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On a multivariate generalization of the covariance

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ABSTRACT

Hoeffding's lemma provides a representation of the covariance of two random variables in terms of the difference between the joint and marginal distributions. This article proposes a multivariate generalization of the covariance between functions of bounded variation in the semialgebra of rectangles on $\mathbb{R}^{2k}$. Some applications include the covariance inequality among functions where the variables are positive orthant dependent.

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1. Introduction

The lemma by Hoeffding (1940) proves that the covariance between two random variables can be obtained in terms of its joint and marginal distribution functions. This lemma was used to prove that the correlation ρ between two random variables X, Y with distribution functions $F(x), G(y)$ is not always bounded between the values -1 and 1. Indeed, it can be shown that ρ is bounded by two correlations, named minimal ρ^- and maximal ρ^+, also called Hoeffding's correlations, which are strongly related to Fréchet's bounds.

Lehmann (1966) proved Hoeffding's lemma and used it in some concepts of dependence; Jogdeo (1968) studied the multivariate version; Block and Fang (1988) used the cumulative concept of a random vector $\mathbf{X} = (X_1, \ldots, X_k)$ to generalize the covariance with more than two random variables; Mardia (1967) and Mardia and Thompson (1972) obtained the covariance for X^r, Y^s; Yu (1993) obtained a generalization for absolutely continuous functions of the components of a random vector; Cuadras (2002) obtained the covariance for a couple of functions of bounded variation; Quesada-Molina (1992) obtained the covariance for quasi-monotonic functions; and Yu (1993) obtained the covariance for two multivariate functions. Further extensions have been obtained by Beare (2009) and Cuadras (2015).

We propose a multivariate generalization of the covariance between functions of bounded variation, as well as a multivariate extension of the identity proved by Cuadras (2002). We also set up the relation between the results in Quesada-Molina (1992) and Cuadras (2002). Finally, we obtain an inequality for the covariances among functions in the case that the random variables are positively orthant dependent (POD).[1]

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[1] The random variables $X_1, \ldots, X_n$ are positively orthant dependent if, for all $(x_1, \ldots, x_n) \in \mathbb{R}^n$, $P(X_1 \le x_1, \ldots, X_n \le x_n) \ge \prod_{i=1}^n P(X_i \le x_i)$.For the bivariate case, they are called positively quadrant dependent (PQD).

2. Hoeffding's Lemma

Let X, Y be two random variables with joint distribution function $H(x, y)$ and marginal distribution functions $F(x)$, $G(y)$. Hoeffding (1940) found the covariance in terms of the cumulative distribution function

$$\mathrm{Cov}(X, Y) = \int_{\mathbb{R}^2} [H(x, y) - F(x)G(y)]dxdy. \tag{1}$$

Block and Fang (1988) generalized this result for the case of more than two random variables, providing an integral representation of the joint cumulant.[2] Mardia (1967) and Mardia and Thompson (1972) proved that

$$\mathrm{Cov}(X^r, Y^s) = \int_{\mathbb{R}^2} [H(x, y) - F(x)G(y)]rsx^{r-1}y^{s-1}dxdy. \tag{2}$$

Cuadras (2002) proved that if $\alpha(\cdot)$ and $\beta(\cdot)$ are functions of bounded variation and their expected values exist

$$E[\alpha(X)\beta(Y)] - E[\alpha(X)]E[\beta(Y)] = \int_{\mathbb{R}^2} [H(x, y) - F(x)G(y)]d\alpha(x)d\beta(y). \tag{3}$$

Equation (3) was initially discussed by Sen (1994), when $\alpha(x)$, $\beta(y)$ are monotonic functions. Equation (3) provides the covariance $\mathrm{Cov}(\alpha(X), \beta(Y))$ and reduces to Hoeffding's identity for $\alpha(x) = x$, $\beta(y) = y$ and to Mardia and Thompson (1972) identity for $\alpha(x) = x^r$, $\beta(y) = y^s$.

