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Generalized bivariate beta distributions involving Appell's hypergeometric function of the second kind[☆]

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ABSTRACT

Let X_1, X_2 and X_3 be independent random variables, X_1 and X_2 having a confluent hypergeometric function kind 1 distribution with probability density function proportional to $x_i^{\nu_i-1}{}_1F_1(\alpha_i;\beta_i;-x_i)$, $i=1,2$, and X_3 having a standard gamma distribution with shape parameter ν_3. Define $(Y_1,Y_2)=(X_1/X_3,X_2/X_3)$ and $(Z_1,Z_2)=(X_1,X_2)/(X_1+X_2+X_3)$. In this article, we derive probability density functions of (Y_1,Y_2) and (Z_1,Z_2), and study their properties. We use the second hypergeometric function of Appell to express these density functions.

1. Introduction

The random variable X is said to have a beta distribution, denoted by $X\sim \mathrm{B1}(a,b)$, if its probability density function (p.d.f.) is given by

$$\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1},\quad 0<x<1, \tag{1}$$

where $a>0$, $b>0$, and $B(a,b)$ is the beta function defined by

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2}$$

The random variable Y with the p.d.f.

$$\mathrm{B2}(y;a,b)=\{B(a,b)\}^{-1}y^{a-1}(1+y)^{-(a+b)},\quad y>0, \tag{3}$$

where $a>0$ and $b>0$, is said to have a beta type 2 distribution with parameters (a,b). Since (3) can be obtained from (1) by the transformation $Y=X/(1-X)$ some authors call the distribution of Y an *inverted beta distribution*. The inverted beta distribution arises from a linear transformation of the F distribution. The beta type 1 and beta type 2 are very flexible distributions for positive random variables and have wide applications in statistical analysis, e.g., see Johnson et al. [1]. Systematic treatment of matrix variate generalizations of the beta type 1 and the beta type 2 distributions is given in Gupta and Nagar [2].

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It is well known that if U_1 and U_2 are independent random variables each having a standard gamma distribution with respective shape parameters a_1 and a_2, then $U_1/(U_1+U_2) \sim \text{B1}(a_1, a_2)$ and $U_1/U_2 \sim \text{B2}(a_1, a_2)$.

Let X_1, X_2, X_3 be independent random variables and define

$$Z_1 = \frac{X_1}{X_1+X_2+X_3}, \qquad Z_2 = \frac{X_2}{X_1+X_2+X_3} \tag{4}$$

and

$$Y_1 = \frac{X_1}{X_3}, \qquad Y_2 = \frac{X_2}{X_3}. \tag{5}$$

If X_i follows a standard gamma distribution with shape parameter a_i, $i = 1, 2, 3$, then Z_1 and Z_2 each has a beta type 1 distribution, $Z_1 \sim \text{B1}(a_1, a_2+a_3)$ and $Z_2 \sim \text{B1}(a_2, a_1+a_3)$. However, they are correlated so that (Z_1, Z_2) has a bivariate beta type 1 distribution. It is well known that (Z_1, Z_2) follows a Dirichlet type 1 distribution with the p.d.f.

$$\text{D1}(z_1, z_2; a_1, a_2, a_3) = \frac{z_1^{a_1-1} z_2^{a_2-1} (1-z_1-z_2)^{a_3-1}}{B(a_1, a_2, a_3)}, \quad z_1 > 0,\ z_2 > 0,\ z_1+z_2 < 1, \tag{6}$$

where

$$B(a, b, c) = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+b+c)}.$$

Similarly, Y_1 and Y_2 each has a beta type 2 distribution, $Y_1 \sim \text{B2}(a_1, a_3)$ and $Y_2 \sim \text{B2}(a_2, a_3)$. However, they are correlated so that (Y_1, Y_2) has a bivariate beta type 2 distribution. It is well known that (Y_1, Y_2) follows a Dirichlet type 2 distribution with the p.d.f.

$$\text{D2}(y_1, y_2; a_1, a_2, a_3) = \frac{y_1^{a_1-1} y_2^{a_2-1} (1+y_1+y_2)^{-(a_1+a_2+a_3)}}{B(a_1, a_2, a_3)}, \quad y_1 > 0,\ y_2 > 0. \tag{7}$$

The distributions defined by the densities (6) and (7) and their matrix variate generalizations have been studied extensively. For example see Kotz et al. [3] and Gupta and Nagar [2].

There are several other bivariate beta distributions that are available in the literature. For example, see, Bekker et al. [4,5], Cardeño et al. [6], Connor and Mosimann [7], Lee [8], Mihram and Hultquist [9], Nadarajah and Kotz [10,11], Nadarajah [12,13], and Olkin and Liu [14].

