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REPRESENTATION OF EQUIPPED POSETS TO GENERATE DELANNOY NUMBERS

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Abstract

In this paper, Delannoy numbers are interpreted as dimensions of suitable representations of some equipped posets induced by compositions of integer numbers.

1. Introduction

Delannoy numbers were introduced by Henri-Auguste Delannoy (1833-

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1915). He investigated the different moves on a chessboard and observed that such numbers arise by investigating the queen movement *la marche de la Reine* [6, 8, 9].

For integer numbers i and j, Delannoy numbers satisfy the recurrence relation:

$$d(i,\ j) = d(i-1,\ j) + d(i,\ j-1) + d(i-1,\ j-1), \tag{1}$$

$d(0,\ 0) = 0$, $d(i,\ j) = 0$ if $i < 0$ and $j < 0$.

The central Delannoy numbers

$$d(i,\ i) = \{1,\ 3,\ 13,\ 63,\ 321,\ 1683,\ 8989,\ \ldots\}$$

appear as the sequence A001850 in the OEIS.

According to Sulanke, very few combinatorial elements are known counted by these numbers. Actually, he describes in [8] 29 configurations which are counted by central Delannoy numbers.

On the other hand, we also recall that equipped posets were introduced and classified by Zabarilo and Zavadskij and their students [10-13]. Actually, the last Zavadskij's published work was devoted to a generalization of the theory of representation of this kind of posets [13].

The theory of representation of equipped posets arises as a generalization of the classical theory of representation of posets introduced and developed by Nazarova and Roiter and their students in Kiev [7]. Nowadays, we know that according to Bautista and Dorado, the theory of classification of such posets can be considered as a particular case of the theory of representation of the so called algebraically equipped posets [3].

In this paper, we interpret Delannoy numbers as dimensions of representations of some suitable equipped posets.

This paper is organized as follows: in Section 2, we define equipped posets. Section 3 describes an open problem in the theory of partitions posed by Andrews. In Section 4, we give a solution to the Andrews's problem regarding integer compositions, while Section 5 deals with the category of lattice representations. Finally, in Section 6, a categorification of Delannoy numbers is described by using dimensions of some lattice representations of equipped posets.

2. Equipped Posets

A poset $(\mathcal{P}, \leq)$ is called *equipped* if all the order relations between its points $x \leq y$ are separated into strong (denoted $x \trianglelefteq y$) and weak (denoted $x \preceq y$) in such a way that

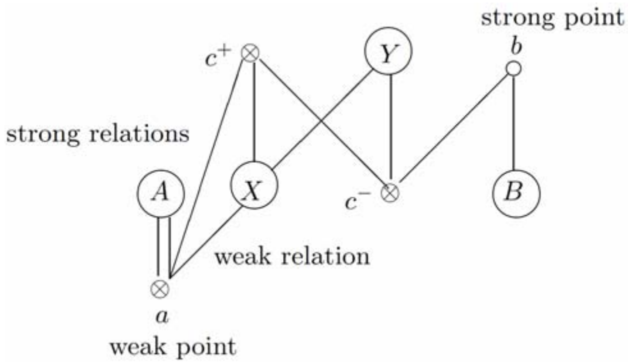
$$x \leq y \trianglelefteq z \text{ or } x \trianglelefteq y \leq z \text{ implies } x \trianglelefteq z, \tag{2}$$

i.e., a composition of a strong relation with any other relation is strong [5, 10-13].

We let $x \leq y$ denote an arbitrary relation in an equipped poset $(\mathcal{P}, \leq)$. The order $\leq$ on an equipped poset $\mathcal{P}$ gives rise to the relations $\prec$ and $\triangleleft$ of *strict inequality*: $x \prec y$ (respectively, $x \triangleleft y$) in $\mathcal{P}$ if and only if $x \preceq y$ (respectively, $x \trianglelefteq y$) and $x \neq y$.

A point $x \in \mathcal{P}$ is called *strong* (*weak*) if $x \trianglelefteq x$ (respectively, $x \preceq x$). These points are denoted $\circ$ (respectively, $\otimes$) in diagrams. We also denote $\mathcal{P}^{\circ} \subseteq \mathcal{P}$ (respectively, $\mathcal{P}^{\otimes} \subseteq \mathcal{P}$) the subset of strong points (respectively, weak points) of $\mathcal{P}$. If $\mathcal{P}^{\otimes} = \varnothing$, then the equipment is *trivial* and the poset $\mathcal{P}$ is ordinary.

