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A GENERALIZATION OF DIRICHLET TYPE 1 DISTRIBUTION

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Abstract: In this article, we give a generalization of the Dirichlet type 1 distribution. This generalization is based on the Lauricella's type B hypergeometric function. We also study several of its properties.

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1. Introduction

The random variables $U_1, \ldots, U_n$ are said to have a Dirichlet type 1 distribution with parameters $a_1, \ldots, a_n; a_{n+1}$, denoted by $(U_1, \ldots, U_n) \sim \text{D1}(a_1, \ldots, a_n; a_{n+1})$, if their joint probability density function (p.d.f.) is given by

$$\frac{\Gamma(\sum_{i=1}^{n+1} a_i)}{\prod_{i=1}^{n+1} \Gamma(a_i)} \prod_{i=1}^{n} u_i^{a_i-1} \left(1 - \sum_{i=1}^{n} u_i\right)^{a_{n+1}-1},$$

$$u_i > 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} u_i < 1. \quad (1.1)$$

The Dirichlet type 1 distribution is primarily used as a conjugate prior distribution for the multinomial distribution parameters. Dirichlet random variables are good candidates for random weights since their beautiful stochastic representation in terms of gamma random variables and their sum is automatically

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one, which has applications in Bayes bootstrap method. The Dirichlet type 1 distribution has been studied extensively, for example, see Kotz, Balakrishnan and Johnson [1], and Gupta and Nagar [2].

In this article, we give a generalization of the Dirichlet type 1 distribution. This generalization is based on the Lauricella type B hypergeometric function and thus will be called Dirichlet-Lauricella type B distribution. In Section 2, we give definitions of Lauricella type B and type D hypergeometric functions. We define Dirichlet-Lauricella type B distribution in Section 3. Section 4, deals with several properties such as marginal densities and joint moment.

2. Preliminaries

The Pochhammer symbol $(a)_n$ is defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$ and $(a)_0 = 1$. The Appell-Lauricella hypergeometric functions of several variable are generalizations of the classical hypergeometric function of one variable. Appell's name is usually associated to the two variable case, whereas the case of more variables is usually ascribed to Lauricella. In this section, we define the Lauricella hypergeometric functions $F_B^{(n)}$ and $F_D^{(n)}$ of several variables. For further results and properties of these functions the reader is referred to Srivastava and Karlsson [8], and Prudnikov, Brychkov and Marichev [7, Sec. 7.2.4]. The Lauricella hypergeometric functions $F_B^{(n)}$ and $F_D^{(n)}$ are defined in multiple series as

$$F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c;z_1,\ldots,z_n)$$
$$= \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{(a_1)_{j_1}\cdots(a_n)_{j_n}(b_1)_{j_1}\cdots(b_n)_{j_n}}{(c)_{j_1+\cdots+j_n}} \frac{z_1^{j_1}\cdots z_n^{j_n}}{j_1!\cdots j_n!}, \ \max\{|z_1|,\ldots,|z_n|\} < 1 \tag{2.1}$$

and

$$F_D^{(n)}(a,b_1,\ldots,b_n;c;z_1,\ldots,z_n)$$
$$= \sum_{j_1,\ldots,j_n=0}^{\infty} \frac{(a)_{j_1+\cdots+j_n}(b_1)_{j_1}\cdots(b_n)_{j_n}}{(c)_{j_1+\cdots+j_n}} \frac{z_1^{j_1}\cdots z_n^{j_n}}{j_1!\cdots j_n!}, \quad \max\{|z_1|,\ldots,|z_n|\} < 1, \tag{2.2}$$

respectively.

If $n = 2$, then these functions reduce to Appell hypergeometric functions F_3 and F_1, respectively. For $n = 1$, they reduce to the Gauss hypergeometric function ${}_2F_1$.

