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SOME INTEGER PARTITIONS INDUCED BY ORBITS OF DYNKIN TYPE

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Abstract

A categorification in the sense of Ringel and Fahr is given to the sequences A016116 and A000034 in the OEIS by using τ-orbits in the Auslander-Reiten quiver of some Dynkin algebras.

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1. Introduction

This paper deals with the categorification of integer sequences which is a recent line of investigation introduced by Ringel and Fahr. According to them, a categorification of an integer sequence means to consider instead of numbers in the sequence invariants of suitable objects in a given category. These procedures allowed them to obtain a categorification of Fibonacci numbers by using, in particular, the structure of the Auslander-Reiten quiver of the 3-Kronecker quiver [9, 10].

We also recall that categorifications of generalized non-crossing partitions (in the sense of Kreweras) of a given finite set have been studied by Hubery, Krause, Ingalls, Ringel and Thomas amongst others mathematicians [13]. It is worth noting that Catalan numbers can be interpreted as the number of cluster variables of a Dynkin algebra of type A_n, and also as $a(A_n)$ or $t(A_n)$, i.e., the number of antichains or support-tilting modules in $\operatorname{mod} A_n$, respectively. Besides, categorifications of different integer sequences have been obtained by Cañadas et al. by using the number of indecomposable representations of some suitable posets, tiled orders and Kronecker modules in [4-7, 17].

In order to obtain categorifications of the sequences A016116 and A000034 in the OEIS, we count integer partitions induced by τ-orbits in the Auslander-Reiten quiver of some algebras of Dynkin type.

This paper is organized as follows: In Section 2, we recall a combinatorial definition of the Auslander-Reiten quiver of a Dynkin algebra, the definition of τ-orbit and Coxeter number is introduced in this section as well. In Section 3, we count τ-orbit partitions of type A_n, an algorithm to compute length of τ-orbit partitions of type A_n is also introduced in this section by using tiled orders. In Section 4, we count τ-orbit partitions of type D_n. In Section 5, we count τ-orbit partitions of type E_6, E_7 and E_8. Finally, in Section 6, we give some examples of τ-orbit partitions.

2. Preliminaries

2.1. The Auslander-Reiten quiver of a Dynkin algebra and the Coxeter number

In this section, we recall ideas of Riedtmann [16] and Oh [19, 20] to give a combinatorial characterization of the Auslander-Reiten quiver of algebras of Dynkin type.

If Δ is a Dynkin diagram of finite representation type, then a function $\xi : \Delta_0 \to \mathbb{Z}$ such that $\xi_j = \xi_i - 1$ for any edge $\alpha : i \to j \in \Delta_1$ is called a *height function*. Note that, two arbitrary height functions differ by a constant.

The set $\mathbb{Z}\Delta = \{(i, p) \in \Delta_0 \times \mathbb{Z} : p - \xi_i \in 2\mathbb{Z}\}$ is associated to Δ, where $\Delta_0 = \{1, 2, ..., n\}$ in such a way that $\mathbb{Z}\Delta$ can be seen as a quiver with edges of the form $(i, p) \to (j, p+1)$ and $(j, q) \to (i, q+1)$ for any pair of connected vertices $i, j \in \Delta_0$. $\mathbb{Z}\Delta$ is called the *quiver of repetition* of Δ. Note that $\mathbb{Z}\Delta$ does not depend on the orientation of the quiver Δ. It is well-known that the quiver $\mathbb{Z}\Delta$ itself has an isomorphism with the AR-quiver of $D^b(\mathbb{C}Q)$ [12]. According to Oh [19], the injective module $I(i)$ is located at the vertex (i, ξ_i) of $\mathbb{Z}\Delta$.

We denote by $S_i\Delta$ the quiver obtained from Δ by reversing the orientation of each arrow that ends at i or starts at i. A reduced expression $w = S_{i_1}S_{i_2}\cdots S_{i_l}$ of an element $w \in W_0$ is called *adapted to* Δ if i_k is source of $S_{i_{k-1}}\cdots S_{i_2}S_{i_1}\Delta$ for all $1 \leq k \leq l$, where W_0 is the group of Weyl associated to Δ.

We denote $\Pi_n = \{\alpha_i : i \in \Delta_0\}$ the set of simple roots and $\Phi_n(\Phi_n^+, \Phi_n^-)$ the set of (positive, negative) roots.

Let $\hat{\Phi}_n := \Phi_n^+ \times \mathbb{Z}$. For $i \in \Delta_0$, we define

$$\gamma_i = \sum_{j \in B(i)} \alpha_j \quad \text{and} \quad \theta_i = \sum_{j \in C(i)} \alpha_j,$$

where $B(i)(C(i))$ is the set of vertices $j \in Q_0$ such that there exists a path from *j* to *i* (from *i* to *j*).

By Gabriel's theorem, the map $[M] \to \underline{\dim}[M]$ gives a bijection from the set $\operatorname{Ind}\Delta$ of indecomposable modules over the path algebra $k\Delta$ (Δ is of finite representation type) to Φ_n^+. Then $\operatorname{Ind}\Delta = \{M(\beta) : \beta \in \Phi_n^+ \text{ and } \underline{\dim}(M(\beta)) = \beta\}$.

Following Hernandez and Leclere [15], the bijection $\phi : \mathbb{Z}Q \to \hat{\Phi}_n$ defined by $M(\beta)[m] \mapsto (\beta, m)$ is described combinatorially as follows:

(1) $\phi(i, \xi_i) = (\gamma_i, 0)$.

(2) For $\beta \in \Phi_n^+$ with $\phi(i, p) = (\beta, m)$ we have:

(a) If $\tau(\beta) \in \Phi_n^+$, we set $\phi(i, p-2) = (\tau(\beta), m)$.