The diagram of an equipped poset $(\mathcal{P}, \leq)$ may be obtained via its Hasse diagram (with strong $(\circ)$ and weak points $(\otimes)$). In this case, a new line is added to the line connecting two points $x, y \in \mathcal{P}$ with $x \triangleleft y$ if and only if such a relation cannot be deduced from any other relations in $\mathcal{P}$. The following figure is an example of this type of diagram:



$$a^{\nabla} = A; \quad a^{\curlyvee} = \{a, c^+\} + X + Y$$

Figure 1

If $\mathcal{P}$ is an equipped poset, then a *chain* $C = \{c_i \in \mathcal{P} | 1 \leq i \leq n, c_{i-1} < c_i$ if $i \geq 2\} \subseteq \mathcal{P}$ is a *weak chain* if and only if $c_{i-1} \prec c_i$ for each $i \geq 2$. If $c_1 \prec c_n$, then we say that C is a *completely weak chain*.

The *category of representations of an equipped poset* over a pair of fields (F, G) (where G is a quadratic extension of F) is defined as a system of the form

$$U = (U_0; U_x \,|\, x \in \mathcal{P}), \tag{3}$$

where U_0 is a finite dimensional F-space and for each $x \in \mathcal{P}$, U_x is a G-subspace of the complexification $\widetilde{U_0}$ of U_0 such that

$$x \leq y \Rightarrow U_x \subset U_y,$$

$$x \trianglelefteq y \Rightarrow \widetilde{Re\, U_0} = F(U_x) \subset U_y.$$

The *sum* of two representations $U, V \in \text{rep}\, \mathcal{P}$ is given by the formula:

$$U \oplus V = (U_0 \oplus V_0; U_x \oplus V_x \,|\, x \in \mathcal{P}).$$

A representation U is indecomposable if $U \simeq U_1 \oplus U_2$ implies that $U_1 = 0$ or $U_2 = 0$.

Matrix problem

Each equipped poset $\mathcal{P}$ naturally defines a matrix problem of mixed type over the pair (F, G). Consider a rectangular matrix M separated into vertical stripes M_x, $x \in \mathcal{P}$, with M_x being over F (over G) if the point x is strong (weak):

$$\begin{array}{cccccc} & & x & \to & y & & \\ & & \otimes & \otimes & \bigcirc & \bigcirc \\ M & = & \boxed{G} & \boxed{G} & \boxed{F} & \boxed{F} \end{array}$$

Such partitioned matrices M are called *matrix representations* of $\mathcal{P}$ over (F, G). Their *admissible transformations* are as follows:

(a) F-elementary row transformations of the whole matrix M.

(b) F-elementary (G-elementary) column transformations of a stripe M_x if the point x is strong (weak).

(c) In the case of a weak relation $x \prec y$, additions of columns of the stripe M_x to the columns of the stripe M_y with coefficients in G.

(d) In the case of a strong relation $x \lhd y$, additions are independent both in real and imaginary parts of columns of the stripe M_x to real and imaginary parts (in any combinations) of columns of the stripe M_y with coefficients in F (assuming that, for y strong, there are no additions to the zero imaginary part of M_y).

Two representations are said to be *equivalent* or *isomorphic* if they can be turned into each other with the help of the admissible transformations. The corresponding *matrix problem* of mixed type over the pair (F, G) consists of classifying the indecomposable in the natural sense matrices M, up to equivalence.

The main problem regarding equipped posets consists of giving a complete description of indecomposable representations and irreducible morphisms of the category of representations of a given equipped poset $\mathcal{P}$.

Examples of indecomposable representations

Here, we describe some examples of indecomposable representations in the category rep $\mathcal{P}$, where $\mathcal{P}$ is an equipped poset. Henceforth, we let $\mathbb{R}$ $(\mathbb{C})$ denote the set of real numbers (complex numbers).

If $\mathcal{P}$ is an equipped poset and $A \subset \mathcal{P}$, then $P(A) = P(\min A) = (F = \mathbb{R}; P_x \mid x \in \mathcal{P})$, $P_x = G = \mathbb{C}$ if $x \in A^{\vee}$ and $P_x = 0$ otherwise. In particular, $P(\varnothing) = (F; 0, ..., 0)$.