Quesada-Molina (1992) proved that if $K(x, y)$ is a real quasi-monotonic function and continuous to the right in the sense of

$$\Delta_{(x_1,y_1)}^{(x_2,y_2)} K(x, y) = K(x_1, y_1) - K(x_2, y_1) - K(x_1, y_2) + K(x_2, y_2) \geq 0, \tag{4}$$

for all $x_1 \leq x_2$ and $y_1 \leq y_2$, the distribution function of X, Y is $H(x, y)$ and the distribution function of X^*, Y^* is $H^*(x, y) = F(x)G(y)$, then

$$E[K(X, Y)] - E[K(X^*, Y^*)] = \int_{\mathbb{R}^2} [H(x, y) - F(x)G(y)]dK(x, y). \tag{5}$$

In particular, if $K(X, Y) = XY$, Equation (5) reduces to Hoeffding's identity (1).

Definition 1. (Vitali) A function $\phi(x, y)$ is of bounded variation in the rectangle $[a, b] \times [c, d]$, if for all points $a = x_0 < x_1 < \cdots < x_m = b$, $c = y_0 < y_1 < \cdots < y_n = d$, the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \Delta_{(x_{i-1},y_{j-1})}^{(x_i,y_j)} \phi(x, y), \tag{6}$$

is bounded.

The next example shows that not all functions of bounded variation are quasi-monotone, as Dewan and Rao (2005) said when considering Equation (3) as a particular case of Equation (5).

[2] The rth-order joint cumulant of $(X_1, \ldots, X_n)$ is defined by

$$\sum (-1)^{p-1}(p-1)! \left(E \prod_{j \in \nu_1} X_j \right) \times \cdots \times \left(E \prod_{j \in \nu_p} X_j \right),$$

where the sum is extended over all partitions $(\nu_1, \ldots \nu_p)$, $p = 1, 2, \ldots, n$, of $\{1, \ldots, n\}$.

Example 1. Let the function $\phi(x,y) = (x-1/2)^2(y-1/2)^2$ be defined in $[0,1]^2$. Let us consider the function $\alpha(x) = (x-1/2)^2$ in $[0,1]$ of bounded variation since it is differentiable with a bounded derivative; besides, the product $\phi(x,y) = \alpha(x)\alpha(y)$ is of bounded variation. Let us show that $\phi(x,y)$ is not a quasi-monotonic function; that is to say, the inequality (4) is not satisfied. For instance, take the rectangular region $[1/2,1] \times [0,1/2]$ contained in the domain of $\phi(x,y)$. Then

$$\begin{aligned}\Delta_{(1,1/2)}^{(0,1/2)}\phi(x,y) &= \phi(1,1/2) - \phi(1/2,1/2) - \phi(1,0) + \phi(1/2,0)\\ &= -\phi(1,0) = -1/16.\end{aligned}$$

3. Generalization of the covariance

In the next theorem, we extend Quesada-Molina (1992) result to the class of functions of bounded Vitali variation of two variables in the rectangle $[a,b]\times[c,d]$. This theorem is related to the result by Cuadras (2002). It is worth noting that Beare (2009) also obtained a generalization for functions of bounded Hardy–Krause variation, instead of bounded Vitali variation considered here.

Theorem 1. *Let X, Y be random variables with support in the intervals $[a,b]$, $[c,d]$, with joint distribution function $H(x,y)$ and marginal distribution functions $F(x)$, $G(y)$, respectively, and let X^*, Y^* be random variables with joint distribution function $H^*(x,y) = F(x)G(y)$. Suppose that $\phi(x,y)$ is a function of bounded variation in the rectangle $[a,b]\times[c,d]$ and that $E[\phi(X,Y)]$ and $E[\phi(X^*,Y^*)]$ exist and are finite. Then*

$$E[\phi(X,Y)] - E[\phi(X^*,Y^*)] = \int_a^b\int_c^d [H(x,y) - F(x)G(y)]d\phi(x,y). \tag{7}$$