By writing $(c)_{j_1+\cdots+j_n} = (c)_{j_{r+1}+\cdots+j_n}(c+j_{r+1}+\cdots+j_n)_{j_1+\cdots+j_r}$, $1 \le r \le n$, in (2.1), the Appell hypergeometric function $F_B^{(n)}$ can also be expressed as

$$\begin{aligned} &F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c;z_1,\ldots,z_n)\\ &\quad = \sum_{j_{r+1},\ldots,j_n=0}^{\infty} \frac{(a_{r+1})_{j_{r+1}}\cdots(a_n)_{j_n}(b_{r+1})_{j_{r+1}}\cdots(b_n)_{j_n}}{(c)_{j_{r+1}+\cdots+j_n}} \frac{z_{r+1}^{j_{r+1}}\cdots z_n^{j_n}}{j_{r+1}!\cdots j_n!}\\ &\qquad \times F_B^{(r)}(a_1,\ldots,a_r,b_1,\ldots,b_r;c+j_{r+1}+\cdots+j_n;z_1,\ldots,z_r). \qquad (2.3)\end{aligned}$$

The integral representation of $F_B^{(n)}$ is given by

$$\begin{aligned} &F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c;z_1,\ldots,z_n) = \qquad (2.4)\\ &\frac{\Gamma(c)}{\prod_{i=1}^n\Gamma(a_i)\Gamma(c-\sum_{i=1}^n a_i)} \underset{\substack{u_1>0,\ldots,u_n>0\\ 1-\sum_{i=1}^n u_i>0}}{\int\cdots\int} \frac{\prod_{i=1}^n u_i^{a_i-1}(1-\sum_{i=1}^n u_i)^{c-\sum_{i=1}^n a_i-1}}{\prod_{i=1}^n(1-z_iu_i)^{b_i}} \prod_{i=1}^n \mathrm{d}u_i,\end{aligned}$$

where $\mathrm{Re}(a_i) > 0$, $i = 1,\ldots,n$, $\mathrm{Re}(c-a_1-\cdots-a_n) > 0$ and $|\arg(1-z_i)| < \pi$, $i = 1,\ldots,n$.

The Lauricalla hypergeometric function F_D has integral representation

$$F_D^{(n)}(a,b_1,\ldots,b_n;c;z_1,\ldots,z_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{\prod_{i=1}^n(1-uz_i)^{b_i}}\mathrm{d}u, \quad (2.5)$$

where $\mathrm{Re}(c) > \mathrm{Re}(a) > 0$ and $|\arg(1-z_i)| < \pi$, $i = 1,\ldots,n$.

3. The Dirichlet-Lauricella Type B Distribution

We define the Dirichlet-Lauricella type B distribution as follows.

The random variables $U_1,\ldots,U_n$ are said to have a Dirichlet-Lauricella type B distribution with parameters $a_1,\ldots,a_n;c;d_1,\ldots,d_n;\theta_1,\ldots,\theta_n$, denoted by $(U_1,\ldots,U_n) \sim \mathrm{DLB}(a_1,\ldots,a_n;c;d_1,\ldots, d_n;\theta_1,\ldots,\theta_n)$, if their joint p.d.f. is given by

$$K_B \frac{\prod_{i=1}^n u_i^{a_i-1}(1-\sum_{i=1}^n u_i)^{c-\sum_{i=1}^n a_i-1}}{\prod_{i=1}^n(1-\theta_iu_i)^{d_i}}, \; u_i > 0,\; i = 1,\cdots,n,\; \sum_{i=1}^n u_i < 1, \quad (3.1)$$

where $a_1 > 0, \ldots, a_n > 0$, $c - \sum_{i=1}^n a_i > 0$ and $|\theta_i| < 1$, $i = 1, \ldots, n$. The normalizing constant K_B in (3.1) is given by

$$\begin{aligned} K_B^{-1} &= \underset{\substack{u_1>0,\ldots,u_n>0 \\ \sum_{i=1}^n u_i<1}}{\int\cdots\int} \frac{\prod_{i=1}^n u_i^{a_i-1}\left(1-\sum_{i=1}^n u_i\right)^{c-\sum_{i=1}^n a_i-1}}{\prod_{i=1}^n (1-\theta_i u_i)^{d_i}} \prod_{i=1}^n \mathrm{d}u_i \\ &= \frac{\prod_{i=1}^n \Gamma(a_i)\Gamma(c-\sum_{i=1}^n a_i)}{\Gamma(c)} F_B^{(n)}(a_1,\ldots,a_n,d_1,\ldots,d_n;c;\theta_1,\ldots,\theta_n), \end{aligned} \tag{3.2}$$