If $a, b \in \mathcal{P}^{\otimes}$, then $T(a)$ and $T(a, b)$ denote indecomposable objects with matrix representation of the following form:

$$T(a) = \begin{array}{c} a \\ \boxed{\begin{matrix}1\\ i\end{matrix}} \\ \otimes \end{array}, \quad a \in \mathcal{P}^{\otimes}, \qquad T(a, b) = \begin{array}{cc} a & b \\ \hline \multicolumn{1}{|c|}{\begin{matrix}1\\ i\end{matrix}} & \multicolumn{1}{c|}{\begin{matrix}0\\ 1\end{matrix}} \\ \hline \otimes & b \\ | & \\ \otimes & a \end{array}, \text{ with } a \prec b.$$

We note that, in the sense of Caldero et al., there is a bijective correspondence between denominators of cluster variables of type A_n (linearly oriented) and indecomposable representations of a completely weak chain $C = \{c_1 \prec \cdots \prec c_{n-1}\}$. Actually, the number of indecomposable representations of C is the nth triangular number [4].

The *dimension vector* is a sequence of nonnegative integers

$$\underline{\dim}\, U = (d_0; d_x \mid x \in \mathcal{P}),$$

where $d_x = \dim_G U_x / \sum_{y<x} U_y$.

3. Integer Partitions

A *partition* λ of a positive integer n is a nonincreasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that

$$n = \sum_{i=1}^{n} \lambda_i.$$

A *composition* is a partition for which the order of its parts matters [1].

$\{1, 1, 1\}$, $\{2, 1\}$ and $\{3\}$ are the three partitions of 3 whereas $\{1, 1, 1\}$, $\{2, 1\}$, $\{1, 2\}$ and $\{3\}$ are the four compositions of 3.

Regarding partitions and compositions, there are numerous open problems. For instance, in 1987, Andrews proposed the following problems [2]:

(1) For what sets of positive integers *S* and *T* is $P(S, n) = P(T, n - a)$ for $n \geq a$ with *a* fixed?

(2) For each pair *S* and *T* which answer question (1), can a bijection be found between the partitions of *n* into the elements of *S* and the partitions of $n - a$ into the elements of *T*?

For $a = 1$, some identities introduced by Gessel and Stanton imply the solutions:

$$S = \{n \mid n \text{ odd or } n \equiv \pm 4, \pm 6, \pm 8, \pm 10 \bmod 32\},$$

$$T = \{n \mid n \text{ odd or } n \equiv \pm 2, \pm 8, \pm 12, \pm 14 \bmod 32\}, \tag{4}$$

$$S = \{n \mid n \equiv \pm 1, \pm 4, \pm 5, \pm 6, \pm 7, \pm 9, \pm 10, \pm 11, \pm 13, \pm 15, \pm 16, 19\},$$

$$T = \{n \mid n \equiv \pm 1, \pm 3, \pm 4, \pm 5, \pm 9, \pm 10, \pm 11, \pm 14, \pm 15, \pm 16, \pm 17, \pm 19\} \tag{5}$$

all of them mod 40.

The problem is still open, if we consider integer compositions.

4. An Advancement to the Andrews's Problem

In this section, we define ordered compositions whose structure allows giving an advancement to the problem posed by Andrews.

Let $(\mathcal{D}, \trianglelefteq)$ be a partially ordered set of integer compositions $\{x_1, x_2, x_3, x_4\}$ such that

(1) $x_i \geq 0, 1 \leq i \leq 4,$

(2) at least two of its elements are positive,

(3) $x_2 = x_4$, and the difference $x_3 - x_1 \geq 0.$

Besides $\{x_1, x_2, x_3, x_4\} \trianglelefteq \{x_1', x_2', x_3', x_4'\}$ if and only if $x_1' \leq x_1$, $x_3' \leq x_3$, $x_2 \leq x_2'$ and $x_4 \leq x_4'$.