(b) If $\tau(\beta) \in \Phi_n^-$, we set $\phi(i, p-2) = (-\tau(\beta), m-1)$.

(c) If $\tau^{-1}(\beta) \in \Phi_n^+$, we set $\phi(i, p+2) = (\tau^{-1}(\beta), m)$.

(d) If $\tau^{-1}(\beta) \in \Phi_n^-$, we set $\phi(i, p+2) = (-\tau^{-1}(\beta), m+1)$.

The *Auslander-Reiten quiver* (AR quiver) Γ_Δ is the full subquiver of $\mathbb{Z}\Delta$ whose set of vertices is $\phi^{-1}(\Phi_n^+ \times \{0\})$. Here the vertex $\phi^{-1}(\beta, 0)$ corresponds to the indecomposable module $M(\beta)$ in Ind Q and the arrow $\phi^{-1}(\beta, 0) \to \phi^{-1}(\beta', 0)$ is associated to an *irreducible morphism* from $M(\beta)$ to $M(\beta')$. In particular, the injective envelope $I(i)$ of S_i corresponds to the vertex $\phi^{-1}(\gamma_i, 0)$ and the projective cover $P(i)$ of S_i is associated to the vertex $\phi^{-1}(\theta_{i^*}, 0)$.

It is well known that

$$\theta_i = \tau^{m_{i^*}}(\gamma_{i^*}), \text{ where } m_i = \max\{k \geq 0 : \tau^k(\gamma_i) \in \Phi_n^+\} \tag{1}$$

and $* : \Delta_0 \to \Delta_0$ is the *involution induced by* w_0 (the unique longest element in W_0) given by $w_0\alpha_i = -\alpha_{i^*}$ [2].

For $\beta \in \Phi_n^+$ with $\tau(\beta) \in \Phi_n^+$, we set $\tau M(\beta) := M(\tau(\beta))$. In the AR quiver Γ_Δ, this map τ is called the *Auslander-Reiten translation* (AR translation). The dimension vector is an *additive function* on Γ_Δ with respect to the map τ; that is, for each vertex $X \in \Gamma_\Delta$ such that $X = \phi^{-1}(\beta, 0)$ and $\tau(\beta) \in \Phi_n^+$,

$$\underline{\dim}\, X + \underline{\dim}\, \tau X = \sum_{Z \in X^-} \underline{\dim}\, Z.$$

Here X^- is the set of vertices $Z \in \Gamma_\Delta$ such that there exists an arrow from Z to X. It is also well-known that for $\beta \in \Phi_n^+$, $\tau(\beta) \in \Phi_n^-$ if and only if $\beta = \theta_i$ for some $i \in \Delta_0$.

The following description is one of the characterizations of Γ_Δ inside $\mathbb{Z}\Delta$:

$$\phi^{-1}(\Phi_n^+ \times \{0\}) = \{(i,\ p) \in \mathbb{Z}\Delta : \xi_i - 2m_i \leq p \leq \xi_i\}.$$

In [11], Gabriel introduced the *Nakayama permutation* ϑ of $\mathbb{Z}\Delta$ which is defined as follows:

$$\vartheta(i,\ p) = (i^*,\ p + h_n - 2), \tag{2}$$

where h_n is the *Coxeter number* associated to Δ.

We also recall that the well known Nakayama functor is related to the Nakayama permutation by the formula

$$\mathscr{V}(P(i)) = I(i). \tag{3}$$

Note that, the formula (3) allows to conclude that $\vartheta(\phi^{-1}(\underline{\dim}P(i), 0)) = \phi^{-1}(\underline{\dim}I(i), 0)$, therefore $\vartheta(\phi^{-1}(\tau^{m_{i^*}}(\gamma_{i^*}), 0)) = \phi^{-1}(\gamma_i, 0) = (i, \xi_i)$ as a consequence of formula (1). Since $\tau^{m_{i^*}}(\gamma_{i^*}) \in \Phi_n^+$, we obtain

$$(i, \xi_i) = \vartheta(i^*, \xi_{i^*} - 2m_{i^*}) = (i^*, \xi_{i^*} - 2m_{i^*} + h_n - 2).$$

That is

$$\xi_i = \xi_{i^*} - 2m_{i^*} + h_n - 2. \tag{4}$$

This formula allows us to know m_{i^*} by using the involution $*$ associated to the Dynkin diagram and a suitable height function.

If $P(i)$ is the projective cover of the simple representation S_i in the category rep Δ, then the set $\mathscr{O}_i = \{M \in \text{Ind}Q : \tau^k P(i) = M \text{ for some } k \in \mathbb{Z}\}$ is called the τ-*orbit* of $P(i)$. According to Schiffler [18], each τ-orbit in an AR quiver of Dynkin type contains exactly one projective representation and one injective representation.

It is well known that the injective envelope $I(i^*)$ of the simple representation S_{i^*} belongs to $\mathscr{O}_i$. Formulas (1) and (4) allow us to obtain the cardinality of the τ-orbit $\mathscr{O}_i$ as follows:

$$|\mathscr{O}_i| = m_{i^*} + 1. \tag{5}$$

2.2. Partitions induced by orbits

A *partition* λ of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_t$ such that $n = \sum_{i=1}^{t} \lambda_i$.

