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An identity involving invariant polynomials of matrix arguments

Daya K. Nagar[a], Arjun K. Gupta[b,*]

[a] *Departamento de Matemáticas, Universidad de Antioquia, Medellín, A. A. 1226, Colombia*
[b] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403-0221, USA*

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Abstract

The purpose of the present paper is to establish an identity involving invariant polynomials of two matrix arguments. This identity is a generalization of a well known identity that gives evaluation of the Gauss hypergeometric function when the argument matrix is identity. Applications of the identity derived in this article are also given.

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1. Introduction

Distributional results of random matrices are often derived in terms of functions of matrix arguments. Constantine [1] gave the power series representation of hypergeometric functions of matrix arguments in series involving zonal polynomials. The theory of zonal polynomials was developed in a series of papers by James and Constantine. For applications and properties of zonal polynomials and functions of matrix arguments the reader is referred to [2–5]. Davis [6,7] defined a class of invariant polynomials with two matrix arguments extending the zonal polynomials. These invariant polynomials facilitated multivariate analysts to derive distributions of latent roots of the non-central Wishart matrix, the non-central quadratic form and the doubly non-central F-matrix that they were unable to obtain using zonal polynomials.

* Corresponding author. Tel.: +1 419 372 2636; fax: +1 419 372 6092.
E-mail address: gupta@bgnet.bgsu.edu (A.K. Gupta).

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In the present paper, in Section 2, we will establish an identity involving invariant polynomials of two matrix arguments. The result is derived by equating two different solutions of an integral involving zonal polynomials and the Gauss hypergeometric function. The first solution conjectured by Subrahmaniam [2] and proved by Kabe [8] is in terms of zonal polynomials. The second solution, derived in this article, is in series involving invariant polynomials of two matrix arguments. Finally, using this identity and several results on invariant polynomials, two infinite series have been expressed as simple elementary functions.

2. Main result

Before establishing the identity we will introduce some notations. We adhere to standard notations. See, for example, [1,9,6,7,10]. Throughout, κ, λ, ϕ and ρ are partitions of the nonnegative integers $k, \ell, f = k+\ell$ and r, respectively. The zonal polynomial of the symmetric $m \times m$ matrix X corresponding to the partition κ will be denoted by $C_\kappa(X)$. The invariant polynomial of $m \times m$ symmetric matrix arguments X and Y will be denoted by $C_\phi^{\kappa,\lambda}(X, Y)$. Some of the properties and results on invariant polynomials are given below:

$$C_\phi^{\kappa,\lambda}(X, X) = \theta_\phi^{\kappa,\lambda} C_\phi(X) \tag{2.1}$$

where

$$\theta_\phi^{\kappa,\lambda} = \frac{C_\phi^{\kappa,\lambda}(I_m, I_m)}{C_\phi(I_m)},$$

$$C_\phi^{\kappa,\lambda}(X, I_m) = \theta_\phi^{\kappa,\lambda} \frac{C_\phi(I_m, I_m) C_\kappa(X)}{C_\phi(I_m)}, \tag{2.2}$$

$$C_\kappa^{\kappa,0}(X, Y) \equiv C_\kappa(X),$$

$$C_\kappa(X) C_\lambda(Y) = \sum_{\phi \in \kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda} \theta_\phi^{\kappa,\lambda}(X, Y), \tag{2.3}$$

where $\phi \in \kappa \cdot \lambda$ denotes that irreducible representation of $Gl(m, R)$, the group of $m \times m$ real invertible matrices, indexed by 2ϕ, appears in the decomposition of the tensor product $2\kappa \otimes 2\lambda$ of the irreducible representation indexed by 2κ and 2λ, and

$$\int_{0<R<I_m} \det(R)^{t-(m+1)/2} \det(I_m - R)^{u-(m+1)/2} C_\phi^{\kappa,\lambda}(R, I_m - R) \mathrm{d}R$$
$$= \frac{\Gamma_m(t, \kappa) \Gamma_m(u, \lambda)}{\Gamma_m(t+u, \phi)} \theta_\phi^{\kappa,\lambda} C_\phi(I_m). \tag{2.4}$$