It is clear that if $\mathcal{D}_n$ denotes the set of compositions of type $\mathcal{D}$ of a fixed integer $n \geq 2$, then:

$$\mathcal{D} = \bigcup_{n \geq 2} \mathcal{D}_n.$$

Theorem 1. *The poset of compositions* $\mathcal{D}_n$ *of type* $\mathcal{D}$ *of a fixed integer* $n \geq 2$ *is a sum of* $\left\lfloor \frac{n}{2} \right\rfloor$ *chains.*

Proof. The set of minimal points of $\mathcal{D}_n$ consists of all compositions $\{x, y, z, w\}$ such that $y = w = 0$, $x + z = n$. Thus, $x \in \left\{1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor\right\}$. □

As an example, we note that $\{2, 0, 4, 0\} \trianglelefteq \{1, 1, 3, 1\} \trianglelefteq \{0, 2, 2, 2\}$ is a chain of compositions of type $\mathcal{D}$ of 6. In the following figure, we show examples of compositions of type $\mathcal{D}_n$, we let $\mathcal{D}_n(k_0)$ denote the set of all compositions $\{x_1, x_2, x_3, x_4\}$ with fixed difference $x_3 - x_1 = k_0$.

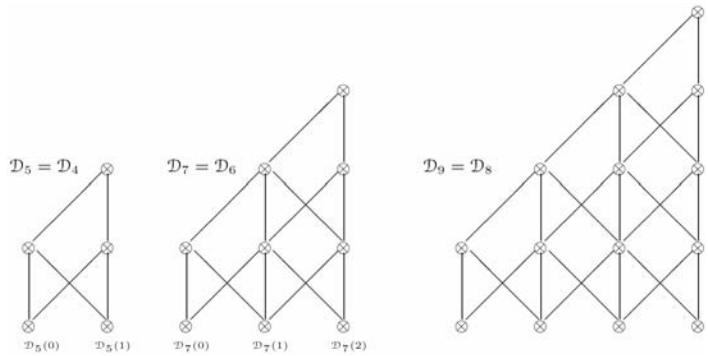


Figure 2

Regarding the number of antichains in $\mathcal{D}_n$, we have the following result:

Theorem 2. *The number* $\mathcal{D}_n^2$ *of two-point antichains contained in* D_n *is given by the formula*:

$$\sum_{i=2}^{n+1} \sum_{j=0}^{\left\lceil \frac{i}{2} \right\rceil - 1} h_{ij}(t_i - 2t_j),$$

where

$$h_{ij} = \begin{cases} 0, & \text{if } i = n+1 \text{ and } j = 0, \\ n - i + 2, & \text{if } j = \left\lceil \frac{i}{2} \right\rceil - 1 > 0, \\ 1, & \text{otherwise}, \end{cases}$$

$$\mathcal{D}_n^2 = \{2, 10, 29, 66, 129, 228, 374, ...\}$$

do not appear in the OEIS.

The structure of posets $\mathcal{D}_n$ allows us to give the following result regarding the Andrews's problems:

Corollary 3. *Let* $C(n, \mathcal{D})$ *be the number of compositions of type* $\mathcal{D}$ *of the positive integer* n. *Then* $C(2n+1, \mathcal{D}) = C(2n, \mathcal{D})$ *for any* $n \geq 1$.

Proof. Since for each $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, there are $i+1$ compositions $\{x, y, z, y\}$ of type $\mathcal{D}$ with $x + y = i$,

$$C(2n+1, \mathcal{D}) = t_{\left\lfloor n+\frac{1}{2} \right\rfloor + 1} - 1 = C(2n, \mathcal{D}). \qquad \square$$

5. The Category of Lattice Representations

In this section, we associate to each composition $\{x_1, x_2, x_3, x_4\}$ of type $\mathcal{D}$ a pair of points (x_1, x_2) and (x_3, x_4) in the usual lattice $\mathbb{N}^2$.

A *weak lattice path* $\mathcal{L} \in \mathcal{D}$ from $\{x, 0, y, 0\}$ to $\{0, x, k_0, x\}$ containing all the points in $\mathcal{D}_n(k_0)$ is defined in such a way that two adjacent vertices have the form:

$$\{x, y, z, y\}, \{x-1, y+1, z-1, y+1\}.$$

Thus, for each vertex, there are two directions to get the next vertex, that is, $\mathcal{L}$, $(-1, 0), (0, 1)$ or $(0, 1), (-1, 0)$. Henceforth, we let $\mathcal{L}_n(k_0)$ denote the set of all weak lattice paths linking out all the points in $\mathcal{D}_n(k_0)$.