We recall that according to Dlab and Ringel [8], the possible values for the global dimension of the endomorphism ring of a generator-cogenerator depend on the maximal length of the τ-orbits. Let us stress that the maximal length d of the τ-orbits depends not only on the Dynkin type of the diagram Δ, but on the given orientation. In fact, the following (optimal) bounds $d' \leq d \leq d''$ for the length of τ-orbits are well known (for the simply laced cases):

Dynkin type	A_n	D_{2m-1}	D_{2m}	E_6	E_7	E_8
d'	$\left\lceil \frac{n}{2} \right\rceil$	$2m-2$	$2m-1$	6	9	15
d''	n	$2m-1$	$2m-1$	8	9	15

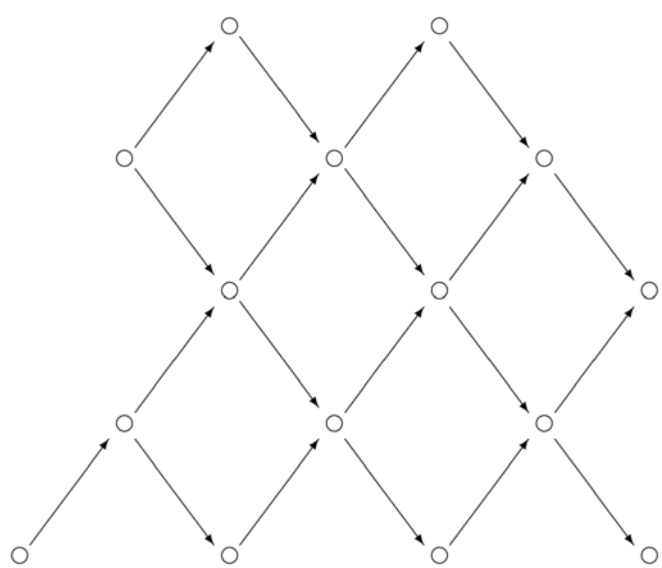
In this paper, we use the length of the τ-orbits in the Auslander-Reiten quiver of algebras of Dynkin type to define suitable integer partitions. For the sake of clarity, we use an example to introduce these partitions whose parts are given by the cardinality of corresponding τ-orbits:

Let us consider the following orientation of $\Delta = A_5$:

$$1 \leftarrow 2 \leftarrow 3 \rightarrow 4 \leftarrow 5.$$

Note that the Auslander-Reiten quiver Γ_Δ has the following shape [18]:

Γ_Δ:



In this case, a partition $\lambda_\Delta = (4, 3, 3, 3, 2)$ is associated to the integer number 15. Note that each part in λ_Δ is given by the cardinality of a τ-orbit ordered in the natural way.

We let $P_\tau(L)$ denote the size of the following set:

$$P_\tau(\Delta) = |\{\lambda_\Delta : \text{where } \lambda_\Delta \text{ is a } \tau\text{-orbit partition}\}|.$$

The main aim of this paper is to find out formulas for $P_\tau(\Delta)$, where Δ is an oriented Dynkin diagram.

3. τ-orbit Partitions

3.1. Cardinality of τ-orbits of type A_n

In this section, we introduce a map which can help us to calculate the cardinality of a τ-orbit in an easy way.

Definition 1. Let Δ be a quiver of type A_n whose vertices and edges are numbering as follows: $1 \overset{\alpha_1}{\text{---}} 2 \overset{\alpha_2}{\text{---}} \cdots \overset{\alpha_{n-2}}{\text{---}} n-1 \overset{\alpha_{n-1}}{\text{---}} n$. An arrow $\alpha_i \in \Delta_1$ is called a *right arrow* (*left arrow*) if $i \overset{\alpha_i}{\to} i+1$ $(i+1 \overset{\alpha_i}{\to} i)$. Henceforth, we call vector $v_Q = \sum_{k=1}^{n} a_k e_k \in \mathbb{Z}^n$ an *orientation vector*. In this case, $a_1 = 0$, $a_k = \sum_{i=1}^{k-1} v(\alpha_i)$ for $k \geq 2$ and $v(\alpha_i) = \begin{cases} 1 & \text{if } \alpha_i \text{ is a right arrow;} \\ 0 & \text{if } \alpha_i \text{ is a left arrow.} \end{cases}$

We recall that for any fixed *n* for a Dynkin diagram $\Delta = A_n$, there exists an associated involution [2],

$$* : \Delta_0 \to \Delta_0 \text{ such that } i \mapsto i^* = n - (i-1) \tag{6}$$

induced by w_0 (the unique longest element in W_0) given by $w_0\alpha_i = -\alpha_{i^*}$.

Theorem 2. *Let Δ be a quiver of type A_n such that $\Delta_0 = \{1, ..., n\}$ with orientation vector of the form $v_Q = \sum_{k=1}^{n} a_k e_k$. Then*

$$|\mathscr{O}_i| = a_{i^*} - a_i + i.$$

Proof. For i fixed, let ξ_i be such that $\xi_i = a_i - a_i^{op}$, where $v_{Q^{op}} = \sum_{k=1}^{n} a_k^{op} e_k$ and Q^{op}. It is easy to see that the function $\xi : Q_0 \to \mathbb{Z}$ with $\xi(i) = \xi_i$ is a height function. According to the formula (4), we have that

$$a_{i^*} - a_i - 2m_{i^*} + h_n - 2 = a_{i^*}^{ap} - a_i^{ap}$$

since for any n, the Coxeter number of A_n is $h_n = n + 1 = i + i^*$. Thus,

$$a_{i^*} - a_i = a_{i^*}^{op} - a_i^{op} + 2m_{i^*} - (i + i^*) + 2$$

since

$$a_{i^*}^{op} - a_i^{op} + i = i^* - (a_{i^*} - a_i),$$

we obtain

$$a_{i^*} - a_i + i = m_{i^*} + 1 = |\mathscr{O}_i|$$

and with this identity, we are done. □

3.2. Applications to tiled orders

A field T is said to be of *discrete norm* or *discrete valuation* if it is endowed with a surjective map

$$\nu : T \to \mathbb{Z} \cup \{\infty\},$$

which satisfies the following conditions:

