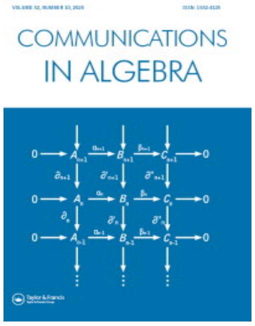

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A note on deformations of finite dimensional modules over $\mathbb{k}$-algebras

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ABSTRACT
Let $\mathbb{k}$ be a field, and let Λ be a (not necessarily finite dimensional) $\mathbb{k}$-algebra. Let V be an indecomposable left Λ-module which is finite dimensional over $\mathbb{k}$ and such that $\dim_{\mathbb{k}} \mathrm{Ext}^1_\Lambda(V,V) \leq 1$. Assume further that V has a weak universal deformation ring $R^w(\Lambda, V)$, which is a complete Noetherian commutative local $\mathbb{k}$-algebra with residue field $\mathbb{k}$. We prove in this note, under certain conditions on the Λ-module V, that $R^w(\Lambda, V)$ is either isomorphic to $\mathbb{k}$, or $\mathbb{k}[\![t]\!]$, or to $\mathbb{k}[\![t]\!]/(t^N)$ for some integer $N \geq 2$.

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1. Introduction

Assume that Λ is a finite dimensional self-injective $\mathbb{k}$-algebra. If V has stable endomorphism ring isomorphic to $\mathbb{k}$, then it follows from the results in [1, Theorem 2.6 (i)] that the weak deformation functor $\widehat{\mathrm{F}}^w_V(-)$ is naturally equivalent to the deformation functor $\widehat{\mathrm{F}}_V(-)$ and that V has a universal deformation ring $R(\Lambda, V)$. Consequently, V has also a weak universal deformation ring $R^w(\Lambda, V)$ that is isomorphic to $R(\Lambda, V)$. Moreover, in [4], the authors obtain the same result for the case when $\Lambda = \widehat{\Gamma}$, which is the repetitive algebra of a finite-dimensional $\mathbb{k}$-algebra Γ (as defined in [8]). It should be noted that in this particular situation, Λ is an infinite-dimensional $\mathbb{k}$-algebra. Furthermore, in [1, 3, 4, 7], several combinatorial methods from the representation theory of algebras were employed to compute universal deformation rings of modules whose stable endomorphism ring is isomorphic to $\mathbb{k}$. It is worth mentioning that many of these calculations use similar methods or involve adjustments from one situation to the other. For instance, we refer to [3, Proof of Claim 4.5] and [4, Proof of Theorem 1.2] for examples of such similarities and adaptations.

In this note, we present a general criterion and a unified technique for computing universal deformation rings under broad assumptions, encompassing the aforementioned results as special cases. Moreover, these calculations can be applied to other scenarios as well. Our objective in this note is to prove the following result.

Theorem 1.1. *Let Λ be a (not necessarily finite dimensional) $\mathbb{k}$-algebra and V be an indecomposable left Λ-module with $\dim_{\mathbb{k}} V < \infty$ and $\dim_{\mathbb{k}} \mathrm{Ext}^1_\Lambda(V,V) = 1$. Assume that V has a weak universal deformation*

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ring $R^w(\Lambda, V)$, and that there exists an ordered sequence of indecomposable finite dimensional left Λ-modules (up to isomorphism) $\mathscr{L}_V = \{V_0, V_1, \dots\}$ with $V_0 = V$ and such that for each $\ell \geq 1$, there exist a surjective morphism $\epsilon_\ell : V_\ell \to V_{\ell-1}$, and an injective morphism $\iota_\ell : V_{\ell-1} \to V_\ell$ such that the composition $\sigma_\ell = \iota_\ell \circ \epsilon_\ell$ satisfies that $\ker \sigma_\ell = V_0$, $\operatorname{im} \sigma_\ell^\ell \cong V_0$, $\sigma_\ell^{\ell+1} = 0$, and $\mathscr{L}_V$ is maximal in the sense that if there is another ordered sequence of indecomposable left Λ-modules $\mathscr{L}'_V$ with these properties, then $\mathscr{L}'_V \subseteq \mathscr{L}_V$.

(i) If $\mathscr{L}_V$ is finite, and its last element, say V_N, satisfies $\dim_{\Bbbk} \operatorname{Hom}_\Lambda(V_N, V) = 1$ and $\operatorname{Ext}^1_\Lambda(V_N, V) = 0$, then $R^w(\Lambda, V) \cong \Bbbk[\![t]\!]/(t^{N+1})$.
(ii) If $\mathscr{L}_V$ is infinite, then $R^w(\Lambda, V) \cong \Bbbk[\![t]\!]$.

