transformaciones de Lorentz sobre los campos electromagnéticos. Se introduce una derivada temporal covariante para tratar con sistemas no inerciales, para mostrar la utilidad de este método se presentan también ejemplos tales como el sistema rotante y la precesión de Thomas.

Descriptores: Relatividad especial; transformaciones de Lorentz.

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1. Introduction

Special relativity was first introduced nearly a century ago in order to explain the massive experimental evidence against ether as the medium for propagating electromagnetic waves. As a consequence of special relativity an unexpected space-time structure was discovered. The pure Lorentz transformations called boosts relate the changes of the space distances and time intervals when they are measured from two different inertial frames. Rotations and boost transformations form the general Lorentz group (The properties of the Lorentz group can be found in other references such as [1–4]).

We show how one can understand boost transformations, which follow from the postulates of special relativity, as corresponding to deformations of the classical Galilean transformations. Also we introduce a covariant temporal derivative to deal with non-inertial systems. This article is arranged as follows. In Sec. 2 we show a simple way to generate and write the matrices associated with the rotation of three dimensional vectors and present some applications of our notation. In Sec. 3 we find the matrices of the boost transformations starting from Galileo's only by imposing the constance of the velocity of light. Finally, in Sec. 4 we show how the electromagnetic fields transform under general Lorentz transformations in the same fashion we introduced before. An appendix deal with non-inertial system.

2. Rotations

2.1. Rotations of the co ordinate frame

Under rotations the Cartesian coordinates of a specific vector transform linearly according to

$$\vec{x} \to \vec{x}' = R\vec{x}, \tag{1}$$

so that

$$\vec{x} \cdot \vec{x} = \vec{x}' \cdot \vec{x}'. \tag{2}$$

In a three dimensional space, R corresponds to a 3×3 orthogonal matrix and the array $\vec{x}$ is written as a column. In order to find explicitly the R matrix we analyze infinitesimal rotations and, as usual, then construct a finite transformation, made of an infinite number of infinitesimal ones. If an infinitesimal transformation is represented by

$$\vec{x} \to \vec{x}' = \vec{x} - \delta\vec{x}, \tag{3}$$

then, from (2), $\delta\vec{x}$ in first approximation satisfies

$$\delta\vec{x} \cdot \vec{x} = 0 \tag{4}$$

for all $\vec{x}$. The solution of this equation is given by

$$\delta\vec{x} = \delta\vec{\theta} \times \vec{x}, \tag{5}$$

the infinitesimal vector $\delta\vec{\theta}$ physically carries the total information about of the rotation: $|\delta\vec{\theta}|$ gives the magnitude of the rotation angle and $\hat{\theta} \equiv \delta\vec{\theta}/|\delta\vec{\theta}|$ are the coordinates of the unit vector, parallel to the rotation axis. From this the (infinitesimally) transformated coordinates are written as

$$\vec{x}' = (1 - \delta\vec{\theta}\times)\vec{x}. \tag{6}$$

The expression in brackets corresponds to the infinitesimal rotation matrix $R(\delta\vec{\theta})$. The quantity $\delta\vec{\theta}\times$ is a (matrix) operator which can be defined as follows:

$$(\delta\vec{\theta}\times)\vec{x} \equiv \delta\vec{\theta} \times \vec{x}, \tag{7}$$

or more explicitly,

$$\delta\vec{\theta}\times = \begin{pmatrix} 0 & -\delta\theta_3 & \delta\theta_2 \\ \delta\theta_3 & 0 & -\delta\theta_1 \\ -\delta\theta_2 & \delta\theta_1 & 0 \end{pmatrix}. \tag{8}$$

Writting

$$\delta\vec{\theta} = \lim_{N\to\infty} \vec{\theta}/N$$

the matrix for a finite angle $\vec{\theta}$ rotation corresponds to

$$R(\vec{\theta}) = \lim_{N\to\infty} \left[R(\vec{\theta}/N)\right]^N = \lim_{N\to\infty} \left(1 - \frac{\vec{\theta}\times}{N}\right)^N$$
$$= \mathrm{e}^{-\vec{\theta}\times}. \tag{9}$$