(1) $\nu(x) = \infty$ if and only if $x = 0$,

(2) $\nu(xy) = \nu(x) + \nu(y)$,

(3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

We let $\mathbb{O}$ denote, the *normalization ring* of the field T, such that

$$\mathbb{O} = \{x \in T \mid \nu(x) \geq 0\}.$$

An element $\pi \in \mathbb{O}$ such that $\nu(\pi) = 1$ is a *prime element* of $\mathbb{O}$. For each $x \in \mathbb{O}$, we have that $x \in \mathbb{O}$ if and only if $x = \varepsilon\pi^m$, for some $m \geq 0$ and $\varepsilon \in \mathbb{O}^*$. Moreover, $x \in T$ if and only if $x = \varepsilon\pi^m$ for some $m \in \mathbb{Z}$ and $\varepsilon \in \mathbb{O}^*$.

Ring $\mathbb{O}$ is such that $\mathbb{O} \supset \pi\mathbb{O}$, where $\pi\mathbb{O}$ is the unique maximal ideal, therefore ideals of $\mathbb{O}$ generate a chain of the form

$$\mathbb{O} \supset \pi\mathbb{O} \supset \pi^2\mathbb{O} \supset \cdots \supset \pi^m\mathbb{O} \supset \cdots.$$

A *tiled order or semimaximal ring* Λ is a subring of the matrix algebra $T^{n\times n}$ with the form

$$\Lambda = \sum_{i,\, j=1}^{n} e_{ij}\pi^{\lambda_{ij}}\mathbb{O} = \begin{bmatrix} \mathbb{O} & \pi^{\lambda_{12}}\mathbb{O} & \cdots & \pi^{\lambda_{1n}}\mathbb{O} \\ \pi^{\lambda_{21}}\mathbb{O} & \mathbb{O} & \cdots & \pi^{\lambda_{2n}}\mathbb{O} \\ \vdots & \vdots & \vdots & \vdots \\ \pi^{\lambda_{n1}}\mathbb{O} & \pi^{\lambda_{n2}}\mathbb{O} & \cdots & \mathbb{O} \end{bmatrix}.$$

Λ consists of all matrices whose entries *ij* belong to $\pi^{\lambda_{ij}}\mathbb{O}$, in this case the $e_{ij} \in T^{n\times n}$ are unit matrices such that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ ($\delta_{jk} = 1$, if $j = k$, $\delta_{jk} = 0$ otherwise). Numbers λ_{ij} are integers which satisfy the following conditions:

(1) $\lambda_{ii} = 0$, for each *i*,

(2) $\lambda_{ij} + \lambda_{jk} \geq \lambda_{ik}$ for all *i*, *j*, *k*.

An order Λ is said to be *Morita reduced* or *reduced* if it satisfies the additional condition:

$$\lambda_{ij} + \lambda_{ji} > 0, \text{ for each } i \neq j.$$

In such a case, projective modules are pairwise non-isomorphic, that is, in the decomposition of $\Lambda = P_1 \oplus P_2 \oplus \cdots \oplus P_n$ via projective modules (i.e., the rows of Λ) all indecomposable projective summands are pairwise not isomorphic, i.e., $P_i \not\simeq P_j$ if $i \neq j$.

In this paper, we assume that tiled orders are reduced.

We denote $\Lambda = (\lambda_{ij})_{i,\, j=1\dots n}$, note that $\Lambda \subset T^{n \times n} = Q = \Lambda \otimes_{\mathbb{O}} T$, where Q is the rational hull of Λ, $\operatorname{Rad} Q = 0$ and Λ has a unique right simple T-module (up to isomorphism) denoted $S_R = (T, T, \dots, T) = \sum_{i=1}^{n} e_i T$, $\{e_i \mid 1 \leq i \leq n\}$ is the standard basis such that $e_i e_{jk} = \delta_{ij} e_k$. We assume the notation $S_L = (T, T, \dots, T)^t$ for left modules.

The main problem in this case consists of describing all finitely generated torsionless Λ-modules which are called *admissible modules*.

A Λ-admissible right module (not null) is said to be *irreducible* if it is a submodule of the unique simple module (up to isomorphism). For instance, any indecomposable projective module P_i is irreducible. Thus,

$$P_i = (\pi^{\lambda_{i1}}\mathbb{O}, \pi^{\lambda_{i2}}\mathbb{O}, \dots, \pi^{\lambda_{in}}\mathbb{O})$$

is a finitely generated irreducible Λ-module without $\mathbb{O}$-torsion.

Any irreducible right Λ-module A has the form

$$A = (\pi^{\alpha_1}\mathbb{O}, \pi^{\alpha_2}\mathbb{O}, \dots, \pi^{\alpha_n}\mathbb{O}),$$

where $\alpha_i + \lambda_{ij} \geq \alpha_j$, $\alpha_i \in \mathbb{Z}$, $1 \leq i \leq n$. If A is a left module, then we have $\lambda_{ij} + \alpha_j \geq \alpha_i$.

Henceforth, we denote a right (left) module A in the form $A = (\alpha_1, \alpha_2, \dots, \alpha_n)((\alpha_1, \alpha_2, \dots, \alpha_n)^t$, respectively).

Note that, $A \simeq A'$ if and only if $\alpha_i = \alpha'_i + k$, for some $k \in \mathbb{Z}$ and any $1 \leq i \leq n$.