2. Preliminaries

As before, we assume that $\Bbbk$ is a fixed field of arbitrary characteristic. We denote by $\widehat{\mathcal{C}}$ the category of all complete local commutative Noetherian $\Bbbk$-algebras with residue field $\Bbbk$. In particular, the morphisms in $\widehat{\mathcal{C}}$ are continuous $\Bbbk$-algebra homomorphisms that induce the identity map on $\Bbbk$. Let Λ be a fixed and not necessarily finite dimensional $\Bbbk$-algebra, and let R be a fixed but arbitrary object in $\widehat{\mathcal{C}}$. We denote by $R\Lambda$ the tensor product of $\Bbbk$-algebras $R \otimes_{\Bbbk} \Lambda$. Let V be a fixed left Λ-module with $\dim_{\Bbbk} V < \infty$. Following [1, Section 2], a *lift* of V over R is a pair (M, ϕ), where M is a finitely generated left $R\Lambda$-module which is free over R together with an isomorphism of left Λ-modules $\phi : \Bbbk \otimes_R M \to V$; two lifts (M, ϕ) and (M', ϕ') are said to be *isomorphic* if there exists an isomorphism of left $R\Lambda$-modules $f : M \to M'$ such that $\phi' \circ (\mathrm{id}_{\Bbbk} \otimes f) = \phi$; a *deformation* of V over R is an isomorphism class $[M, \phi]$ of V over R; we denote by $\operatorname{Def}_\Lambda(V, R)$ the set of all deformations of V over R. The *deformation functor* of V is the covariant functor $\widehat{\mathrm{F}}_R : \widehat{\mathcal{C}} \to$ Sets defined as follows: for all objects R in $\widehat{\mathcal{C}}$, we let $\widehat{\mathrm{F}}_V(R) = \operatorname{Def}_\Lambda(V, R)$, and if $\theta : R \to R'$ is a morphism in $\widehat{\mathcal{C}}$, then $\widehat{\mathrm{F}}(\theta) : \operatorname{Def}_\Lambda(V, R) \to \operatorname{Def}_\Lambda(V, R')$ that sends $[M, \phi]$ to $[M', \phi_\theta]$, where $M' = R' \otimes_{R,\theta} M$ and $\phi_\theta : \Bbbk \otimes'_R M' \to V$ is the composition of ϕ with the natural isomorphism $\Bbbk \otimes_R M' \cong \Bbbk \otimes_R M$. Assume that (M, ϕ) is a lift of V over R. Then the isomorphism class $[M]$ of M as a left $R\Lambda$-module is called a *weak deformation* of V over R. We denote by $\operatorname{Def}^w_\Lambda(V, R)$ the set of all weak deformations of V over R and thus we obtain a *weak deformation functor* of V as the covariant functor $\widehat{\mathrm{F}}^w_V : \widehat{\mathcal{C}} \to$ Sets that sends every object R in $\widehat{\mathcal{C}}$ to $\operatorname{Def}^w_\Lambda(V, R)$, and which sends any morphism $\theta : R \to R'$ in $\widehat{\mathcal{C}}$ to the morphism $\widehat{\mathrm{F}}^w_V(\theta) : \operatorname{Def}^w_\Lambda(V, R) \to \operatorname{Def}^w_\Lambda(V, R')$ that is defined as $\widehat{\mathrm{F}}^w_V([M]) = [R' \otimes_{R,\theta} M]$. Observe that there is a natural transformation $\widehat{\mathrm{F}}_V \to \widehat{\mathrm{F}}^w_V$. As mentioned in Section 1, it follows from [1, Theorem 2.6 (i)] that the weak deformation functor $\widehat{\mathrm{F}}^w_V(-)$ is naturally equivalent to the deformation functor $\widehat{\mathrm{F}}_V(-)$ when Λ is a Frobenius $\Bbbk$-algebra and $\underline{\operatorname{End}}_\Lambda(V) = \Bbbk$ (see also [4, Lemma 3.10]).