where the last line has been obtained by using (2.4). For $n = 1$, the above p.d.f. reduces to a generalized beta type 1 p.d.f. given by

$$\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(c-a_1)\,{}_2F_1(a_1,d_1;c;\theta_1)} \frac{u_1^{a_1-1}(1-u_1)^{c-a_1-1}}{(1-\theta_1 u_1)^{d_1}},\ 0 < u_1 < 1,$$

where $c > a_1 > 0$, $-1 < \theta_1 < 1$ and ${}_2F_1$ is the Gauss hypergeometric function. The generalized beta type 1 distribution has been studied by Nagar and Rada-Mora [5], Libby and Novic [3], Pham-Gia and Duong [6]. Further, for $n = 2$, the p.d.f. in (3.1) slides to a generalized bivariate beta type 1 p.d.f. defined by (Nadarajah [4]),

$$\begin{aligned} &\frac{\Gamma(c)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c-a_1-a_2)\,F_3(a_1,a_2,d_1,d_2;c;\theta_1,\theta_2)} \\ &\times \frac{u_1^{a_1-1}u_2^{a_2-1}(1-u_1-u_2)^{c-a_1-a_2-1}}{(1-\theta_1 u_1)^{d_1}(1-\theta_2 u_2)^{d_2}},\ u_1 > 0,\ u_2 > 0,\ u_1+u_2 < 1, \end{aligned}$$

where $a_1 > 0$, $a_2 > 0$, $c > a_1 + a_2$, $-1 < \theta_1 < 1$, $-1 < \theta_2 < 1$ and F_3 is the third hypergeometric function of Appell.

4. Properties

This section gives several properties of the Dirichlet-Lauricella type B distribution.

Consider the transformation $Z_i = (1 - \sum_{j=1}^s U_j)^{-1} U_i$, $i = s+1, \ldots, n$. Then, $u_i = (1 - \sum_{j=1}^s u_j) z_i$, $i = s+1, \ldots, n$ with the Jacobian $J(u_{s+1}, \ldots, u_n \to z_{s+1}, \ldots, z_n) = (1 - \sum_{j=1}^s u_j)^{n-s}$. Substituting appropriately in (3.1), the joint p.d.f. of $U_1, \ldots, U_s, Z_{s+1}, \ldots, Z_n$ is given by

$$K_B \frac{\prod_{i=1}^s u_i^{a_i-1}(1-\sum_{i=1}^s u_i)^{c-\sum_{i=1}^s a_i-1}}{\prod_{i=1}^s (1-\theta_i u_i)^{d_i}} \frac{\prod_{i=s+1}^n z_i^{a_i-1}(1-\sum_{i=s+1}^n z_i)^{c-\sum_{i=1}^n a_i-1}}{\prod_{i=s+1}^n [1-\theta_i z_i(1-\sum_{j=1}^s u_j)]^{d_i}}, \tag{4.1}$$

where $u_i > 0, i = 1, \dots, s, \sum_{i=1}^{s} u_i < 1, z_i > 0, i = s+1, \dots, n$ and $\sum_{i=s+1}^{n} z_i < 1$.

First, we find the marginal p.d.f. of $U_1, \dots, U_s$ by integrating out $z_{s+1}, \dots, z_n$ from the joint p.d.f. of $U_1, \dots, U_s, Z_{s+1}, \dots, Z_n$ as

$$K_B \frac{\prod_{i=1}^{s} u_i^{a_i-1}(1-\sum_{i=1}^{s} u_i)^{c-\sum_{i=1}^{s} a_i-1}}{\prod_{i=1}^{s}(1-\theta_i u_i)^{d_i}} \times \int \cdots \int_{\substack{z_{s+1}>0,\dots,z_n>0 \\ \sum_{i=s+1}^{n} z_i<1}} \frac{\prod_{i=s+1}^{n} z_i^{a_i-1}(1-\sum_{i=s+1}^{n} z_i)^{c-\sum_{i=1}^{n} a_i-1}}{\prod_{i=s+1}^{n}[1-\theta_i z_i(1-\sum_{j=1}^{s} u_j)]^{d_i}} \prod_{i=s+1}^{n} \mathrm{d}z_i. \tag{4.2}$$