A *segment* $\overline{p_0 p_1}$ in a subset $U_0 \subset \mathbb{N}^2$ is a two-point set whose elements have the form:

$$p_0 = (x_0, y_0) \text{ and } p_1 = (x_1, y_0), x_0 < x_1.$$

A *weak lattice path* $W\dfrac{\overline{z_0 z_0'}}{\overline{z_k z_k'}}$ from a segment $\overline{z_k z_k'}$ to a segment $\overline{z_0 z_0'}$ is a path of the form:

$$\{\overline{(x_k, y_k), (x_k + t, y_k)},$$

$$\overline{(x_k + \varepsilon_{x,k}, y_k + \varepsilon_{y,k}), (x_k + t + \varepsilon_{x,k-1}, y_k + \varepsilon_{y,k-1})}, \ldots,$$

$$\overline{(x_k + k, y_k + \varepsilon_{x,0}), (x_k + t + k, y_k + \varepsilon_{y,0})}\},$$

where $\varepsilon_{x,s} = 1$ if and only if $\varepsilon_{y,s} = 0$ or $\varepsilon_{x,s} = 0 = \varepsilon_{x+s,k}$ if and only if $\varepsilon_{y,s} = -1.$

The following figure shows an example of a weak lattice path:

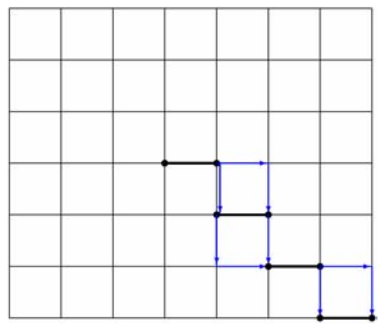


Figure 3

Strong lattice paths belong to one of the following classes:

(i) Lattice paths $S_{p_0}^{p_k} = \{(x_0, y_0), (x_1, y_1), ..., (x_k, y_k)\}$ from $p_0 = (x_0, y_0)$ to $p_k = (x_k, y_k)$, where for a given point (x_i, y_i) it holds that either $x_i = x_{i-1}$ and $y_i = y_{i-1} + 1$ or $x_i = x_{i-1} + 1$ and $y_i = y_{i-1}$.

(ii) Lattice paths $S_{p_k}^{p_0} = \{(x_k, y_k), (x_{k-1}, y_{k-1}), ..., (x_0, y_0)\}$ from $p_k = (x_k, y_k)$ to $p_0 = (x_0, y_0)$, where for a given point (x_j, y_j) it holds that either $x_j = x_j + 1$ and $y_j = y_{i-1}$ or $x_j = x_{j-1}$ and $y_j = y_j - 1$.

(iii) Products $(P)(Q)$, where P is a lattice path of type (I) and Q is a lattice path of type (II).

The following is an example of a product of strong lattice paths:

Figure 4

Regarding the number of weak lattice paths, we have the following result:

Theorem 4. *For* $k_0 \geq 1$ *fixed, the number of weak lattice paths from* $\{x, 0, x + k_0, 0\}$ *to* $\{0, x, k_0, x\}$ *containing all the points in* $\mathcal{D}_n(k_0)$ *equals* 2^x.

Proof. For each y, $0 \leq y \leq x$, the ways to connect two adjacent vertices $\{\{x, y, z, y\}, \{x-1, y+1, z-1, y+1\}\}$ are $\{\{x, y, z, y\}, \{x-1, y, z-1, y\}, \{x-1, y+1, z-1, y+1\}\}$ and $\{\{x, y, z, y\}, \{x, y+1, z, y+1\}, \{x-1, y+1, z-1, y+1\}\}$. And the sequences of points in this case consist of the points $\{\{x, 0, k_0 + x, 0\}, \{x-1, 1, k_0 + x - 1, 1\}, ..., \{0, x, k_0, x\}\}$. □

Lattice path products

Products of lattice paths (strong or weak) are defined as follows:

A *weak product* P_w in the sublattice $U_0 \subset \mathbb{N}^2$ is defined in such a way that if $\overline{z_k z'_k}$ is a segment, $S^{z_k}_{p_0}$ and $S^{z'_k}_{p_0}$ are strong lattice paths and $W \frac{\overline{z_0 z'_0}}{z_k z'_k}$ is a weak lattice path, then

$$P_w = (S^{z_k}_{p_0}, S^{z'_k}_{p_0}) W \frac{\overline{z_0 z'_0}}{z_k z'_k} = (S^{z_k}_{p_0} W^{z_0}_{z_k}, S^{z'_k}_{p_0} W^{z'_0}_{z'_k}).$$

In such a case, we write $\overline{z_k z'_k} \lhd P_w$.