Assume that there exists an object $R^w(\Lambda, V)$ in $\widehat{\mathcal{C}}$ that represents $\widehat{\mathrm{F}}^w_V(-)$ in the sense that there is a natural equivalence between the functors $\widehat{\mathrm{F}}^w_V(-)$ and $\operatorname{Hom}_{\widehat{\mathcal{C}}}(R^w(\Lambda, V), -)$. In this situation, we call $R^w(\Lambda, V)$ the *weak universal deformation ring* of V. If R is the ring of dual numbers $\Bbbk[\epsilon]$ with $\epsilon^2 = 0$, then $t_V = \mathrm{F}_V(\Bbbk[\epsilon])$ is called the *tangent space* of F_V. Assume further that there exists an isomorphism of $\Bbbk$-vector spaces $t_V \cong \operatorname{Ext}^1_\Lambda(V, V)$. If $\dim_{\Bbbk} t_V = n$, then it follows by using the same arguments as those in [2, First paragraph on p. 223] that $R^w(\Lambda, V)$ is a quotient of the ring of formal power series $\Bbbk[\![t_1, \dots, t_n]\!]$. In particular, if $\operatorname{Ext}^1_\Lambda(V, V) = 0$, then $R^w(\Lambda, V) = \Bbbk$. In this note, we are interested in finite dimensional Λ-modules V such that $\dim_{\Bbbk} \operatorname{Ext}^1_\Lambda(V, V) = 1$ and which have a weak deformation ring $R^w(\Lambda, V)$ which is then a quotient of $\Bbbk[\![t]\!]$.

Remark 2.1. In order to prove the main result in this note, we need the following definition and property of morphisms between objects in $\widehat{\mathcal{C}}$. Following [6], if R is an object in $\widehat{\mathcal{C}}$, we denote by t^*_R the quotient $\mathfrak{m}_R/\mathfrak{m}^2_R$ and call it the *Zariski cotangent space* of R over $\Bbbk$. Let $\theta : R \to R'$ be a morphism in $\widehat{\mathcal{C}}$. It follows by [6, Lemma 1.1] that θ is surjective if and only if the induced map of cotangent spaces $\theta^* : t^*_R \to t^*_{R'}$ is surjective.

3. Proof of main result

Proof of Theorem 1.1. Let $\ell \geq 1$ be a fixed integer and assume that V_ℓ and σ_ℓ are as in the hypothesis of Theorem 1.1. It follows that the Λ-module V_ℓ is naturally a $\Bbbk[\![t]\!]/(t^{\ell+1}) \otimes_{\Bbbk} \Lambda$-module by letting t act on $x \in V_\ell$ as $t \cdot x = \sigma_\ell(x)$. In particular, $tV_\ell \cong V_{\ell-1}$. Assume that $d = \dim_{\Bbbk} V$ and let $\{\bar{r}_1, \dots, \bar{r}_d\}$ be a fixed basis of V over $\Bbbk$. By using the isomorphism $V_\ell/tV_\ell \cong V_0$, we can lift the elements $\bar{r}_1, \dots, \bar{r}_d$ to corresponding elements $r_1, \dots, r_d \in V_\ell$ that are linearly independent over $\Bbbk$ and such that $\{t^s r_1, \dots, t^s r_d : 1 \leq s \leq \ell\}$ is a $\Bbbk$-basis of $tV_\ell \cong V_{\ell-1}$, which implies that $\{r_1, \dots, r_d\}$ is a $\Bbbk[\![t]\!]/(t^{\ell+1})$-basis of V_ℓ, i.e. V_ℓ is free over $\Bbbk[\![t]\!]/(t^{\ell+1})$. Note also that V_ℓ lies in a short exact sequence of Λ-modules

$$0 \to tV_\ell \to V_\ell \to \Bbbk \otimes_{\Bbbk[\![t]\!]/(t^{\ell+1})} V_\ell \to 0,$$

which implies that there exists an isomorphism of Λ-modules $\phi_\ell : \Bbbk \otimes_{\Bbbk[\![t]\!]/(t^{\ell+1})} V_\ell \to V_0$, which implies that V_ℓ induces a weak lift of V_0 over $\Bbbk[\![t]\!]/(t^{\ell+1})$. Moreover, for each $\ell \geq 1$, we also have short exact sequences of $\Bbbk[\![t]\!]/(t^{\ell+1})\Lambda$-modules

$$0 \to V_0 \to V_\ell \xrightarrow{\psi_{\ell,\ell-1}} V_{\ell-1} \to 0, \tag{1}$$

where V_0 is considered as a left $\Bbbk[\![t]\!]/(t^{\ell+1})\Lambda$-module with trivial t-action and $\psi_{n,n-1}$ is the canonical projection.