Now, using the integral representation of $F_B^{(n)}$ given in (2.4), we derive the marginal p.d.f. of $U_1, \dots, U_s$ as

$$K_{B1} \frac{\prod_{i=1}^{s} u_i^{a_i-1} (1-\sum_{i=1}^{s} u_i)^{c-\sum_{i=1}^{s} a_i-1}}{\prod_{i=1}^{s}(1-\theta_i u_i)^{d_i}} F_B^{(n-s)}\Bigg(a_{s+1}, \dots, a_n, d_{s+1}, \dots, d_n; c-\sum_{i=1}^{s} a_i; \theta_{s+1}\left(1-\sum_{i=1}^{s} u_i\right), \dots, \theta_n\left(1-\sum_{i=1}^{s} u_i\right)\Bigg), \tag{4.3}$$

where $u_i > 0$, $i = 1, \dots, s$, $\sum_{i=1}^{s} u_i < 1$ and

$$K_{B1}^{-1} = \frac{\prod_{i=1}^{s} \Gamma(a_i)\Gamma(c-\sum_{i=1}^{s} a_i)}{\Gamma(c)} F_B^{(n)}(a_1, \dots, a_n, d_1, \dots, d_n; c; \theta_1, \dots, \theta_n).$$

It is interesting to note that the marginal p.d.f. of $U_1, \dots, U_s$ does not belong to the Dirichlet-Lauricella type B family of distributions and differs by an additional factor containing the Lauricella hypergeometric function F_B.

Further, making the transformation $Z_i = (1-U_s)^{-1}U_i$, $i = 1, \dots, s-1$ with the Jacobian $J(u_1, \dots, u_{s-1} \to z_1, \dots, z_{s-1}) = (1-u_s)^{s-1}$ in (4.3), the joint p.d.f. of $Z_1, \dots, Z_{s-1}$ and U_s is derived as

$$K_{B1} \frac{u_s^{a_s-1}(1-u_s)^{c-a_s-1}}{(1-\theta_s u_s)} \frac{\prod_{i=1}^{s-1} z_i^{a_i-1}(1-\sum_{i=1}^{s-1} z_i)^{c-\sum_{i=1}^{s} a_i-1}}{\prod_{i=1}^{s-1}[1-\theta_i z_i(1-u_s)]^{d_i}} \times F_B^{(n-s)}\Bigg(a_{s+1}, \dots, a_n, d_{s+1}, \dots, d_n; c-\sum_{i=1}^{s} a_i; \theta_{s+1}(1-u_s)\left(1-\sum_{i=1}^{s-1} z_i\right), \dots, \theta_n(1-u_s)\left(1-\sum_{i=1}^{s-1} z_i\right)\Bigg).$$

where $0 < u_s < 1$, $z_i > 0$, $i = 1, \ldots, s-1$, and $\sum_{i=1}^{s-1} z_i < 1$. Now, expanding $F_B^{(n-s)}$ using (2.1) and integrating $u_1, \ldots, u_{s-1}$ by applying (2.4) in the above p.d.f., the marginal p.d.f. of U_s is derived as

$$K_{B2} \frac{u_s^{a_s-1}(1-u_s)^{c-a_s-1}}{(1-\theta_s u_s)} \sum_{j_{s+1},\ldots,j_n=0}^{\infty} \frac{(a_{s+1})_{j_{s+1}} \cdots (a_n)_{j_n} (d_{s+1})_{j_{s+1}} \cdots (d_n)_{j_n}}{(c-a_s)_{j_{s+1}+\cdots+j_n}}$$
$$\times \frac{\theta_{s+1}^{j_{s+1}} \cdots \theta_n^{j_n}}{j_{s+1}! \cdots j_n!} (1-u_s)^{j_{s+1}+\cdots+j_n} F_B^{(s-1)}(a_1, \ldots, a_{s-1}, d_1, \ldots, d_{s-1};$$
$$c + j_{s+1} + \cdots + j_n - a_s; \theta_1(1-u_s), \ldots, \theta_n(1-u_s)),$$