Proof. Let us consider the function

$$\Delta_{(a,c)}^{(u,v)}\phi(x,y) = \phi(u,v) - \phi(a,v) - \phi(u,c) + \phi(a,c),$$

where $a \le u \le b$, $c \le v \le d$. Thus

$$\Delta_{(a,c)}^{(b,d)}\phi(x,y) = \sum \Delta_{(a,c)}^{(u,v)}\phi(x,y).$$

If the sum $\sum|\Delta_{(a,c)}^{(u,v)}\phi(x,y)|$ is less than any fixed positive number and besides $\phi(x,y)$ is for each value of x a function of bounded variation with respect to y, and for each value of y a function of bounded variation with respect to x, then $\phi(x,y)$ is a function of bounded variation in the rectangle $[a,b]\times[c,d]$.

Let $V_{(a,c)}^{(b,d)}\phi(x,y)$ be the upper bound of the sum $\sum|\Delta_{(a,c)}^{(u,v)}\phi(x,y)|$. If $\sum$ is expressed in two parts $\sum_1$ and $\sum_2$, where $\sum_1$ denotes the sum of all those terms for which Δ is positive and $\sum_2$ denotes the sum of all those terms for which Δ is negative, we have that $\sum_1 \Delta\phi(x,y)$ and $-\sum_2\Delta\phi(x,y)$ have upper finite bounds denoted by $P_{(a,c)}^{(b,d)}\phi(x,y)$ and $N_{(a,c)}^{(b,d)}\phi(x,y)$, respectively.

The functions $P(x,y) = P_{(a,c)}^{(x,y)}\phi(x,y)$ and $N(x,y) = N_{(a,c)}^{(x,y)}\phi(x,y)$ are monotonic functions in the sense that if $x \le x'$, $y \le y'$, then $P(x,y) \le P(x',y')$ and $N(x,y) \le N(x',y')$.

Hence

$$\phi(x,y) = \phi(a,y) + \phi(x,c) - \phi(a,c) + P(x,y) - N(x,y).$$

If ϕ is a function of x, of bounded variation for all $x \in (a,b)$, then $\phi(x) = f(x) - g(x)$, where $f(x)$, $g(x)$ are increasing monotonic functions of x. Similarly, if ϕ is a function of y, of bounded variation for all $y \in (c, d)$, then it can be represented as the difference $\phi(y) = r(y) - s(y)$ of two increasing monotonic functions of y.

Setting

$$\bar{P}(x, y) = P(x, y) + f(x, c) + r(a, y)$$

and

$$\bar{N}(x, y) = N(x, y) + g(x, c) + s(a, y).$$

Then

$$\phi(x, y) = \bar{P}(x, y) - \bar{N}(x, y) - \phi(a, c), \tag{8}$$

where $\bar{P}$, $\bar{N}$ are quasi-monotonic functions, since they are the sum of a two-variable monotonic function and of separable variable increasing monotonic functions (see Hobson, 1927, p. 347). This shows that a function $\phi(x, y)$ of bounded variation in the rectangle $[a, b] \times [c, d]$ can be expressed as the difference of two quasi-monotonic functions.

From the definition provided by Quesada-Molina (1992) for each one of the quasi-monotonic functions $\bar{P}$, $\bar{N}$ in the rectangle $[a, b] \times [c, d]$, we have that

$$E[\bar{P}(X, Y)] - E[\bar{P}(X^*, Y^*)] = \int_a^b \int_c^d [H(x, y) - F(x)G(y)]d\bar{P}(x, y), \tag{9}$$

and

$$E[\bar{N}(X, Y)] - E[\bar{N}(X^*, Y^*)] = \int_a^b \int_c^d [H(x, y) - F(x)G(y)]d\bar{N}(x, y). \tag{10}$$

Combining Equations (9) and (10) with (8) and considering Lebesgue integration induced in the rectangle $[a, b] \times [c, d]$ by the function of bounded variation $\phi(x, y)$, we have

$$E[\phi(X, Y)] - E[\phi(X^*, Y^*)] = \int_a^b \int_c^d [H(x, y) - F(x)G(y)]d\phi(x, y).$$

□

In a similar way to the proof of this theorem, we can obtain an extension to the multivariate case of even dimension $2k$.