The expansion of the exponential in (9) gives us the R matrix explicitly,

$$\mathrm{e}^{-\vec{\theta}\times} = \hat{\theta}\hat{\theta}\cdot - \sin\theta\,\hat{\theta}\times - \cos\theta(\hat{\theta}\times)^2, \tag{10}$$

which applied to the coordinates gives the conventional expression of coordinate rotations [4]. For arriving to (10) we have used the properties of the triple vector product to obtain

$$(\vec{\theta}\times)(\vec{\phi}\times) = \vec{\theta}\vec{\phi}\cdot - \vec{\theta}\cdot\vec{\phi}; \tag{11}$$

the last term is understood to be the coefficient of an identity matrix. In this notation the period after a vector implies its transposition: $\vec{\theta}\cdot \equiv \vec{\theta}^T$.

2.2. Rotations algebra

As is well known a group is a set of operators with a multiplication law which satisfies four basic properties: closure, associativity, existence of the identity and the existence of a unique inverse for each element. The set of rotation matrices R represents a group: the rotation group. The elements of the rotation group are labeled by the set of continuos parameters θ_i. The antisymmetric matrix $\vec{\theta}\times$ generates the rotation matrix $R(\vec{\theta})$, this is why it is called “generator”. Generators form a vector space as well. The rotations algebra is the commutation relations among the elements of the generators vector space basis.

The closure property it is nothing more than the statement that the composition of two rotations is again a rotation. This is implemented in group theory language by saying that the commutator between two generators is a generator. For the generators of the rotation group we obtain

$$[\vec{\theta}\times, \vec{\phi}\times] = (\vec{\theta}\times\vec{\phi})\times, \tag{12}$$

where we have used the Jacobi identity for the triple vector product.

If the $\hat{e}_i$ form the standard basis of the coordinate space, they satisfy the algebra

$$\hat{e}_i\cdot\hat{e}_j = \delta_{ij}, \quad \hat{e}_i\times\hat{e}_j = \epsilon_{ijk}\hat{e}_k, \tag{13}$$

where ϵ_{ijk} is the totally antisymmetric Levi-Civita tensor. (The sum over the repeated indexes is understood.) Writing

$$\mathcal{J}_i = \mathrm{i}\hat{e}_i\times, \tag{14}$$

we find that the generators can be re-written as

$$\vec{\theta}\times = -\mathrm{i}\vec{\theta}\cdot\vec{\mathcal{J}}; \tag{15}$$

that is, $\vec{\mathcal{J}}$ corresponds to a hermitian base for the generator space. According to (12) and (13) the $\mathcal{J}$'s then satisfy

$$[\mathcal{J}_i, \mathcal{J}_j] = \mathrm{i}\epsilon_{ijk}\mathcal{J}_k. \tag{16}$$

The relation (16) corresponds to the algebra of rotations.

2.3. Rotating systems

All of the subsection 2.2. is standard, however in connection with subsection 2.1. we can obtain interesting results. As an example of the usefulness of the notation introduced in (9) for the rotation matrix, let's find the velocity and acceleration of a particle observed from a rotating system. Let a vector $\vec{x}$ be the coordinates of a particle in an inertial system and $\vec{x}'$ the coordinates of the same particle observed from a rotating system, with angular velocity $\vec{\omega}$; the origins of these two systems are located at the same geometrical point so that the coordinates satisfy the relation

$$\vec{x}' = \mathrm{e}^{-\vec{\theta}\times}\vec{x}, \tag{17}$$

where $\vec{\theta}$ is a time-dependent function. In the inertial system the velocity and acceleration of one particle are the first and second time-derivative of the coordinates, respectively. Assuming that the components of a force, acting over the particle, transform according to (17) we conclude that, in the rotating system, the second Newton law $\vec{F} = m\vec{a}$ does not have this form, unless we change the time-derivative to a covariant time-derivative given by