where

$$K_{B2}^{-1} = \frac{\Gamma(a_s)\Gamma(c-a_s)}{\Gamma(c)} F_B^{(n)}(a_1, \ldots, a_n, d_1, \ldots, d_n; c; \theta_1, \ldots, \theta_n).$$

Finally, rewriting the infinite series involving the Lauricella function by using (2.3), the p.d.f. of U_s is derived as

$$K_{B2} \frac{u_s^{a_s-1}(1-u_s)^{c-a_s-1}}{(1-\theta_s u_s)^{d_s}} F_B^{(n-1)}(a_1, \ldots, a_{s-1}, a_{s+1}, \ldots, a_n, d_1, \ldots, d_{s-1}, d_{s+1}, \ldots, d_n;$$
$$c - a_s; \theta_1(1-u_s), \ldots, \theta_{s-1}(1-u_s), \theta_{s+1}(1-u_s), \ldots, \theta_n(1-u_s)).$$

The marginal joint p.d.f. of $Z_{s+1}, \ldots, Z_n$ is given by

$$K_B \prod_{i=s+1}^{n} z_i^{a_i-1} \left(1 - \sum_{i=s+1}^{n} z_i\right)^{c-\sum_{i=1}^{n} a_i - 1}$$
$$\times \int_{\substack{u_1>0,\ldots,u_s>0 \\ \sum_{i=1}^{s} u_i<1}} \cdots \int \frac{\prod_{i=1}^{s} u_i^{a_i-1}(1-\sum_{i=1}^{s} u_i)^{c-\sum_{i=1}^{s} a_i - 1}}{\prod_{i=1}^{s}(1-\theta_i u_i)^{d_i} \prod_{i=s+1}^{n} \left[1-\theta_i z_i \left(1-\sum_{i=1}^{s} u_i\right)\right]^{d_i}} \prod_{i=1}^{s} \mathrm{d}u_i.$$

Now, writing

$$\prod_{i=s+1}^{n} \left[1 - \theta_i z_i \left(1 - \sum_{i=1}^{s} u_i\right)\right]^{-d_i}$$
$$= \sum_{j_{s+1},\ldots,j_n=0}^{\infty} \frac{(d_{s+1})_{j_{s+1}} \cdots (d_n)_{j_n} (\theta_{s+1} z_{s+1})^{j_{s+1}} \cdots (\theta_n z_n)^{j_n}}{j_{s+1}! \cdots j_n!} \left(1 - \sum_{i=1}^{s} u_i\right)^{j_{s+1}+\cdots+j_n}$$

and integrating $u_1, \ldots, u_{s-1}$ by applying (2.4) in the above p.d.f., the marginal p.d.f. of $Z_{s+1}, \ldots, Z_n$ is derived as

$$K_{B3} \prod_{i=s+1}^{n} z_i^{a_i-1} \left(1 - \sum_{i=s+1}^{n} z_i\right)^{c-\sum_{i=1}^{n} a_i - 1}$$
$$\times \sum_{j_{s+1},\ldots,j_n=0}^{\infty} \frac{(c - \sum_{i=1}^{s} a_i)_{j_{s+1}+\cdots+j_n} (d_{s+1})_{j_{s+1}} \cdots (d_n)_{j_n}}{(c)_{j_{s+1}+\cdots+j_n}} \frac{(\theta_{s+1} z_{s+1})^{j_{s+1}} \cdots (\theta_n z_n)^{j_n}}{j_{s+1}! \cdots j_n!}$$
$$\times F_B^{(s)}(a_1, \ldots, a_s, d_1, \ldots, d_s; c + j_{s+1} + \cdots + j_n; \theta_1, \ldots, \theta_s),$$

where

$$K_{B3}^{-1} = \frac{\prod_{i=s+1}^{n} \Gamma(a_i) \Gamma(c - \sum_{i=1}^{n} a_i)}{\Gamma(c - \sum_{i=1}^{s} a_i)} F_B^{(n)}(a_1, \ldots, a_n, d_1, \ldots, d_n; c; \theta_1, \ldots, \theta_n).$$