Theorem 2. *Let $\mathbf{X} = (X_1, \ldots, X_{2k})$ be a random vector with joint distribution function H and marginal distribution functions $F_i(x_i)$, $1 \leq i \leq 2k$, defined in the intervals $[a_i, b_i]$, $1 \leq i \leq 2k$. Let $\mathbf{X}^* = (X_1^*, \ldots, X_{2k}^*)$ be a random vector with joint distribution function $H^* = \prod_{i=1}^{2k} F_i(x_i)$. Let $\phi(x_1, \ldots, x_{2k})$ be a function of bounded variation in the semialgebra of rectangles on $\mathbb{R}^{2k}$. If $\mathcal{A}$ is the class of non empty subsets of $\{1, 2, \ldots, 2k\}$, suppose that $E[\phi(X_1, \ldots, X_{2k})]$ and $E[\phi(X^*_{i_{i\in\mathcal{A}}}; X_{i_{i\notin\mathcal{A}}})]$ exist and are finite. Then*

$$2E\left[\phi(X_1, \ldots, X_{2k})\right] + \sum_{\mathcal{A}} (-1)^{card(\mathcal{A})} E\left[\phi(X^*_{i_{i\in\mathcal{A}}}; X_{i_{i\notin\mathcal{A}}})\right]$$

$$= \int_{\mathbb{R}^{2k}} \sum_{\mathcal{A}} (-1)^{card(\mathcal{A})} E\left[\textstyle\prod_{i\in\mathcal{A}} I(u_i, x_i)\right] E\left[\textstyle\prod_{i\notin\mathcal{A}} I(u_i, x_i)\right] d\phi(x_1, \ldots, x_{2k}), \tag{11}$$

where $card(\mathcal{A})$ denotes the cardinality of $\mathcal{A}$ and $I(u, x) = 1$, if $u \leq x$, and 0 otherwise.

Proof. Any function of bounded variation $\phi(x_1, \ldots, x_n)$ in the semialgebra of rectangles on $\mathbb{R}^n$ can be expressed as the difference of two functions K_1 and K_2, which have non negative differences of the order n; this class of functions is called n-positive, namely

$$\Delta_{(x_1,\ldots,x_n)}^{(x_1',\ldots,x_n')} K_j \geq 0; \;\; j = 1, 2,$$

for all $x_i \leq x_i'$, $1 \leq i \leq n$. In particular, 1-positive functions are the increasing monotonic functions, and 2-positive functions are called quasi-monotonic functions or 2-increasing.

Following the same procedures for the proof of Theorem 1 in a space of even dimension $2k$, we have that

$$\phi(x_1, \ldots, x_{2k}) \equiv K_1(x_1, \ldots, x_{2k}) - K_2(x_1, \ldots, x_{2k}). \tag{12}$$

Prakasa Rao (1998) extended the identity of Quesada-Molina (1992) to the multivariate case; based on this result, for every n-positive function K_j; $j = 1, 2$

$$2E[K_j(X_1, \ldots, X_{2k})] + \sum_{\mathcal{A}} (-1)^{\mathrm{card}(\mathcal{A})} E[K_j(X^*_{i_{i\in\mathcal{A}}}; X_{i_{i\notin\mathcal{A}}})]$$

$$= \int_{\mathbb{R}^{2k}} \sum_{\mathcal{A}} (-1)^{\mathrm{card}(\mathcal{A})} E\left[\prod\nolimits_{i\in\mathcal{A}} I(u_i, x_i)\right] E\left[\prod\nolimits_{i\notin\mathcal{A}} I(u_i, x_i)\right] dK_j(x_1, \ldots, x_{2k}). \tag{13}$$

where $\sum$ is evaluated over all the non empty proper sets $\mathcal{A}$ of $\{1, 2, \ldots, 2k\}$ and $\{X^*_{i_{i\in\mathcal{A}}}\}$ identically distributes as $\{X_{i_{i\in\mathcal{A}}}\}$ and independently from $\{X_{i_{i\notin\mathcal{A}}}\}$.