In order to prove (i), assume next that $\mathscr{L}_V$ is finite and that the last term of this sequence, say V_N, satisfies $\dim_{\Bbbk} \mathrm{Hom}_\Lambda(V_N, V_0) = 1$ and $\mathrm{Ext}^1_\Lambda(V_N, V_0) = 0$. Consider the weak lift V_N of V_0 over $\Bbbk[\![t]\!]/(t^{N+1})$. Since $\widehat{\mathrm{F}}^w_V(-)$ is representable, it follows that there exists a unique morphism $\theta : R^w(\Lambda, V_0) \to \Bbbk[\![t]\!]/(t^{N+1})$ in $\widehat{\mathcal{C}}$ such that

$$V_N \cong \Bbbk[\![t]\!]/(t^{N+1}) \otimes_{R^w(\Lambda,V_0),\theta} U(\Lambda, V_0),$$

where $U(\Lambda, V_0)$ is the weak lift corresponding to the weak deformation of V_0 over $R^w(\Lambda, V_0)$. On the other hand, since V_1 is a weak lift of V_0 over $\Bbbk[\![t]\!]/(t^2)$, there exists a unique morphism $\theta' : R^w(\Lambda, V_0) \to \Bbbk[\![t]\!]/(t^2)$ in $\widehat{\mathcal{C}}$ such that

$$V_1 \cong \Bbbk[\![t]\!]/(t^2) \otimes_{R^w(\Lambda,V_0),\theta'} U(\Lambda, V_0).$$

By considering the natural projection $\pi_{N+1,2} : \Bbbk[\![t]\!]/(t^{N+1}) \to \Bbbk[\![t]\!]/(t^2)$ and the weak lift $(U', \phi_{U'})$ of V_0 over $\Bbbk[\![t]\!]/(t^2)$ corresponding to the morphism $\pi_{N+1,2} \circ \theta$, we obtain

$$\begin{aligned} U' &\cong \Bbbk[\![t]\!]/(t^2) \otimes_{R(\Lambda_N,V_0),\pi_{N+1,2}\circ\theta} U(\Lambda_N, V_0) \\ &\cong \Bbbk[\![t]\!]/(t^2) \otimes_{\Bbbk[\![t]\!]/(t^{N+1}),\pi_{N+1,2}} \left(\Bbbk[\![t]\!]/(t^N) \otimes_{R(\Lambda_N,V_0),\theta} U(\Lambda_N, V_0)\right) \\ &\cong \Bbbk[\![t]\!]/(t^2) \otimes_{\Bbbk[\![t]\!]/(t^{N+1}),\pi_{N+1,2}} V_N \\ &\cong V_N/t^2V_N \cong V_1. \end{aligned}$$

The uniqueness of θ' implies that $\theta' = \pi_{N+1,2} \circ \theta$, and since θ' is surjective, it follows that θ is also surjective. We claim that θ is an isomorphism. If this is false, then there exists a surjective $\Bbbk$-algebra homomorphism $\theta_0 : R^w(\Lambda, V_0) \to \Bbbk[\![t]\!]/(t^{N+2})$ in $\widehat{\mathcal{C}}$ such that $\pi_{N+2,N+1} \circ \theta_0 = \theta$, where $\pi_{N+2,N+1} : \Bbbk[\![t]\!]/(t^{N+2}) \to \Bbbk[\![t]\!]/(t^{N+1})$ is the natural projection. Let M_0 be a weak lift of V_0 over $\Bbbk[\![t]\!]/(t^{N+2})$ corresponding to θ_0. Since the kernel of $\pi_{N+2,N+1}$ is $(t^{N+1})/(t^{N+2})$, it follows that $M_0/t^{N+1}M_0 \cong V_N$. Consider the $\Bbbk[\![t]\!]/(t^{N+2}) \otimes_{\Bbbk} \Lambda$-module homomorphism $g : M_0 \to t^{N+1}M_0$ defined as $g(x) = t^{N+1}x$ for all $x \in M_0$. Since M_0 is free over $\Bbbk[\![t]\!]/(t^{N+2})$ it follows that the kernel of g is isomorphic to tM_0. Thus, $M_0/tM_0 \cong t^{N+1}M_0$ for g is a surjection. Therefore $t^{N+1}M_0 \cong V_0$, and thus we obtain a short exact sequence of $\Bbbk[\![t]\!]/(t^{N+2}) \otimes_{\Bbbk} \Lambda$-modules

$$0 \to V_0 \to M_0 \to V_N \to 0. \tag{2}$$

Since by assumption we have $\mathrm{Ext}^1_\Lambda(V_N, V_0) = 0$, it follows that the sequence (2) splits as a sequence of Λ-modules. Hence, $M_0 = V_0 \oplus V_N$ as Λ-modules. Identifying the elements of M_0 as (v, x) with