$$D_t \equiv \mathrm{e}^{-\vec{\theta}\times}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\vec{\theta}\times} = \frac{\mathrm{d}}{\mathrm{d}t} + \vec{\omega}\times + \frac{1}{2}(\vec{\omega}\times\vec{\theta})\times$$
$$+ \frac{1}{3!}\Big((\vec{\omega}\times\vec{\theta})\times\vec{\theta}\Big)\times + \cdots, \tag{18}$$

where we have used (12) in the known relation

$$\mathrm{e}^{-A}B\mathrm{e}^{A} = B + [B,A] + \frac{1}{2}[[B,A],A]$$
$$+ \frac{1}{3!}[[[B,A],A]],A] + \cdots$$

Thus we can define a covariant velocity $\vec{v}'$ of the particle, seen in the rotating system, as the covariant derivative of the coordinates; in the simple case in which $\vec{\omega}$ is paralell to $\vec{\theta}$ we have

$$\vec{v}' = \frac{\mathrm{d}\vec{x}'}{\mathrm{d}t} + \vec{\omega}\times\vec{x}'. \tag{19}$$

In the same way the covariant acceleration is then given by

$$\vec{a}' = D_t\vec{v}' = \frac{\mathrm{d}^2\vec{x}'}{\mathrm{d}t^2} + 2\vec{\omega}\times\frac{\mathrm{d}\vec{x}'}{\mathrm{d}t} + \vec{\omega}\times(\vec{\omega}\times\vec{x}') + \vec{\alpha}\times\vec{x}',. \quad (20)$$

where $\vec{\alpha}$ is the angular acceleration of the system. In the second term of the RHS we recognize the Coriolis acceleration [4–7], and the centrifugal acceleration in the third term. In this way the primed vectors are related with the un-primed quantities by a relation similar to (17).

3. Lorentz transformations

Lorentz transformations are the rules that relate space-time coordinates of any event in two different inertial systems. Basically, Lorentz transformations can be classified in two types, rotations and boosts. A general Lorentz transformation is a mixing between them. Boosts are the Lorentz transformations when the systems have parallel spatial axis with spatial origin in relative movement. As we will see, Lorentz transformations are the generalization of the classical rotations to 4-dimensional space-time.

3.1. Boost transformations

In order to deduced how to transform the coordinates of any event after a boost let us take S' to be an inertial system in relative movement with respect to another inertial system S. The respective axes in both systems are parallel. Take also their spatial origin as coincident at time zero for both systems. We get that the space-time origin of the two systems is the same. According to the Galilean transformations, in that case, the coordinates t' and $\vec{x}'$ of a event,as observed from S', are related with the t and $\vec{x}$ coordinates of S given by

$$t' = t,$$
$$\vec{x}' = \vec{x} - \vec{u}t; \quad (21)$$

where $\vec{u}$ is the velocity of S' relative to S. As a result of these relations the velocity of one particle observed in S' is the velocity observed by S minus the relative velocity $\vec{u}$. Clearly this is in contradiction with the postulate of special relativity that the speed of the light is constant independently of the choice of coordinates, because that relation of velocities remains true even when a light pulse is considered instead of a particle.

According to the special relativity principles if we suppose that a light pulse is emitted from the origin the space-time coordinates, the pulse must satisfy

$$c^2t^2 - \vec{x}^2 = c^2t'^2 - \vec{x}'^2 = 0. \quad (22)$$

One can, however, try to modify the Galilean transformations to make it compatible with the relativity principles, let us proceed like this; for the $u/c \to 0$ approximation take the deformed Galilean transformations to be

$$t' = t - \delta t$$
$$\vec{x}' = \vec{x} - \vec{u}t, \quad (23)$$

introducing a factor δt. In order to satisfy (22) in first approximation we obtain

$$\delta t = \frac{\vec{u}\cdot\vec{x}}{c^2}. \quad (24)$$

Notice that (22) together with (23) satisfy the first equation in (21) even if $c^2t^2 - \vec{x}^2$ not vanishes. That is, even if $\vec{x}$ and t represent the coordinates of any arbitrary event. These so deformed Galilean transformations correspond to infinitesimal boost transformations.