The following result characterizes isomorphic orders via matrix problems [3, 14]:

Theorem 3. *Two orders* Λ *and* Λ' *are isomorphic if the corresponding exponent matrices* λ_{ij} *and* λ'_{ij} *can be turned into each other with the help of the following admissible t-transformations*:

(1) *To add an integer n to each entry of a given row i and simultaneously subtract n to each entry of the column i.*

(2) *To transpose simultaneously rows i and j and columns i and j.*

Let $\mathcal{O}$ be a discrete valuation ring with prime element π. Then we define the reduced tiled order $A = \sum_{k=1}^{\left\lceil \frac{n}{2} \right\rceil} A_k$, where A_k is the matrix ring

$$\begin{bmatrix} \mathcal{O} & \pi^k \mathcal{O} \\ \pi^{k^*} \mathcal{O} & \mathcal{O} \end{bmatrix}$$

whose adjacency matrix is

$$\Lambda_k = (\lambda_{ij}^k) = \begin{bmatrix} 0 & k \\ k^* & 0 \end{bmatrix}.$$

Theorems 2 and 3 define the following algorithm to calculate the cardinality of τ-orbits of type A_n:

Algorithm 1. Given a diagram of type A_n

Input: n: = number of vertices v: = an orientation vector of the form [“r”, “l”, “r”, $\cdots$, “r”, “l”] of length $n - 1$, where symbols “l” or “r” in the ith coordinate denotes, respectively, the orientation $((1, 0)$ or $(-1, 0))$ of the corresponding edge α_i.

Output: Cardinality of the τ-orbits: $|\mathcal{O}_k|$ for each $k = 1, 2, \ldots, n$.

Step 1: Find out the vector orientation $v_Q = \sum_{k=1}^{n} a_k e_k \in \mathbb{Z}^n$.

Step 2: The admissible transformation $a_{k^*} - a_k$ on row and column one is applied to the matrix Λ_k to obtain an isomorphic tiled order $\Lambda'_k = (\lambda'^{k}_{ij})$.

Step 3: Define $|\mathscr{O}_k| = \lambda'^{k}_{12}$ and $|\mathscr{O}_{k^*}| = \lambda'^{k}_{21}$ for each $k = 1, 2, \ldots, \left\lceil \frac{n}{2} \right\rceil$.

Remark 4. If Q and Q' are isomorphic quivers, then so are the corresponding partitions. Note that the reciprocal statement is in general not true. As an example, let Q and Q' be the oriented quivers $Q := 1 \to 2 \leftarrow 3 \to 4$ and $Q' := 1 \to 2 \to 3 \leftarrow 4$. $v_Q = (0, 0, 1, 1)$ and $v_{Q'} = (0, 0, 0, 1)$ where according to the algorithm the corresponding isomorphic tiled orders have the following forms (taking into account admissible transformations on vectors v_Q and $v_{Q'}$):

$$\Lambda_1 = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix},$$

$$\Lambda_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix},$$

$$\Lambda_1 \sim \Lambda'_1 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda_2 \sim \Lambda'_2 = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix},$$

$$\Lambda_1 \sim \Lambda''_1 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} \quad \text{and} \quad \Lambda_2 \sim \Lambda''_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}.$$

Thus quivers Q and Q' have associated the same partition $\lambda_Q = \lambda_{Q'} = (3, 3, 2, 2)$ of 10. However, Q and Q' are not isomorphic.

3.3. Counting τ-orbit partitions of type A_n

We note that the length of a τ-orbit defined in a natural way partitions the number $t_n = \frac{n(n+1)}{2}$ of indecomposable representations of an algebra of Dynkin type A_n into n parts. Since for fixed n, each indecomposable projective module in the Auslander-Reiten quiver of such algebra has solely one τ-orbit. In this case, such partitions λ_i are defined in such a way that $\lambda_i = |\mathscr{O}_{\sigma(i)}|$, where σ is a permutation satisfying the condition

$$|\mathscr{O}_{\sigma(1)}| \geq \cdots \geq |\mathscr{O}_{\sigma(n)}|.$$

Proposition 5. *An integer partition* $\lambda = (\lambda_1, ..., \lambda_n)$ *of an integer n is a* τ*-orbit of type* A_n *if and only if it satisfies the following conditions*:

(a) $\lambda_i + \lambda_{i^*} = h_n$ *for each integer* $1 \leq i \leq n$,

(b) $0 \leq \lambda_i - \lambda_{i+1} \leq 1$ *for each integer* $1 \leq i \leq n-1$.

Proof. Suppose that $\lambda_\Delta = (\lambda_1, ..., \lambda_n)$ is the τ-orbit partition induced by the quiver Δ of type A_n, without loss of generality, we can suppose that $\lambda_i = |\mathscr{O}_i|$ for each integer $1 \leq i \leq n$. We suppose that $v_Q = \sum_{k=1}^{n} a_k e_k$ is the orientation vector associated to the quiver then Theorem 2 allows us to establish that

$$\lambda_i + \lambda_{i^*} = |\mathscr{O}_i| + |\mathscr{O}_{i^*}| = (a_{i^*} - a_i + i) + (a_i - a_{i^*} + i^*) = i + i^* = h_n.$$

Further,

$$\lambda_i - \lambda_{i+1} = |\mathscr{O}_i| - |\mathscr{O}_{i+1}| = (a_{i^*} - a_i + i) - (a_{(i+1)^*} - a_{i+1} + i + 1)$$

$$= (a_{i+1} - a_i) + (a_{i^*} - a_{i^*-1}) - 1.$$

Thus, $a_{i+1} - a_i \leq 1$ and $a_{i^*} - a_{i^*-1} \leq 1$, therefore $\lambda_i - \lambda_{i+1} \leq 1$. Since $\lambda_i \geq \lambda_{i+1}$, it follows that λ_Δ satisfies the conditions (a) and (b).