It is well known that if $(U_1, \ldots, U_n) \sim \mathrm{D1}(a_1, \ldots, a_n; c - \sum_{i=1}^{n} a_i)$, then

$$\left(\frac{U_1}{\sum_{i=1}^{n} U_i}, \ldots, \frac{U_{n-1}}{\sum_{i=1}^{n} U_i}\right) \sim \mathrm{D1}(a_1, \ldots; a_n)$$

and the sum $\sum_{i=1}^{n} U_i$ follows a beta type 1 distribution with parameters $\sum_{i=1}^{n} a_i$ and $c - \sum_{i=1}^{n} a_i$. In the next theorem, we derive similar result for the Dirichlet-Lauricells type B distribution.

Theorem 4.1. *Let $(U_1, \ldots, U_n) \sim \mathrm{DLB}(a_1, \ldots, a_n; c; d_1, \ldots, d_n; \theta_1, \ldots, \theta_n)$ and define $U = \sum_{i=1}^{n} U_i$ and $X_i = U_i/U$, $i = 1, \ldots, n-1$. Then, the joint p.d.f. of $X_1, \ldots, X_{n-1}$ is given as*

$$K_B \frac{\Gamma(\sum_{i=1}^{n} a_i) \Gamma(c - \sum_{i=1}^{n} a_i)}{\Gamma(c)} \prod_{i=1}^{n-1} x_i^{a_i-1} \left(1 - \sum_{i=1}^{n-1} x_i\right)^{a_n-1}$$
$$\times F_D^{(n)}\left(\sum_{i=1}^{n} a_i, d_1, \ldots, d_n; c; \theta_1 x_1, \ldots, \theta_{n-1} x_{n-1}, \theta_n \left(1 - \sum_{i=1}^{n-1} x_i\right)\right),$$

where $x_i > 0$, $i = 1, \ldots, n-1$, $\sum_{i=1}^{n-1} x_i < 1$. Further, the p.d.f. of U is derived as

$$K_B \frac{\prod_{i=1}^{n} \Gamma(a_i)}{\Gamma(\sum_{i=1}^{n} a_i)} u^{\sum_{i=1}^{n} a_i - 1} (1-u)^{c - \sum_{i=1}^{n} a_i - 1}$$
$$\times F_B^{(n)}\left(a_1, \ldots, a_n, d_1, \ldots, d_n; \sum_{i=1}^{n} a_i; \theta_1 u, \ldots, \theta_n u\right), \quad 0 < u < 1.$$

Proof. Substituting $u_i = ux_i$, $i = 1, \dots, n-1$ and $u_n = u(1 - \sum_{i=1}^{n-1} x_i)$ with the Jacobian $J(u_1, \dots, u_n \to x_1, \dots, x_{n-1}, u) = u^{n-1}$ in the joint p.d.f. of $(U_1, \dots, U_n)$, we get the joint p.d.f. of $(X_1, \dots, X_{n-1})$ and U as

$$K_B \prod_{i=1}^{n-1} x_i^{a_i-1} \left(1 - \sum_{i=1}^{n-1} x_i\right)^{a_n-1} \frac{u^{\sum_{i=1}^n a_i - 1}(1-u)^{c-\sum_{i=1}^n a_i - 1}}{\prod_{i=1}^{n-1}(1-\theta_i x_i u)^{d_i}[1-\theta_n u(1-\sum_{i=1}^{n-1} x_i)]^{d_n}},$$

where $x_i > 0$, $i = 1, \dots, n-1$, $\sum_{i=1}^{n-1} x_i < 1$ and $0 < u < 1$. Now, integrating u in the above expression using the integral representation of F_D given in (2.5), we get the desired result. Further, writing