A *strong product* P_s is defined in such a way that

$$P_s = (S^{z_k}_{p_0}, \varnothing) W \frac{\overline{z_0 z'_0}}{z_k z'_k} = (S^{z_k}_{p_0} W^{z_0}_{z_k}, W^{z'_0}_{z'_k}) \text{ or}$$

$$P_s = (\varnothing, S^{z_k'}_{p_0}, \varnothing) W \frac{\overline{z_0 z'_0}}{z_k z'_k} = (W^{z_0}_{z_k}, S^{z'_k}_{p_0} W^{z'_0}_{z'_k}).$$

If $z = (x_0, y_0)$, then $P_{z,k}$ denote the set of all the products passing by the segment $\overline{(x_0, y_0), (x_0 + k, y_0)}$.

The *derivative* $\delta(P)$ of a given product P is defined as follows:

$$\delta(P_w) = \{(S_{p_0}^{z_k} W_{z_k}^{z_0}, W_{z_0}^{z_k})\} \bigcup \{(W_{z_k}^{z_0}, S_{p_0}^{z'_k} W_{z'_k}^{z'_0})\},$$

$$\delta(P_s) = \{(S_{p_0}^{z_k} W_{z_k}^{z_0}, W_{z_0}^{z_k})\} \bigcup \{(W_{z_k}^{z_0}, \varnothing)\}. \tag{6}$$

Figure 5 shows the examples of these kinds of products.

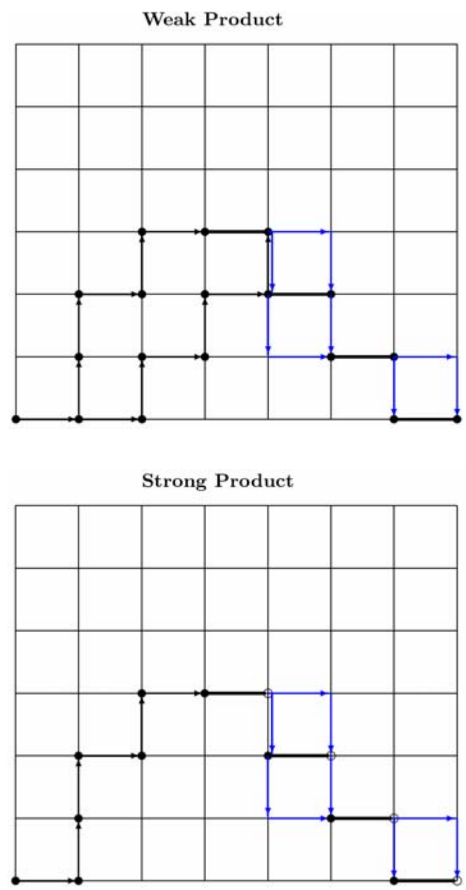


Figure 5

Given the fixed points $(x_0, y_0), (x_0 + m, y_0) \in \mathbb{N}^2$, $m > 1$, a *lattice representation* U of an equipped poset $\mathcal{P}$ is a system of the form:

$$U = (U_0; U_x \mid x \in \mathcal{P}),$$

where U_0 is an $m \times n$ order sublattice of $\mathbb{N}^2$, where (x_0, y_0) is the minimum of U_0. For each $x \in \mathcal{P}$, U_x is a system of the form:

$$(\mathcal{D}_x^u; P_{x_1, k_1}^u, P_{x_2, k_2}^u, \ldots, P_{x_J, k_j}^u),$$

where $D_x^u \subset U_0$ is an $m \times n_x$ order sublattice of D_0 containing all the products $P_{x_1, k_1}^u, P_{x_2, k_2}^u, \ldots, P_{x_J, k_j}^u$ from (x_t, y_t) to $(x_t + k_t, y_t)$ with $x_t = x_0 + r, r, k \geq 0, r + k_t \leq m$.