Now suppose that $\lambda = (\lambda_1, ..., \lambda_n)$ satisfies (a) and (b), let a_n be such that $a_n = n - \lambda_n$, we define $a_{i-1} = a_i - 1$ for each integer $\left\lceil \frac{n}{2} \right\rceil \leq i \leq n$ and $a_i = a_{i^*} - v_{i^*}$ for each integer $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$ with $u_i = i - \lambda_i$. Given the vector $v = \sum_{k=1}^{n} a_k e_k$, we define a quiver Q with orientation vector $v_Q = v$.

We set $Q_0 = \{1, ..., n\}$ and $Q_1 = \{\alpha_1, ..., \alpha_{n-1}\}$, where α_i is an arrow with vertices i and $i+1$ oriented according to the identities $a_{i+1} - a_i = 0$ $(a_{i+1} - a_i = 1)$. By construction, it is easy to see that $v_Q = v$. Finally, we see that the partition induced by the quiver Q is $\lambda_Q = \lambda$. Moreover, for any integer $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil - 1$, we have

$$|\mathcal{O}_i| = a_{i^*} - a_i + i = u_{i^*} + 1 = i^* - \lambda_{i^*} + i = h_n - \lambda_{i^*} = \lambda_i$$

and with this identity, we are done. □

If $P_\tau(A_n)$ is the number of τ-orbit partitions of type A_n of the triangular number t_n, then we have the following result.

Theorem 6. $P_\tau(A_n) = 2^{\left\lceil \frac{n}{2} \right\rceil - 1}$.

Proof. Firstly, let us to consider the case n odd, that is, $n = 2k - 1$ for some $k \geq 1$. We proceed by induction on k. If $k = 2$, then it is easy to see that there are two τ-orbit partitions which are $\lambda = (3, 2, 1)$ and $\lambda' = (2, 2, 2)$ of type A_3, since $2^{\left\lceil \frac{n}{2} \right\rceil - 1} = 2^{k-1} = 2$, the theorem holds in this case. Now we suppose that the assertion is true for any $s < k$ and j such that $2s - 1 = j \leq 2k - 1$, we will see that the theorem is true for $N = n + 2 = 2(k+1) - 1$.

It is clear that a τ-orbit partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n, \lambda_{n+1}, \lambda_{n+2})$ of the triangular number t_N arises from the τ-orbit partition $\overline{\lambda} = (\lambda_2 - 1, ..., \lambda_n - 1, \lambda_{n+1} - 1)$ of t_n.

On the other hand, if $\lambda' = (\lambda'_1, \lambda'_2, ..., \lambda'_n)$ is an integer partition of t_n and λ is an integer partition of t_N such that $\overline{\lambda} = \lambda'$, then

$$\lambda = (\lambda_1, \lambda'_1 + 1, ..., \lambda'_n + 1, \lambda_{n+2})$$

via condition (b), $0 \leq \lambda_1 - \lambda'_1 + 1 \leq 1$ therefore $\lambda_1 = \lambda'_1 + 2$ or $\lambda_1 = \lambda'_1 + 1$. If $\lambda_1 = \lambda'_1 + 2$, then condition (a) implies $\lambda_1 + \lambda_{n+2} = n + 3$, then $\lambda_{n+2} = \lambda'_n$. Thus

$$\lambda = (\lambda'_1 + 2, \lambda'_1 + 1, \ldots, \lambda'_n + 1, \lambda'_n). \tag{7}$$

On the other hand, if $\lambda_1 = \lambda'_1 + 1$, then via condition (a), we obtain $\lambda_1 + \lambda_{n+2} = \lambda'_1 + 1 + \lambda_{n+2} = n + 3$ therefore $\lambda_{n+2} = n - (\lambda'_1 - 1) + 1 = \lambda'_{1^*} + 1 = \lambda'_n + 1$, then

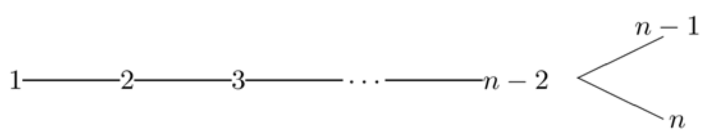
$$\lambda = (\lambda'_1 + 1, \lambda'_1 + 1, \lambda'_2 + 1, \ldots, \lambda'_n + 1, \lambda'_n + 1). \tag{8}$$

Thus, each integer partition of t_n gives place to two partitions of t_{n+2}, that is, $P_\tau(A_{n+2}) = 2P_\tau(n) = 2(2^{\lceil \frac{n}{2} \rceil - 1}) = 2(2^{(k-1)}) = 2^{(k+1)-1} = 2^{\lceil \frac{N}{2} \rceil - 1}$. Since the proof for the case n even follows in a similar way, we are done. □

Remark 7. The integer sequence $(2^{\lceil \frac{n}{2} \rceil - 1})_{n \geq 1}$ is encoded as A016116 in the On-line Encyclopedia of Integer Sequences [22].

4. τ-orbits Partitions of Type D_n

For the rest of this section, a Dynkin diagram D_n with n vertices has the following numbering:



for $n \geq 4$

According to Oh, the Coxeter number h_n is $2n - 2$ whereas the involution $*$ induced by $w_0 \in W_0$ is given by $i^* = i$ for $1 \leq i \leq n - 2$. Note that $(n-1)^* = n - 1$, $n^* = n$ if n is even, whereas $(n-1)^* = n$, $n^* = n - 1$ if n is odd [2, 20].