By combining (13) with (12) we get

$$2E[\phi(X_1, \ldots, X_{2k})] + \sum_{\mathcal{A}} (-1)^{\mathrm{card}(\mathcal{A})} E[\phi(X^*_{i_{i\in\mathcal{A}}}; X_{i_{i\notin\mathcal{A}}})]$$

$$= \int_{\mathbb{R}^{2k}} \sum_{\mathcal{A}} (-1)^{\mathrm{card}(\mathcal{A})} E\left[\prod_{i\in\mathcal{A}} I(u_i, x_i)\right] E\left[\prod_{i\notin\mathcal{A}} I(u_i, x_i)\right] d\phi(x_1, \ldots, x_{2k}).$$

□

As a particular case, when ϕ is a function of separable variables, i.e., $\phi(x_1, \ldots, x_{2k}) = \prod_{i=1}^{2k} \alpha_i(x_i)$, we obtain a multivariate extension in even dimension $2k$ of the identity obtained by Cuadras (2002).

Corollary 3. *Let $X_1, \ldots, X_{2k}$ be independent random variables, H the joint distribution function, and $F_i(X_i)$, $1 \leq i \leq 2k$, the marginal distribution functions. Let $\alpha_i(x_i)$, $1 \leq i \leq 2k$ be functions of bounded variation. If $E\left[\prod_{i\in\mathcal{A}} \alpha_i(X_i)\right]$ and $E\left[\prod_{i\notin\mathcal{A}} \alpha_i(X_i)\right]$ exist and are finite, then*

$$\sum_{\mathcal{A}} (-1)^{card(\mathcal{A})} E\left[\prod_{i\in\mathcal{A}} \alpha_i(X_i)\right] E\left[\prod_{i\notin\mathcal{A}} \alpha_i(X_i)\right]$$

$$= \int_{\mathbb{R}^{2k}} \sum_{\mathcal{A}} (-1)^{card(\mathcal{A})} E\left[\prod_{i\in\mathcal{A}} I(u_i, X_i)\right] E\left[\prod_{i\notin\mathcal{A}} I(u_i, X_i)\right] \prod_{i=1}^{2k} d\alpha_i(u_i). \tag{14}$$

Proof. Let $\mathbf{X} = (X_1, \ldots, X_{2k})$ be a random vector and $\mathbf{X}^* = (X_1^*, \ldots, X_{2k}^*)$ an independent copy of $\mathbf{X}$. Let us consider the non decreasing function $I(u, x) = 1$, if $u \leq x$, and 0 otherwise. Since α_i are functions of bounded variation, $d\alpha_i(x_i)$ is Lebesgue integrable. Then

$$\alpha_i(X_i) - \alpha_i(X_i^*) = \int_{X_i^*}^{X_i} d\alpha_i(u_i) = \int_{\mathbb{R}} [I(u_i, X_i) - I(u_i, X_i^*)] d\alpha_i(u_i).$$

Expanding the product, we have

$$\prod_{i=1}^{2k}[\alpha_i(X_i)-\alpha_i(X_i^*)]=\prod_{i=1}^{2k}\left\{\int_{\mathbb{R}}[I(u_i,X_i)-I(u_i,X_i^*)]d\alpha_i(u_i)\right\}$$
$$=\int_{\mathbb{R}^{2k}}\prod_{i=1}^{2k}[I(u_i,X_i)-I(u_i,X_i^*)]\prod_{i=1}^{2k}d\alpha_i(u_i). \quad (15)$$

The terms of the product can be expressed as a sum, such that

$$\prod_{i=1}^{2k}[\alpha_i(X_i)-\alpha_i(X_i^*)]=\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}\prod_{i\in\mathcal{A}}\alpha_i(X_i)\prod_{i\notin\mathcal{A}}\alpha_i(X_i^*), \quad (16)$$

and

$$\prod_{i=1}^{2k}[I(u_i,X_i)-I(u_i,X_i^*)]=\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}\prod_{i\in\mathcal{A}}I(u_i,X_i)\prod_{i\notin\mathcal{A}}I(u_i,X_i^*), \quad (17)$$

where the sum is performed over all non empty subsets $\mathcal{A}\subset\{1,2,\dots,2k\}$.