It is convenient to define a infinitesimal parameter as

$$\delta\vec{\eta} \equiv \left.\frac{\vec{u}}{c}\right|_{u/c\to 0}. \quad (25)$$

We can write the infinitesimal Lorentz transformation (23), using (24) and (25), as the following matrix equation:

$$\begin{pmatrix} ct' \\ \vec{x}' \end{pmatrix} = \left[1 - \begin{pmatrix} 0 & \delta\vec{\eta}\cdot \\ \delta\vec{\eta} & 0 \end{pmatrix}\right]\begin{pmatrix} ct \\ \vec{x} \end{pmatrix}. \quad (26)$$

Assuming

$$\delta\vec{\eta} = \lim_{N\to\infty}\vec{\eta}/N,$$

one can reconstruct the finite Lorentz transformations, using a procedure similar to the one introduced in (9); performing an infinite number of infinitesimal transformations the result is

$$\begin{pmatrix} ct' \\ \vec{x}' \end{pmatrix} = \lim_{N\to\infty}\left[1 - \frac{1}{N}\begin{pmatrix} 0 & \vec{\eta}\cdot \\ \vec{\eta} & 0 \end{pmatrix}\right]^N\begin{pmatrix} ct \\ \vec{x} \end{pmatrix}$$
$$= \exp\begin{pmatrix} 0 & -\vec{\eta}\cdot \\ -\vec{\eta} & 0 \end{pmatrix}\begin{pmatrix} ct \\ \vec{x} \end{pmatrix}. \quad (27)$$

Expanding the exponential we obtain

$$\exp\begin{pmatrix} 0 & -\vec{\eta}\cdot \\ -\vec{\eta} & 0 \end{pmatrix} = \begin{pmatrix} \cosh\eta & -\hat{u}\cdot\sinh\eta \\ -\hat{u}\sinh\eta & \hat{u}\hat{u}\cdot\cosh\eta - (\hat{u}\times)^2 \end{pmatrix}. \quad (28)$$

From (27) and (28) we can work out the relative velocity between the two coordinate systems

$$\vec{u} = -\left.\frac{\vec{x}'}{t'}\right|_{\vec{x}=\vec{0}} = \hat{u}\tanh\eta, \quad (29)$$

therefore

$$\sinh\eta = \frac{u/c}{\sqrt{1-u^2/c^2}};$$
$$\cosh\eta = \frac{1}{\sqrt{1-u^2/c^2}} \equiv \gamma. \quad (30)$$

Thus (29) gives the relation between the parameter η and the relative velocity u. It is evident that if $u/c \rightarrow 0$ we get $\eta \rightarrow u/c$; for this reason η is called the relative "rapidity".

In general, a Lorentz vector is a 4-vector which transforms according to (27) [with (28) and (30)]. Just by introducing a deformation to the Galilean transformations one can introduce the results of special relativity and motivate the necessity of a constant speed of light (for any observer).

3.2. Lorentz algebra

As in subsection 2.2. once we know the way a vector transforms we can find out about the group algebra that these transformations imply. From the expression (27) one can guess the generators of a boost transformation. The set of boost transformations does not form a group, this can be seeing by the fact that the commutation relation between boost generators is not a boost generator itself,

$$\left[\begin{pmatrix} 0 & \vec{\eta}\cdot \\ \vec{\eta} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\kappa}\cdot \\ \vec{\kappa} & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & \vec{0}\cdot \\ \vec{0} & -(\vec{\eta}\times\vec{\kappa})\times \end{pmatrix}. \quad (31)$$