In the above expression, $\Gamma_m(a, \rho)$ is defined by

$$\Gamma_m(a, \rho) = (a)_\rho \Gamma_m(a), \tag{2.5}$$

with

$$(a)_\rho = \prod_{j=1}^m \Gamma\left(a - \frac{j-1}{2}\right)_{r_j}, \tag{2.6}$$

$$(a)_k = a(a+1)\cdots(a+k-1), k = 1, 2, \ldots \quad \text{and} \quad (a)_0 = 1 \tag{2.7}$$

where $\rho = (r_1, \ldots, r_m), r_1 \geq \cdots \geq r_m \geq 0, r_1 + \cdots + r_m = r$. Note that $\Gamma_m(a, 0) = \Gamma_m(a)$, which is a multivariate gamma function given by

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{j=1}^{m} \Gamma\left[a - \frac{j-1}{2}\right], \qquad \mathrm{Re}(a) > \frac{m-1}{2}. \tag{2.8}$$

Now, consider the following integral involving the Gauss hypergeometric function of matrix argument:

$$f(Z) = \int_{0<X<I_m} \det(X)^{d-(m+1)/2} \det(I_m - X)^{\sigma-(m+1)/2} C_\lambda(Z(I_m - X))_2F_1(a, b; d; X)\mathrm{d}X \tag{2.9}$$

where λ denotes the partition $\lambda = (\ell_1, \ldots, \ell_m)$, $\ell_1 \geq \cdots \geq \ell_m \geq 0$, $\ell_1 + \cdots + \ell_m = \ell$, and $C_\lambda(X)$ is the zonal polynomial of the symmetric $m \times m$ matrix X corresponding to the partition λ. The Gauss hypergeometric function of matrix argument is defined by

$${}_2F_1(a, b; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(X)}{k!} \tag{2.10}$$

where a, b, c are arbitrary complex numbers, $X(m \times m)$ is a complex symmetric matrix and $\sum_\kappa$ denotes summation over all partitions κ. Conditions for convergence of this series are available in the literature.

It can easily be seen that for any $H \in O(m)$, $f(Z) = f(HZH')$. Thus integrating $f(HZH')$ over the orthogonal group, $O(m)$, we obtain

$$f(Z) = \frac{f(I_m)C_\lambda(Z)}{C_\lambda(I_m)}.$$

Subrahmaniam [2] conjectured that

$$f(I_m) = \frac{\Gamma_m(d)\Gamma_m(\sigma, \lambda)\Gamma_m(d + \sigma - a - b, \lambda)}{\Gamma_m(d + \sigma - a, \lambda)\Gamma_m(d + \sigma - b, \lambda)} C_\lambda(I_m). \tag{2.11}$$

Kabe [8] proved this conjecture using certain identities and integrals involving functions of matrix argument. Here we will obtain the solution by expanding ${}_2F_1$ and using the definition of invariant polynomial.

Expanding ${}_2F_1$ in series form, and using the result (2.3), we obtain

$$\begin{aligned} C_\lambda(I_m - X)_2F_1(a, b; d; X) &= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(d)_\kappa k!} C_\lambda(I_m - X) C_\kappa(X) \\ &= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(d)_\kappa k!} \sum_{\phi \in \kappa \cdot \lambda} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X, I_m - X). \end{aligned} \tag{2.12}$$