Theorem 8. *If n is even, then the τ-orbit partition of D_n-type is $\lambda = (n-1, n-1, \ldots, n-1)$. Whereas, if n is odd, then the τ-orbit partitions of D_n-type are either*

$$\lambda = (n-1, n-1, \ldots, n-1) \text{ or } \lambda = (n, n-1, \ldots, n-1, n-2).$$

Proof. Suppose that n is an even number, since $\xi_i = \xi_{i^*} - 2m_{i^*} + h_n - 2$ and $i^* = i$, we have that $2m_{i^*} = h_n - 2$, that is, $m_i = n-2$. Therefore, $|\mathcal{O}_i| = n-1$. According to this fact, we see that to each quiver D_n with n even, there is an associated partition $\lambda = (n-1, n-1, \ldots, n-1)$ which does not depend on orientation. On the other hand, if n is odd, then we have that if $i \neq n, n-1$, then $i^* = i$, thus $m_i = n-2$, that is $|\mathcal{O}_i| = n-1$ for $1 \leq i \leq n-2$. It remains to compute $|\mathcal{O}_{n-1}|$ and $|\mathcal{O}_n|$. Since $n^* = n-1$ and $(n-1)^* = n$, $\xi_n = \xi_{n-1} - 2m_{n-1} + h_n - 2$ and $\xi_{n-1} = \xi_n - 2m_n + h_n - 2$. Definition of height function allows us to conclude that $|\xi_{n-1} - \xi_n| = 2$ or 0. Indeed, if $\alpha_{n-2} : n-2 \to n-1$ and $\alpha_{n-1} : n-2 \to n$ or if α_{n-2} : $n-2 \leftarrow n-1$ and $\alpha_{n-1} : n-2 \leftarrow n$, then $\xi_{n-1} = \xi_{n-2} - 1$ and $\xi_n = \xi_{n-2} - 1$ or $\xi_{n-2} = \xi_{n-2} - 1$ and $\xi_{n-2} = \xi_n - 1$. Therefore, $\xi_{n-1} = \xi_n$, moreover, if $\alpha_{n-2} : n-2 \to n-1$ and $\alpha_{n-1} : n \to n-2$ or if α_{n-2} : $n-2 \leftarrow n-1$ and $\alpha_{n-1} : n \leftarrow n-2$, then $\xi_n - \xi_{n-1} = 2$ or $\xi_n - \xi_{n-1} = -2$.

Now, if $|\xi_{n-1} - \xi_n| = 0$, then since $\xi_n = \xi_{n-1} - 2m_{n-1} + (2n-2) - 2$ and $\xi_{n-1} = \xi_n - 2m_n + (2n-2) - 2$, we conclude that $m_n = m_{n-1} = n-2$, that is, $|\mathcal{O}_{n-1}| = |\mathcal{O}_n| = n-1$ thus the τ-orbit partition induced is

$$\lambda = (n-1, \ldots, n-1).$$

Finally, if $|\xi_{n-1} - \xi_n| = 2$, then we take into account that $\xi_n = \xi_{n-1} - 2m_{n-1} + (2n-2) - 2$ and $\xi_{n-1} = \xi_n - 2m_n + (2n-2) - 2$ to observe that

$\xi_{n-1} - \xi_n = 2$ or -2 thus $m_n = n-1$ and $m_{n-1} = n-3$ or $m_n = n-3$ and $m_{n-1} = n-1$, that is, $|\mathcal{O}_{n-1}| = n$ and $|\mathcal{O}_n| = n-2$ or $|\mathcal{O}_{n-1}| = n-2$ and $|\mathcal{O}_n| = n$ therefore the τ-orbit partition induced has the form

$$\lambda = (n, n-1, \ldots, n-1, n-2).$$

We are done. □

If we let $P_\tau(D_n)$ denote the number of τ-orbit partitions of type $D_{n(n-1)}$, then we have the following result.

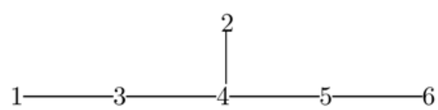
Corollary 9. $P_\tau(D_n) = 1 + \overline{n}_{\text{mod } 2}$.

Remark 10. The integer sequence $P_\tau(D_n)$ is encoded as A000034 in the OEIS [23].

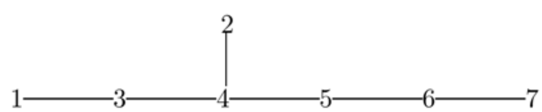
5. τ-orbit Partitions of Type E_6, E_7 and E_8

In this section, we define the following numbering for Dynkin diagrams of type E_6, E_7, and E_8.

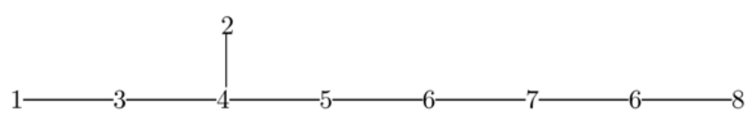
E_6



E_7



E_8



We recall that if h_n is the Coxeter number and w_0 is the unique longest element in the Weyl group W_0 associated to a Dynkin diagram such that

$w_0\alpha_i = -\alpha_{i^*}$, then it is possible to define an involution $*$ on the corresponding vertices. In cases E_6, E_7 and E_8 we have that:

Dynkin type	E_6	E_7	E_8
h_n	12	18	30
Involution $*$	$w_0\alpha_1 = -\alpha_6$, $w_0\alpha_2 = -\alpha_2$, $w_0\alpha_3 = -\alpha_5$, $w_0\alpha_4 = -\alpha_4$, $w_0\alpha_5 = -\alpha_3$, $w_0\alpha_6 = -\alpha_1$.	$w_0 = -1$	$w_0 = -1$

Theorem 11. $P_\tau(E_6) = 5$, $P_\tau(E_7) = P_\tau(E_8) = 1$.