$\mathcal{D}_x^u \subseteq \mathcal{D}_y^u$ if $x \leq y$ ($\mathcal{D}_y^u$ covers $\mathcal{D}_x^u$ and $\mathcal{A}_x < \mathcal{A}_y$ if $y \in x^+$).

Moreover, x is a weak point if and only if the products in U_x up to the radical $\bigcup_{y \in x_\blacktriangle} U_y$ are all weak. Also,

$$D_x \subset D_y \text{ if } x \leq y,$$

$$U_x \subset U_y \text{ if } x \preceq y,$$

$$\delta(P_x^u) \subset P_y^u \text{ if } x \trianglelefteq y. \tag{7}$$

For $z \in \mathcal{P}$,

$$\mathcal{A}_z = \{t \in \mathbb{N} \mid \overline{(x, t), (x', t')} \triangleleft P\}_{P \in U_z} - \bigcup_{w \in z_\blacktriangle} U_w.$$

Remark 5. $P_x^u = \bigcup_j P_{x_j, k_j}^u$ is the set of products associated to a given point x in a lattice representation U of a poset $\mathcal{P}$.

The presentation $\varnothing$ with $U_0 = \varnothing$ is the only lattice representation with no associated products in $\text{rep}_{\mathcal{L}}\, \mathcal{P}$.

A *morphism* $\varphi : U \to V$ between two lattice representations is an order lattice homomorphism $\varphi : U_0 \to V_0$ such that $\varphi(U_x) \subseteq V_x$ for each $x \in \mathcal{P}$.

A morphism $\varphi : U \to V$ is an *isomorphism* if and only if $\varphi : U_0 \to V_0$ is an isomorphism such that for each $x \in \mathcal{P}$ the restriction $\varphi_x = \varphi|_x :$ $U_x \to V_x$ is an isomorphism as well.

The usual meet $\wedge$ and join $\vee$ operations on sets allow defining the sum and intersection in $\text{rep}_{\mathcal{L}}\, \mathcal{P}$ as follows:

$$U \bigcap V = (U_0 \wedge V_0;\, U_x \wedge V_x \,|\, x \in \mathcal{P}),$$

$$U \oplus V = (U_0 \oplus V_0;\, U_x \oplus V_x \,|\, x \in \mathcal{P}),$$

where

$$U_0 \oplus V_0 = U_0 \vee V_0,$$

$$D_x^u \oplus D_x^v = D_x^u \vee D_x^v,$$

$$U_x \oplus V_x = \left(D_x^u \vee D_x^v;\, P_x^u \bigcup P_x^v\right). \tag{8}$$

A lattice representation U is *decomposable* if there exist lattice representations $U_1, U_2, U_i \neq \varnothing$ such that $U = U_1 \oplus U_2$.

The following are the examples of indecomposable lattice representations:

Figure 6

If x is a weak point, then an indecomposable representation U of x has the form:

$$U = (\mathcal{D}_x;\, (D_x;\, P_{z,k}^u)),$$

where $z = (x_t, y_t)$, and

$$D_x = \{(x, y) \in \mathbb{N}^2 \mid 0 \leq x \leq (y_t + 1)k + x_t, 0 \leq y \leq y_t\}.$$

Henceforth, we let $\text{rep}_{\mathcal{L}} \mathcal{P}$ to be the category of lattice representations of a given equipped poset $\mathcal{P}$ attached to the sublattice $\mathcal{L} \subset \mathbb{N}^2$.

The size $\| U \|$ or dimension of a lattice representation $U \in \mathcal{P}$ is a sequence of nonnegative integers:

$$(d_x \mid x \in \mathcal{P}),$$

where for each $x \in \mathcal{P}$, $d_x = \left| P_x^u - \sum_{z \in x_\blacktriangle} P_z^u \right|$.

Points and relations in $(\mathcal{D}, \trianglelefteq)$ are either weak or strong. We say that a point $x \in \mathcal{D}$ is *weak* if and only if its lattice representation only has associated weak products. Moreover, a chain $C \subset \mathcal{D}$ is *weak* if all of its points are weak. Further, relations between the points in $\mathcal{D}$ with strong points are also strong.

Now, we consider the lattice representation

$$U_n(k_0) = (U_0; U_{c_i} \mid x \in \mathcal{D}_n(k_0))$$

of the weak chain $\mathcal{D}_n(k_0)$, where the dimension d_p of the subset U_p equals the number of all weak products from $(0, 0)$ to the weak lattice path starting in p and finishing in $p_0 = \{x_0, 0, x_0 + k_0, 0\}$ for some fixed $x_0 > 0$.