When replacing (16) and (17) in (15), we get

$$\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}\prod_{i\in\mathcal{A}}\alpha_i(X_i)\prod_{i\notin\mathcal{A}}\alpha_i(X_i^*)$$
$$=\int_{\mathbb{R}^{2k}}\left\{\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}\prod_{i\in\mathcal{A}}I(u_i,X_i)\prod_{i\notin\mathcal{A}}I(u_i,X_i^*)\right\}\prod_{i=1}^{2k}d\alpha_i(u_i).$$

Supposing that $E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]$ and $E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i)\right]$ exist and that the vectors $\mathbf{X}$, $\mathbf{X}^*$ are independent, from Fubini's theorem, we get

$$E\left[\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}\prod_{i\in\mathcal{A}}\alpha_i(X_i)\prod_{i\notin\mathcal{A}}\alpha_i(X_i^*)\right]$$
$$=\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i^*)\right]$$
$$=\int_{\mathbb{R}^{2k}}\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}I(u_i,X_i)\right]E\left[\prod_{i\notin\mathcal{A}}I(u_i,X_i^*)\right]\prod_{i=1}^{2k}d\alpha_i(u_i).$$

Since $\mathbf{X}$, $\mathbf{X}^*$ are identically distributed,

$$\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i)\right]$$
$$=\int_{\mathbb{R}^{2k}}\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}I(u_i,X_i)\right]E\left[\prod_{i\notin\mathcal{A}}I(u_i,X_i)\right]d\alpha_i(u_i).$$

□

To clarify Theorem 2, let us consider the case $k = 1$. Then $\mathcal{A}$ contains all the non empty subsets of $\{1, 2\}$, thus

$$
\begin{aligned}
&\sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i)\right]\\
&\quad= 2(E[\alpha_1(X_1)\alpha_2(X_2)] - E[\alpha_1(X_1)]E[\alpha_2(X_2)])\\
&\quad= 2\mathrm{Cov}(\alpha_1(X_1), \alpha_2(X_2))\\
&\quad= 2\int_{\mathbb{R}^2}\mathrm{Cov}[I(u_1, X_1), I(u_2, X_2)]d\alpha_1(u_1)d\alpha_2(u_2)\\
&\quad= 2\int_{\mathbb{R}^2}[P(X_1 \le u_1, X_2 \le u_2) - P(X_1 \le u_1)P(X_2 \le u_2)]\, d\alpha_1(u_1)d\alpha_2(u_2)\\
&\quad= 2\int_{\mathbb{R}^2}[H(u_1, u_2) - F_1(u_1)F_2(u_2)]\, d\alpha_1(u_1)d\alpha_2(u_2).
\end{aligned}
$$

This result agrees with the expression for the covariance obtained by Cuadras (2002).

Example 2. Let us consider the discrete case where $\alpha(x) = \mathrm{sgn}(x)$ and $\beta(y) = \mathrm{sgn}(y)$, where $\mathrm{sgn}(x) = 1$, if $x > 0$, 0 if $x = 0$ and -1 if $x < 0$. The differential of $\mathrm{sgn}(x)$ is $\mathrm{dsgn}(x) = 2$, if $x = 0$ and 0 otherwise. Then for any x_0 and y_0

$$\mathrm{Cov}(\mathrm{sgn}(X - x_0), \mathrm{sgn}(Y - x_0)) = 4[H(x_0, y_0) - F(x_0)G(y_0)].$$

4. Some consequences

In this section, we obtain an inequality for the covariance of multidimensional functions of bounded variation of POD random variables.