Nevertheless this generators form a vector space which can be expanded in the basis of $\vec{\mathcal{K}}$, defined by

$$\mathcal{K}_i = \begin{pmatrix} 0 & \hat{e}_i\cdot \\ \hat{e}_i & 0 \end{pmatrix}. \quad (32)$$

The commutation relations (31) for the $\mathcal{K}$'s are

$$[\mathcal{K}_i, \mathcal{K}_j] = \mathrm{i}\epsilon_{ijk}\mathcal{J}_k, \quad (33)$$

where, in this case, the $\mathcal{J}$'s are the rotation generators given in (13) extended to four dimensions,

$$\mathcal{J}_i = \begin{pmatrix} 0 & \vec{0}\cdot \\ \vec{0} & \mathrm{i}\hat{e}_i\times \end{pmatrix}. \quad (34)$$

The generators $\mathcal{K}$ do not form a closed algebra, $\mathcal{K}\oplus\mathcal{J}$'s do, the algebra closes with

$$[\mathcal{J}_i, \mathcal{K}_j] = \mathrm{i}\epsilon_{ijk}\mathcal{K}_k. \quad (35)$$

Relations (16), (34) and (35) form the Lorentz algebra. This algebra is a manifestation of the fact that rotations, together with boosts, form a group, the Lorentz group. The $\mathcal{K}$'s and $\mathcal{J}$'s are a basis for the generator space of this group. We can change the basis, in particular a good choice is the basis compounded by the $\mathcal{N}$'s and their complex conjugate $\mathcal{N}^*$'s defined by

$$\mathcal{N}_i \equiv \mathcal{J}_i + \mathcal{K}_i, \quad (36)$$

which satisfy the algebra

$$[\mathcal{N}_i, \mathcal{N}_j] = 2\mathrm{i}\epsilon_{ijk}\mathcal{N}_k, \quad (37)$$

that is $(1/2)\mathcal{N}_i$ and $-(1/2)\mathcal{N}_i^*$ satisfy independently satisfy the rotation algebra (16), additionally,

$$[\mathcal{N}_i, \mathcal{N}_j^*] = 0. \quad (38)$$

We see that the Lorentz algebra can be splitted into two "rotation" invariant subalgebras.

3.3. Thomas precession

Relation (33) correspond to the application of two consecutive boosts; it shows that a vector is rotated when these two boosts are applied. This phenomena is known as the Thomas precession. Physically the Thomas precession appears when we try to describe the time evolution of quantities asociated to accelerated particles.

In order to analyze the problem of an accelerated particle, the usual thinking is of a non-inertial system as composed of infinite inertial system where the particle is always instantaneously at rest in one of them $[1, 8, 9]$. However, as we will see, this problem is equivalent (at least locally) to considering only one non-inertial rest frame where the "boost" from the laboratory system is characterized by a time depending rapidity $\vec{\eta}(t)$.

For the non-accelerated particle the time derivative used in the laboratory system changes as

$$\frac{\mathrm{d}}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}}{\mathrm{d}t'} = \frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t},$$

when the observer uses the system where the particle is at rest.

Following the procedure of subsection 2.3., for an accelerated particle, we must define a covariant time derivative for an observer in the frame in which the particle is at rest, as with the the rotating system (18),

$$\frac{\mathrm{d}}{\mathrm{d}t} \rightarrow D_t = \mathrm{e}^{-\vec{\eta}\cdot\vec{\mathcal{K}}}\frac{\mathrm{d}}{\mathrm{d}t'}\mathrm{e}^{\vec{\eta}\cdot\vec{\mathcal{K}}}. \quad (39)$$

In the non-relativistic approximation, and considering (39) acting only on 3-vectors (see appendix) we have

$$D_t = \frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{\vec{u}\times\dot{\vec{u}}}{2c^2}\right)\times, \quad (40)$$

where $\vec{u}$ is the velocity the particle seen from the laboratory system. Comparing with (18) we find that this system has a precession frecuency given by

$$\vec{\omega} = \frac{\vec{u}\times\dot{\vec{u}}}{2c^2} \equiv -\vec{\omega}_T; \quad (41)$$