Now, substituting $Z = I$ and (2.12) in (2.9), and integrating out X, we obtain

$$\begin{aligned} f(I_m) = &\sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(d)_\kappa k!} \sum_{\phi \in \kappa \cdot \lambda} \theta_\phi^{\kappa,\lambda} \\ &\times \int_{0<X<I_m} \det(X)^{d-(m+1)/2} \det(I_m - X)^{\sigma-(m+1)/2} C_\phi^{\kappa,\lambda}(X, I_m - X)\mathrm{d}X \end{aligned}$$

$$= \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(d)_\kappa k!} \sum_{\phi \in \kappa \cdot \lambda} \left(\theta_\phi^{\kappa,\lambda}\right)^2 \frac{\Gamma_m(d,\kappa)\Gamma_m(\sigma,\lambda)}{\Gamma_m(d+\sigma,\phi)} C_\phi(I_m)$$

$$= \frac{\Gamma_m(d)\Gamma_m(\sigma,\lambda)}{\Gamma_m(d+\sigma)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\phi \in \kappa \cdot \lambda} \frac{(a)_\kappa (b)_\kappa}{(d+\sigma)_\phi k!} \left(\theta_\phi^{\kappa,\lambda}\right)^2 C_\phi(I_m) \tag{2.13}$$

where the last step has been obtained by using (2.4). Equating (2.11) and (2.13), we get the following result:

$$\sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\phi \in \kappa \cdot \lambda} \frac{(a)_\kappa (b)_\kappa}{(c)_\phi k!} \left(\theta_\phi^{\kappa,\lambda}\right)^2 C_\phi(I_m) = \frac{\Gamma_m(c)\Gamma_m(c-a-b,\lambda)}{\Gamma_m(c-a,\lambda)\Gamma_m(c-b,\lambda)} C_\lambda(I_m) \tag{2.14}$$

where $\mathrm{Re}(c-a) > (m-1)/2 - \ell_m$, $\mathrm{Re}(c-b) > (m-1)/2 - \ell_m$ and $c = d + \sigma$. Note that for $\lambda = 0$, the left-hand side reduces to a Gauss hypergeometric function and we get the following well known result as a corollary of the above identity:

$${}_2F_1(a, b; c; I_m) = \frac{\Gamma_m(c)\Gamma_m(c-a-b)}{\Gamma_m(c-a)\Gamma_m(c-b)}. \tag{2.15}$$

Thus (2.14) can be considered a generalization of (2.15). The identity (2.14) can be used to express several infinite series involving invariant polynomials in terms of elementary functions. Multiplying (2.14) by $(c-a)_\lambda (c-b)_\lambda / (c-a-b)_\lambda \ell!$ and summing over λ and ℓ, we obtain

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\lambda} \sum_{\kappa} \sum_{\phi \in \kappa \cdot \lambda} \frac{(c-a)_\lambda (c-b)_\lambda}{(c-a-b)_\lambda \ell!} \frac{(a)_\kappa (b)_\kappa}{(c)_\phi k!} \frac{[C_\phi^{\kappa,\lambda}(I_m, I_m)]^2}{C_\phi(I_m)}$$
$$= \frac{\Gamma_m(c)\Gamma_m(c-a-b)}{\Gamma_m(c-a)\Gamma_m(c-b)} \sum_{\ell=0}^{\infty} \sum_{\lambda} \frac{C_\lambda(I_m)}{\ell!} = \frac{\Gamma_m(c)\Gamma_m(c-a-b)}{\Gamma_m(c-a)\Gamma_m(c-b)} \exp(m).$$

Further, multiplying (2.14) by $(c-a)_\lambda / \ell!$ and summing over λ and ℓ, we arrive at

$$\sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{\lambda} \sum_{\kappa} \sum_{\phi \in \kappa \cdot \lambda} \frac{(c-a)_\lambda}{\ell!} \frac{(a)_\kappa (b)_\kappa}{(c)_\phi k!} \frac{[C_\phi^{\kappa,\lambda}(I_m, I_m)]^2}{C_\phi(I_m)}$$
$$= \frac{\Gamma_m(c)\Gamma_m(c-a-b)}{\Gamma_m(c-a)\Gamma_m(c-b)} {}_1F_1(c-a-b; c-b; I_m).$$

Finally, it may be remarked here that a number of results can be obtained by suitably multiplying (2.14) and summing over λ and ℓ.

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