We note that $| D_n(k_0) | = 2 + k_0$ and;

$$U_0 = \left\{(x, y) | 0 \leq x \leq \frac{n + k_0}{2}, 0 \leq y \leq \frac{n - k_0}{2}\right\},$$

$$D_{c_i} = \{(x, y) | 0 \leq x \leq n - 1, 0 \leq y \leq i\}, \quad 0 \leq i \leq \frac{n - k_0}{2},$$

$$P_{c_i, k_0} = P_{\left(\frac{n-k_0}{2}-i,\, i\right), k_0}. \tag{9}$$

The following (Figure 7) illustrates the lattice representation of $D_5(1)$ as defined above:

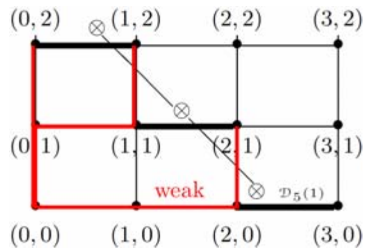


Figure 7

6. Categorification of Delannoy Numbers

In this section, we interpret Delannoy numbers as dimensions of lattice representations of weak chains of type $D_n(k_0)$.

Theorem 6. *For* $k_0 \geq 1$, *the dimension vector* $\| U_n(k_0) \| = (d_p \mid p \in \mathcal{D}_n(k_0))$, *where for* $p = \{x, y, x + k_0, y\} \in \mathcal{D}_n(k_0)$;

$$d_p = 2^x c(x + y, y) c(x + k_0 + y, y),$$

where $c(x, y) = \binom{x+y}{x}$.

Proof. For each $p = \{x, y, x + k_0, y\} \in \mathcal{D}_n(k_0)$, d_p is the number of weak products from $(0, 0)$ to $\{x_0, 0, x_0 + k_0, 0\}$, where the corresponding weak chain has as a starting vertex p. Since the number of weak lattice paths from p to $\{x_0, 0, x_0 + k_0, 0\}$ is 2^x and the number of lattice paths from $(0, 0)$ to the segment $((x, y), (x + k_0, y))$ is $c(x + y, x) \cdot c(x + k_0 + y, y)$, we are done. □

Corollary 7. *For* $x_0,\ k_0 \geq 1$ *fixed*,

$$D(x_0,\ k_0) = \sum_{p \in \mathcal{D}_n(k_0)} d_p = \sum_{y=0}^{x_0} 2^y c(x_0 - y,\ y) c(x_0 + k_0 + y,\ y).$$

Theorem 6 and Corollary 7 allow to obtain a categorification of numbers $D(x_0,\ k_0)$ which can be seen as Delannoy numbers $d(x,\ y)$. As we described in the introduction, such numbers are also obtained by counting the number of lattice paths in $\mathbb{N}^2$ from $(0,\ 0)$ to $(x,\ y)$ considering directions $(1,\ 0)$, $(0,\ 1)$ and $(1,\ 1)$. That is,

$$\begin{aligned} &d(x,\ y) = d(x-1,\ y) + d(x,\ y-1) + d(x-1,\ y-1), \\ &d(0,\ 0) = 1. \end{aligned} \tag{10}$$

The following (Figure 8) shows some of these numbers:

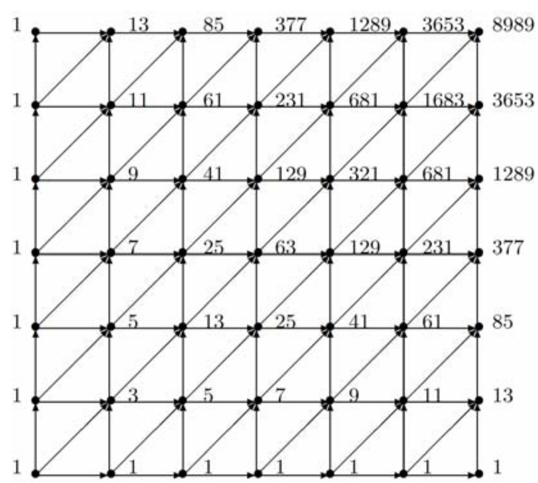


Figure 8

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