$\vec{\omega}_T$ is called Thomas frecuency. For instance, the time evolution of the spin vector of a accelerated particle of mass m, charge e and gyro-magnetic ratio g is not $\mathrm{d}\vec{s}/\mathrm{d}t = g(e/2m)\,\vec{s}\times\vec{B}'$ but

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} - \vec{\omega}_T\times\vec{s} = g\frac{e}{2m}\vec{s}\times\vec{B}', \quad (42)$$

where $\vec{B}'$ is the magnetic field observed in the rest frame of the particle. Once again, following the method introduced in classical mechanics and deforming the Galilean set of transformations one is able to obtain, without too much effort, a fundamental result of relativistic mechanics.

4. Transformations of the electromagnetic field

In the same spirit of this paper, Maxwell equations with sources can be written in a matricial form as

$$\begin{pmatrix} 0 & -\vec{E}\cdot \\ -\vec{E} & \vec{B}\times \end{pmatrix} \begin{pmatrix} \overleftarrow{\partial}_0 \\ -\overleftarrow{\nabla} \end{pmatrix} = \begin{pmatrix} \rho \\ \vec{J} \end{pmatrix}, \qquad (43)$$

where $\leftarrow$ over the derivatives means that they act to the right. We are assuming $c = \epsilon_0 = 1$ for simplicity. (Homogeneous Maxwell equations are obtained by duality, $\vec{E} \to -\vec{B}$, $\vec{B} \to \vec{E}$, $\rho \to 0$.) We can then write the electromagnetic field array as a combination of the generators of the Lorentz group; in our notation

$$\begin{pmatrix} 0 & -\vec{E}\cdot \\ -\vec{E} & \vec{B}\times \end{pmatrix} = -(\vec{E}\cdot\vec{\mathcal{K}} + \mathrm{i}\vec{B}\cdot\vec{\mathcal{J}}), \qquad (44)$$

Under Lorentz transformations the spacetime derivative and the sources in (43) transforms like the coordinates in (27), so the matrix of the electromagnetic fields transform according to

$$\vec{E}'\cdot\vec{\mathcal{K}} + \mathrm{i}\vec{B}'\cdot\vec{\mathcal{J}} = \mathrm{e}^{-\vec{\eta}\cdot\vec{\mathcal{K}}}(\vec{E}\cdot\vec{\mathcal{K}} + \mathrm{i}\vec{B}\cdot\vec{\mathcal{J}})\mathrm{e}^{\vec{\eta}\cdot\vec{\mathcal{K}}}; \qquad (45)$$

taking infinitesimal transformations for the fields we find

$$\vec{E}'\cdot\vec{\mathcal{K}} + \mathrm{i}\vec{B}'\cdot\vec{\mathcal{J}} = \vec{E}\cdot\vec{\mathcal{K}} + \mathrm{i}\vec{B}\cdot\vec{\mathcal{J}} + [(\vec{E}\cdot\vec{\mathcal{K}} + \mathrm{i}\vec{B}\cdot\vec{\mathcal{J}}), \delta\vec{\eta}\cdot\vec{\mathcal{K}}].$$

For the $\mathcal{K}$'s and $\mathcal{J}$'s coefficients we have

$$\vec{E}' = \vec{E} + \delta\vec{\eta}\times\vec{B},$$
$$\vec{B}' = \vec{B} - \delta\vec{\eta}\times\vec{E};$$

these coupled equations can be written in one, using a complexified electromagnetic vector field:

$$(\vec{E} + \mathrm{i}\vec{B})' = (1 - \mathrm{i}\delta\vec{\eta}\times)(\vec{E} + \mathrm{i}\vec{B}), \qquad (46)$$

corresponding to an infinitesimal imaginary rotation of the quantity $\vec{E} + \mathrm{i}\vec{B}$. The finite transformation is therefore