Proof. Suppose that Δ is a quiver of type E and ξ is a height function defined on Δ_0.

When $\overline{\Delta} = E_7$ or E_8, it suffices to take into account the involution $*$ as the identity, therefore for any vertex i, we have $\xi_i = \xi_{i^*}$, and thus as a consequence of the identity (4),

$$2m_i = 2m_{i^*} = h_n - 2.$$

Thus, if Δ is a quiver of type E_7, then $|\mathcal{O}_i| = 9$, $1 \le i \le 7$ by (5), therefore the τ-orbit has the form $\lambda_\Delta = (9, 9, 9, 9, 9, 9, 9)$ which does not depend on orientation.

Analogously, if Δ is a quiver of type E_8, then the τ-orbit partition induced has the form $\lambda_\Delta = (15, 15, 15, 15, 15, 15, 15, 15)$ which does not depend on orientation.

On the other hand, if $\overline{\Delta} = E_6$, then we must compute cardinalities of τ-orbits independently. Note that, if $i = 2$ or $i = 4$, then $*$ is defined in such a way that $\xi_i = \xi_{i^*}$ therefore

$$2m_i = 2m_{i^*} = h_n - 2$$

by (4) which implies that $|\mathcal{O}_2| = |\mathcal{O}_4| = 6$.

Furthermore, if $i = 3$ or $i = 5$, we note that $3^* = 5$ and $5^* = 3$, moreover $|\xi_3 - \xi_5| = 0$ or 2, in the case $\xi_3 = \xi_5$, we obtain $|\mathcal{O}_3| = |\mathcal{O}_5| = 6$ as a consequence of formulas (4) and (5). If $|\xi_3 - \xi_5| = 2$, then we observe that $|\mathcal{O}_5| = 5$ and $|\mathcal{O}_3| = 7$ or $|\mathcal{O}_5| = 7$ and $|\mathcal{O}_3| = 5$.

When $i = 1$ or $i = 6$ implies $|\xi_1 - \xi_6| = 0$, 2 or 4, thus $\xi_1 = \xi_6$ implies $|\mathcal{O}_1| = |\mathcal{O}_6| = 6$, $|\xi_1 - \xi_6| = 2$ implies $|\mathcal{O}_1| = 5$ and $|\mathcal{O}_6| = 7$ or $|\mathcal{O}_1| = 7$ and $|\mathcal{O}_6| = 5$ and $|\xi_1 - \xi_6| = 4$ implies $|\mathcal{O}_1| = 8$ and $|\mathcal{O}_6| = 4$ or $|\mathcal{O}_1| = 4$ and $|\mathcal{O}_6| = 8$. Thus, $\lambda_\Delta = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$ is such that for any $1 \leq i \leq 6$, $\lambda_i = |\mathcal{O}_{\sigma(i)}|$, where σ is the permutation which satisfies the condition $|\mathcal{O}_{\sigma(1)}| \geq \cdots \geq |\mathcal{O}_{\sigma(n)}|$, $|\mathcal{O}_i| + |\mathcal{O}_{i^*}| = h_n = 12$, $|\mathcal{O}_1| \in \{4, 5, 6, 7, 8\}$, $|\mathcal{O}_2| = |\mathcal{O}_4| = 6$ and $|\mathcal{O}_3| \in \{5, 6, 7\}$. □

6. Appendix

6.1. List of τ-orbit partitions of type *A*

The following are examples of τ-orbit partitions of type A_n:

Dynkin diagram	τ-orbit partitions
A_1	(1)
A_2	(2, 1)
A_3	(3, 2, 1), (2, 2, 2)
A_4	(4, 3, 2, 1), (3, 3, 2, 2)
A_5	(5, 4, 3, 2, 1), (4, 4, 3, 2, 2), (4, 3, 3, 3, 2), (3, 3, 3, 3, 3)
A_6	(6, 5, 4, 3, 2, 1), (5, 5, 4, 3, 2, 2), (5, 4, 4, 3, 3, 2), (4, 4, 4, 3, 3, 3)
A_7	(7, 6, 5, 4, 3, 2, 1), (6, 6, 5, 4, 3, 2, 2), (6, 5, 5, 4, 3, 3, 2), (6, 5, 4, 4, 4, 3, 2) (5, 5, 5, 4, 3, 3, 3), (5, 5, 4, 4, 4, 3, 3), (5, 4, 4, 4, 4, 4, 3), (4, 4, 4, 4, 4, 4, 4)
A_8	(8, 7, 6, 5, 4, 3, 2, 1), (7, 7, 6, 5, 4, 3, 2, 2), (7, 6, 6, 5, 4, 3, 3, 2) (7, 6, 5, 5, 4, 4, 3, 2), (6, 6, 6, 5, 4, 3, 3, 3), (6, 6, 5, 5, 4, 4, 3, 3) (6, 5, 5, 5, 4, 4, 4, 3), (5, 5, 5, 5, 4, 4, 4, 4)

6.2. List of τ-orbit partitions of type E_6

The following are the τ-orbit partitions of type E_6:

- (6, 6, 6, 6, 6, 6),
- (7, 6, 6, 6, 6, 5),
- (7, 7, 6, 6, 5, 5),
- (8, 6, 6, 6, 6, 4),
- (8, 7, 6, 6, 5, 4).

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