$v \in V_0$ and $x \in V_N$, we see that t acts on $(v,x) \in M_0$ as $t \cdot (v,x) = (\mu(x), \sigma_N(x))$, where $\mu : V_N \to V_0$ is a surjective Λ-module homomorphism. Since the surjection $\sigma_N^N : V_N \to V_0$ is non-zero and since by hypothesis $\dim_{\Bbbk} \mathrm{Hom}_\Lambda(V_N, V_0) = 1$, it follows that there exists $c \in \Bbbk^*$ such that $\mu = c\sigma_N^N$, which implies that the kernel of μ is tV_N. Therefore $t^{N+1}(v,x) = (\mu(t^N x), \sigma_N^{N+1}(x)) = (0,0)$ for all $v \in V_0$ and $x \in V_N$. This contradicts that $t^{N+1}M_0 \cong V_0$. Thus $\theta : R^w(\Lambda, V_0) \to \Bbbk[[t]]/(t^{N+1})$ is an isomorphism. This proves (i).

Next assume that $\mathscr{L}_V$ is infinite. In the following, we will argue as in e.g. [3, Proof of Claim 4.5] or in the last paragraph of [4, Proof of Theorem 1.2]. Namely, by using the short exact sequences (1), we obtain an inverse system $\{V_\ell, \psi_{\ell,\ell-1}\}_{\ell \geq 1}$ that satisfies the Mittag-Leffler Condition, and by letting $W = \varprojlim_\ell V_\ell$ and using e.g. [5, Prop. III.10.3], we obtain an exact sequence of left $\Bbbk[[t]]\Lambda$-modules

$$0 \to V_0 \to W \xrightarrow{\varprojlim \psi_{\ell,\ell-1}} W \to 0. \tag{3}$$

where t acts on W as $\varprojlim \psi_{\ell,\ell-1}$ and $\Bbbk \otimes_R W \cong W/tW \cong V_0$. By arguing as before, it follows that W is a weak lift of V_0 over $\Bbbk[[t]]$. Therefore, there exists a unique $\Bbbk$-algebra homomorphism $\theta : R^w(\Lambda, V_0) \to \Bbbk[[t]]$ in $\widehat{\mathcal{C}}$ which corresponds to the weak deformation induced by W. Note that since $W/t^2W \cong V_1$ as Λ-modules, we obtain that W/t^2W defines a non-trivial lift of V_0 over $\Bbbk[[t]]/(t^2)$ and thus there exists a unique morphism $\theta' : R^w(\Lambda, V_0) \to \Bbbk[[t]]/(t^2)$. Note that since the cotangent space (as Remark 2.1) of $\Bbbk[[t]]/(t^2)$ is 1-dimensional over $\Bbbk$, it follows that θ' is also surjective. Moreover, if $\theta'' : \Bbbk[[t]] \to \Bbbk[[t]]/(t^2)$ is the canonical projection, it follows by the uniqueness of θ' that $\theta' = \theta'' \circ \theta$. Thus again by Remark 2.1, since $(\theta'')^*$ is an isomorphism, we obtain that θ^* and thus θ is surjective. Therefore by using that $R^w(\Lambda, V_0)$ is a quotient of $\Bbbk[[t]]$, we conclude that θ is an isomorphism. This proves (ii) and finishes the proof of Theorem 1.1. □

Remark 3.1. (i) Assume that Λ is either a finite dimensional self-injective $\Bbbk$-algebra or the repetitive algebra of a finite dimensional $\Bbbk$-algebra. If V is a finite dimensional module such that $\underline{\mathrm{End}}_\Lambda(V) = \Bbbk$, then V has a universal deformation ring $R(\Lambda, V)$ (in the sense of [1]) and moreover, V also has a weak universal deformation ring $R^w(\Lambda, V)$ with $R^w(\Lambda, V) \cong R(\Lambda, V)$. Therefore, Theorem 1.1 recovers the results in e.g. [3, Claims 4.4 and 4.5], [4, Theorem 1.2 (ii), (iii)]), [7, Claims 4.1.1 and 4.1.2], which involve symmetric (thus self-injective) special biserial $\Bbbk$-algebras (in the sense of [8]).

(ii) The condition concerning the weak universal deformation ring $R^w(\Lambda, V)$ being a quotient of the ring of formal series $\Bbbk[[t]]$ is commonly encountered, as demonstrated by the aforementioned results. In fact, a more commonly encountered condition, which implies this, is when the dimension of the tangent space $t_V \cong \mathrm{Ext}^1_\Lambda(V, V)$ is equal to one as a $\Bbbk$-vector space.

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