$$(\vec{E} + \mathrm{i}\vec{B})' = \mathrm{e}^{-\mathrm{i}\vec{\eta}\times}(\vec{E} + \mathrm{i}\vec{B}), \qquad (47)$$

which can be expanded as in (10). Taking the real and imaginary parts we finally obtain

$$\vec{E}' = \hat{u}\hat{u}\cdot\vec{E} + \sinh\eta\,\hat{u}\times\vec{B} - \cosh\eta(\hat{u}\times)^2\vec{E},$$
$$\vec{B}' = \hat{u}\hat{u}\cdot\vec{B} - \sinh\eta\,\hat{u}\times\vec{E} - \cosh\eta(\hat{u}\times)^2\vec{B}, \qquad (48)$$

which correspond to the usual electromagnetic boost transformations.

We now have that the square of transformation (47) gives

$$(\vec{E}' + \mathrm{i}\vec{B}')^2 = (\vec{E} + \mathrm{i}\vec{B})^2 \qquad (49)$$

i.e. $E^2 - B^2$ and $\vec{B}\cdot\vec{E}$ are invariant quantities. So, if $\vec{B}\cdot\vec{E} \neq 0$, the electric an magnetic fields will exist simultaneously in all inertial frames, while the angle between the fields stays acute or obtuse depending on its value in the original coordinate frame.

In the case in which the fields are ortogonal ($\vec{B}\cdot\vec{E} = 0$), it is possible to find an inertial frame where

$$\vec{E}' = 0 \quad \text{if} \quad B^2 > E^2, \quad \text{or} \quad \vec{B}' = 0 \quad \text{if} \quad \mathrm{E}^2 > \mathrm{B}^2.$$

Let us clarify this with an example. Consider a particle moving in an electromagnetic field where $\vec{E}\cdot\vec{B} = 0$ and $B^2 > E^2$ (the case where $B^2 < E^2$ can be obtained from this by duality). As we saw, there is an inertial system where the particle is afected only by a magnetic field $\vec{B}'$. Using the condition $\vec{E}' = 0$ in the first expresion of (48) and taking both the parallel and perpendicular components with respect to $\hat{u}$ we find

$$\hat{u}\cdot\vec{E} = 0,$$
$$\sinh\eta\,\hat{u}\times\vec{B} = \cosh\eta(\hat{u}\times)^2\vec{E};$$

from which we obtain

$$-\vec{u}\times\vec{B} = \vec{E}, \qquad (50)$$

where we have used $\vec{E} = -(\hat{u}\times)^2\vec{E}$ and $\tanh\eta = u$. This equation does not univocally determine $\vec{u}$, so there are many system where the electric field vanishes.

In particular we can choose the velocity to be ortogonal to the magnetic field, obtaining the following expresion for the velocity

$$\vec{u} = \frac{E}{B}\hat{u}. \qquad (51)$$

Because the Eq. (47) corresponds to a rotation, we see that the parallel component to $\vec{u}$ of the electromagnetic field is an invariant, so for our case $\vec{B}$ and $\vec{B}'$ must be parallel. Furthermore, by the invariance of $E^2 - B^2$ we obtain

$$\vec{B}' = \frac{\sqrt{B^2 - E^2}}{B}\vec{B}. \qquad (52)$$

In this example we saw the utility of the relation (49) which is easilly derived from (47) and is not evident from the usual transformations (48). (Usually is derived using tensorial notation).

Another interesting example of Lorentz transformations of the electromanetic field is when we consider the evolution of the spin of a charged particle, moving in a region with an electric field $\vec{E}$. In the system in which the particle is at rest a magnetic field appears. Its value is given by the second expresion in (48) which, in the non relativistic aproximation, is written as

$$\vec{B}' = -\vec{u} \times \vec{E}.$$

The evolution of the spin is given by (42) and (41), where $\dot{\vec{u}} = e\vec{E}/m$, therefore

$$\frac{\mathrm{d}\vec{s}}{\mathrm{d}t} = -(g-1)\frac{e}{2m}\vec{s} \times (\vec{u} \times \vec{E}), \tag{53}$$

which is the Thomas equation [10] with $B = 0$ and $\gamma \to 1$ As it is well known, this equation gives the correct spin-orbit correction in the non relativistic aproximation [11].