$$\left[1 - \theta_n u \left(1 - \sum_{i=1}^{n-1} x_i\right)\right]^{-d_n} = \sum_{j=0}^{\infty} \frac{(d_n)_j (\theta_n u)^j}{j!} \left(1 - \sum_{i=1}^{n-1} x_i\right)^j$$

the joint p.d.f. of $(X_1, \dots, X_{n-1})$ and U is rewritten as

$$K_B u^{\sum_{i=1}^n a_i - 1}(1-u)^{c-\sum_{i=1}^n a_i - 1} \sum_{j=0}^{\infty} \frac{(d_n)_j (\theta_n u)^j}{j!} \frac{\prod_{i=1}^{n-1} x_i^{a_i-1}\left(1 - \sum_{i=1}^{n-1} x_i\right)^{a_n+j-1}}{\prod_{i=1}^{n-1}\left(1-\theta_i x_i u\right)^{d_i}}.$$

Now, integrating $x_1, \dots, x_{n-1}$ in the above expression by using (2.4), we get the marginal p.d.f. of U as

$$K_B u^{\sum_{i=1}^n a_i - 1}(1-u)^{c-\sum_{i=1}^n a_i - 1} \sum_{j=0}^{\infty} \frac{(d_n)_j (\theta_n u)^j}{j!} \frac{\prod_{i=1}^{n-1}\Gamma(a_i)\Gamma(a_n+j)}{\Gamma(\sum_{i=1}^n a_i + j)}$$
$$\times F_B^{(n-1)}\left(a_1, \dots, a_{n-1}, d_1, \dots, d_{n-1}; \sum_{i=1}^n a_i + j; \theta_1 u, \dots, \theta_{n-1} u\right), \; 0 < u < 1.$$

Finally, re-writing the infinite series by applying (2.3), we get the desired result. □

By definition, the product moments are obtained as

$$E\left[\prod_{i=1}^n U_i^{r_i}\right] = \int \cdots \int_{\substack{u_1>0,\dots,u_n>0 \\ \sum_{i=1}^n u_i < 1}} \frac{\prod_{i=1}^n u_i^{a_i+r_i-1}\left(1-\sum_{i=1}^n u_i\right)^{c-\sum_{i=1}^n a_i - 1}}{\prod_{i=1}^n (1-\theta_i u_i)^{d_i}} \prod_{i=1}^n du_i$$
$$= \frac{\Gamma(c)\prod_{i=1}^n \Gamma(a_i+r_i)}{\Gamma(c+r)\prod_{i=1}^n \Gamma(a_i)}$$

$$\times \frac{F_B^{(n)}(a_1 + r_1, \ldots, a_n + r_n, d_1, \ldots, d_n; c + r; \theta_1, \ldots, \theta_n)}{F_B^{(n)}(a_1, \ldots, a_n, d_1, \ldots, d_n; c; \theta_1, \ldots, \theta_n)},$$

where $r = \sum_{i=1}^n r_i$, $\mathrm{Re}(a_i + r_i) > 0$, $i = 1, \ldots, n$ and $\mathrm{Re}(c + r) > 0$. Further

$$E\left[\left(1 - \sum_{i=1}^n U_i\right)^h\right] = \int_{\substack{u_1>0,\ldots,u_n>0\\ \sum_{i=1}^n u_i<1}} \cdots \int \frac{\prod_{i=1}^n u_i^{a_i-1}\,(1 - \sum_{i=1}^n u_i)^{c+h-\sum_{i=1}^n a_i - 1}}{\prod_{i=1}^n (1 - \theta_i u_i)^{d_i}} \prod_{i=1}^n \mathrm{d}u_i$$

$$= \frac{\Gamma(c)\Gamma(c + h - \sum_{i=1}^n a_i)}{\Gamma(c + h)\Gamma(c - \sum_{i=1}^n a_i)} \times \frac{F_B^{(n)}(a_1, \ldots, a_n, d_1, \ldots, d_n; c + h; \theta_1, \ldots, \theta_n)}{F_B^{(n)}(a_1, \ldots, a_n, d_1, \ldots, d_n; c; \theta_1, \ldots, \theta_n)}.$$

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