Theorem 4. *Let $X_1, \dots, X_{2k}$ be POD random variables with finite variance, let $\alpha_i(x_i)$ be functions of bounded variation on $\mathbb{R}$, whose derivative $\alpha_i'(x_i)$ exists and is bounded. Then*

$$
\begin{aligned}
&\sum_{\mathcal{A}}(-1)^{card(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i)\right]\\
&\quad\le \prod_{i=1}^{2k}|\max\{\alpha_i'(x_i)\}|\sum_{\mathcal{A}}(-1)^{card(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}X_i\right]E\left[\prod_{i\notin\mathcal{A}}X_i\right]. \qquad (18)
\end{aligned}
$$

Proof. If $X_1, \dots, X_{2k}$ are POD random variables with finite variance, then each $I(u_i, X_i)$; $1 \le i \le 2k$ is non decreasing in x_i for each fixed u_i, thus

$$\sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}I(u_i, X_i)\right]E\left[\prod_{i\notin\mathcal{A}}I(u_i, X_i)\right] \ge 0. \qquad (19)$$

Replacing (19) in (14), we have

$$\sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i)\right] \ge 0. \qquad (20)$$

Assuming that $\alpha_i(x_i)$, $1 \le i \le 2k$ are functions of bounded variation on $\mathbb{R}$, whose derivative $\alpha_i'(x_i)$ exists and is bounded, we have

$$\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}\alpha_i(X_i)\right]E\left[\prod_{i\notin\mathcal{A}}\alpha_i(X_i)\right]$$

$$= \int_{\mathbb{R}^{2k}}\prod_{i=1}^{2k}\alpha_i'(u_i)\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}I(u_i,X_i)\right]E\left[\prod_{i\notin\mathcal{A}}I(u_i,X_i)\right]du_1\ldots du_{2k}$$

$$\le \int_{\mathbb{R}^{2k}}\prod_{i=1}^{2k}|\max\{\alpha_i'(u_i)\}|\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}I(u_i,X_i)\right]E\left[\prod_{i\notin\mathcal{A}}I(u_i,X_i)\right]du_1\ldots du_{2k}$$

$$\le \prod_{i=1}^{2k}|\max\{\alpha_i'(x_i)\}|\int_{\mathbb{R}^{2k}}\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}I(u_i,X_i)\right]E\left[\prod_{i\notin\mathcal{A}}I(u_i,X_i)\right]du_1\ldots du_{2k}$$

$$= \prod_{i=1}^{2k}|\max\{\alpha_i'(x_i)\}|\sum_{\mathcal{A}}(-1)^{\text{card}(\mathcal{A})}E\left[\prod_{i\in\mathcal{A}}X_i\right]E\left[\prod_{i\notin\mathcal{A}}X_i\right].$$

□

If $k = 1$, then

$$\text{Cov}(\alpha_1(X_1),\alpha_2(X_2)) \le |\max\{\alpha_1'(x_1)\}\max\{\alpha_2'(x_2)\}|\text{Cov}(X_1,X_2). \tag{21}$$

Example 3. Let X, Y be uniform random variables in the interval $(0, 1)$, PQD with joint distribution function $H(x,y) = \min(x,y)^{\theta}(xy)^{1-\theta}; \theta \in [0,1]$ (see Cuadras and Augé, 1981). Let us consider the functions of bounded variation $\alpha(x) = (x-1/2)^2$, $\beta(y) = (y-1/2)^2$ with derivatives $\alpha'(x) = 2(x-1/2)$, $\beta'(y) = 2(y-1/2)$. Then

$$\begin{aligned}\text{Cov}(X,Y) &= \int_{I^2}(H(x,y)-F(x)G(y))dxdy \\ &= \int_{I^2}[\min(x,y)^{\theta}(xy)^{1-\theta}-xy]dxdy = \frac{\theta}{4(4-\theta)},\end{aligned} \tag{22}$$

and

$$\begin{aligned}\text{Cov}(\alpha(X),\beta(Y)) &= \int_{I^2}[H(x,y)-F(x)G(y)]d\alpha(x)d\beta(y) \\ &= 4\int_{I^2}[\min(x,y)^{\theta}(xy)^{1-\theta}-xy](x-1/2)(y-1/2)dxdy \\ &= \frac{8}{3(6-\theta)}+\frac{1}{(4-\theta)}-\frac{10}{3(5-\theta)}-\frac{1}{36}.\end{aligned} \tag{23}$$