5. Conclusions

We have introduced a way of writing the coordinates of a rotated vector and deduced the Coriolis acceleration in a straightforward way. The generators of the rotation group are given a compact form. In the same spirit we have obtained Lorentz transformations for 4-vectors and show how the Thomas precession appears in a non-inertial system after the introduction the covariant time derivative.

Using a matrix construction we write the non-homogeneous Maxwell equations in a compact form and, starting from this, we deduce the Lorentz transformations of the electromagnetic fields using the notation introduced before. We show that the Lorentz transformation of the electromagnetic fields can be seen as a rotations of the complexified electromagnetic vector $\vec{E} + \mathrm{i}\vec{B}$.

Appendix: Non-inertial system

In this appendix we will explitly find the time covariant derivative given in (40) for non-inertial system. We can express this derivative written in terms of the $\mathcal{N}$'s, defined in (36), as

$$D_t = \mathrm{e}^{-\frac{1}{2}\vec{\eta}\cdot\vec{\mathcal{N}}}\frac{\mathrm{d}}{\mathrm{d}t'}\mathrm{e}^{\frac{1}{2}\vec{\eta}\cdot\vec{\mathcal{N}}} + \mathrm{c.c.}, \tag{54}$$

where we have used the fact that $\mathcal{N}_i$ and $\mathcal{N}_j^*$ commute (Eq. (38)). The $\mathcal{N}$'s satisfy the simple relation

$$\mathcal{N}_i\mathcal{N}_i = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\mathcal{N}_k,$$

so we have

$$\mathrm{e}^{\frac{1}{2}\vec{\eta}\cdot\vec{\mathcal{N}}} = \cosh\frac{\eta}{2} + \hat{\eta}.\vec{\mathcal{N}}\sinh\frac{\eta}{2},$$

and therefore

$$\mathrm{e}^{-\frac{1}{2}\vec{\eta}\cdot\mathcal{N}}\frac{\mathrm{d}}{\mathrm{d}t'}\mathrm{e}^{\frac{1}{2}\vec{\eta}\cdot\mathcal{N}} = \frac{\mathrm{d}}{\mathrm{d}t'} + \frac{1}{2}\left(\hat{\eta}\frac{\mathrm{d}\eta}{\mathrm{d}t'} + \sinh\eta\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}t'} - \mathrm{i}(\cosh\eta - 1)\hat{\eta}\times\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}t'}\right)\cdot\vec{\mathcal{N}}. \tag{55}$$

Finally, returning to the $\mathcal{J}$'s and $\mathcal{K}$'s we write

$$D_t = \frac{1}{\gamma}\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{\gamma}\left(\frac{\mathrm{d}\eta}{\mathrm{d}t}\hat{\eta} + \sinh\eta\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}t}\right)\cdot\vec{\mathcal{K}} - \frac{\mathrm{i}}{\gamma}(\cosh\eta - 1)\hat{\eta}\times\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}t}\cdot\vec{\mathcal{J}}. \tag{56}$$

In a non-relativistic approximation, $\gamma \to 1$, we have

$$D_t = \frac{\mathrm{d}}{\mathrm{d}t} + \frac{\dot{\vec{u}}}{c}\cdot\vec{\mathcal{K}} - \mathrm{i}\frac{\vec{u}\times\dot{\vec{u}}}{2c^2}\cdot\vec{\mathcal{J}}. \tag{57}$$

Considering the covariant derivative acting only on 3-vectors and using the definitions of the $\mathcal{J}$'s given in (16) we obtain

$$D_t = \frac{\mathrm{d}}{\mathrm{d}t} + \left(\frac{\vec{u}\times\dot{\vec{u}}}{2c^2}\right)\times \tag{58}$$

which is the result (42).

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