Since $\max\{\alpha'(x)\} = \max\{\beta'(y)\} = 1$, when replacing (22) and (23) in (21), we get

$$\frac{8}{3(6-\theta)}+\frac{1}{(4-\theta)}-\frac{10}{3(5-\theta)}-\frac{1}{36} \le \frac{\theta}{4(4-\theta)}.$$

Simplifying, we obtain the following inequality

$$-\frac{2\theta(8-\theta)}{9(6-\theta)(5-\theta)} \le 0,$$

valid for all $\theta \in [0,1]$.

Corollary 5. *Let* $\mathbf{X} = (X_1, \dots, X_{2k})$ *and* $\mathbf{X}^* = (X_1^*, \dots, X_{2k}^*)$ *be two POD random vectors with finite variance, independent and identically distributed. Then*

$$E\left[\prod_{j=1}^{2k}(e^{t_jX_j} - e^{t_jX_j^*})\right] \le \prod_{j=1}^{2k}|t_j| \times E\left[\prod_{j=1}^{2k}(X_j - X_j^*)\right]. \tag{24}$$

Proof. Let $\alpha_j(x_j) = e^{t_jx_j}$, $1 \le j \le 2k$, the derivative $\alpha_j'(x_j) = t_je^{t_jx_j}$ exists. From (18) we obtain

$$E\left[\prod_{j=1}^{2k}(e^{t_jX_j} - e^{t_jX_j^*})\right] = \sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{j\in\mathcal{A}}e^{t_jX_j}\right]E\left[\prod_{j\notin\mathcal{A}}e^{t_jX_j}\right]$$

$$= \int_{\mathbb{R}^{2k}}\prod_{j=1}^{2k}\{t_je^{t_ju_j}\}\sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{j\in\mathcal{A}}I(u_j, X_j)\right]E\left[\prod_{j\notin\mathcal{A}}I(u_j, X_j)\right]du_1\dots du_{2k}$$

$$\le \prod_{j=1}^{2k}|t_j|\int_{\mathbb{R}^{2k}}\sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{j\in\mathcal{A}}I(u_j, X_j)\right]E\left[\prod_{j\notin\mathcal{A}}I(u_j, X_j)\right]du_1\dots du_{2k}$$

$$\le \prod_{j=1}^{2k}|t_j|\sum_{\mathcal{A}}(-1)^{\mathrm{card}(\mathcal{A})}E\left[\prod_{j\in\mathcal{A}}X_j\right]E\left[\prod_{i\notin\mathcal{A}}X_j\right] = \prod_{j=1}^{2k}|t_j| \times E\left[\prod_{j=1}^{2k}(X_j - X_j^*)\right].$$

□

In particular, taking $k = 1$ in (24), we get

$$\begin{aligned}
E\left[\prod_{j=1}^{2}(e^{t_jX_j} - e^{t_jX_j^*})\right] &= E\big[(e^{t_1X_1} - e^{t_1X_1^*})(e^{t_2X_2} - e^{t_2X_2^*})\big] \\
&= 2\left\{E\left[e^{(t_1X_1+t_2X_2)}\right] - E\left[e^{t_1X_1}\right]E\left[e^{t_2X_2}\right]\right\} \\
&= 2\mathrm{Cov}(e^{t_1X_1}, e^{t_2X_2}) \le 2|t_1||t_2|E\left[(X_1 - X_1^*)(X_2 - X_2^*)\right] \\
&= 2|t_1t_2|\{E[X_1X_2] - E[X_1]E[X_2]\} = 2|t_1t_2|\mathrm{Cov}(X_1, X_2).
\end{aligned}$$

Therefore,

$$\mathrm{Cov}\left(e^{t_1X_1}, e^{t_2X_2}\right) \le |t_1t_2|\mathrm{Cov}(X_1, X_2).$$

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