Bivariate beta distributions are tractable lifetime models in many areas, including life testing and telecommunications. These distributions have attracted useful applications in the modeling of the proportion of substances in a mixture, brand shares, proportion of electorate voting for the candidate in a two candidate election and the dependence between two soil strength parameters.

For results on bivariate distributions the reader is referred to Mardia [15], Arnold et al. [16], Balakrishnan and Lai [17], Kotz et al. [3], Hutchinson and Lai [18], Gupta and Wong [19].

The objective of this work is to give generalizations of the bivariate beta distributions defined by the densities (6) and (7) by taking X_1 and X_2 as confluent hypergeometric function kind 1 and X_3 as standard gamma variable in (4) and (5).

The random variable X is said to have a confluent hypergeometric function kind 1 distribution, denoted by $X \sim \text{CH}(\nu, \alpha, \beta, \text{kind 1})$, if its p.d.f. is given by (Gupta and Nagar [2], Nagar and Sepúlveda-Murillo [20]),

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu)}{\Gamma(\nu)\Gamma(\beta)\Gamma(\alpha-\nu)} x^{\nu-1} {}_1F_1(\alpha; \beta; -x), \quad x > 0, \tag{8}$$

where $\beta \geq \alpha > \nu > 0$, and ${}_1F_1$ is the confluent hypergeometric function kind 1 (see Luke [21]). Mathai and Saxena [22] have shown that the above density can be obtained as a limiting case of a generalized hypergeometric density involving the Gauss hypergeometric function. The confluent hypergeometric function kind 1 distribution occurs as the distribution of the ratio of independent gamma and beta variables. For $\alpha = \beta$, the density (8) reduces to a standard gamma density with shape parameter ν and in this case we write $X \sim \text{Ga}(\nu)$. The gamma distribution has been used to model amounts of daily rainfall [23]. In neuroscience, the gamma distribution is often used to describe the distribution of inter-spike intervals [24]. The gamma distribution is widely used as a conjugate prior in Bayesian statistics. It is the conjugate prior for the precision (i.e. inverse of the variance) of a normal distribution. It is also the conjugate prior for the exponential distribution. It is, therefore, reasonable to say that the confluent hypergeometric function kind 1 distribution, which is a generalization of the gamma distribution, can be used as an alternative to gamma quite effectively in analyzing many lifetime data. Therefor, to explore the variables-in-common (or trivariate reduction) technique for introducing and investigating new bivariate distributions using independent gamma and confluent hypergeometric function kind 1 distributions, develop some seemingly useful properties of these bivariate beta distributions suggested herein will further enrich existing literature on bivariate distributions.

The trivariate reduction technique using confluent hypergeometric function variables enables us to generate two new bivariate distributions which are more general than Dirichlet distributions. These bivariate distributions are more flexible due to increased number of parameters thereby giving a wide variety of forms of their densities and extending the spectrum of applications. Further, because of mathematical tractability of the confluent hypergeometric function and Appell's functions, the bivariate beta distributions considered in this article will advance the literature on bivariate distributions and may serve as an alternative to many existing distributions.

In this article, we derive the densities of (Y_1, Y_2) and (Z_1, Z_2) when $X_i \sim \text{CH}(\nu_i, \alpha_i, \beta_i, \text{kind } 1)$, $i = 1, 2$ and $X_3 \sim \text{Ga}(\nu_3)$ and study their properties. The densities of (Z_1, Z_2) and (Y_1, Y_2) and their properties are studied in Sections 3 and 4, respectively.

2. Some definitions and preliminary results

In this section, we give some definitions and preliminary results which are used in subsequent sections. Throughout this work we will use the Pochhammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\dots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \dots,$ and $(a)_0 = 1$. The generalized hypergeometric function of scalar argument is defined by

$$ {}_pF_q\left(a_1, \dots, a_p; b_1, \dots, b_q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots \left(a_p\right)_k}{(b_1)_k \cdots \left(b_q\right)_k} \frac{z^k}{k!}, \tag{9} $$

where $a_i,\ i = 1, \dots, p;\ b_j,\ j = 1, \dots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (9) are available in the literature, see Luke [21].

From (9) it is easy to see that

$$ {}_1F_1(a; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{z^k}{k!}, \tag{10} $$

and

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad |z| < 1. \tag{11} $$

Also, under suitable conditions,

$$ \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}{}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; zy)\,dz = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}{}_{p+1}F_{q+1}(a_1, \dots, a_p, \alpha; b_1, \dots, b_q, \alpha+\beta; y). \tag{12} $$

The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$ {}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \exp(zt)\,dt, \tag{13} $$

and

$$ {}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,dt, \quad |\arg(1-z)| < \pi, \tag{14} $$

respectively, where $\text{Re}(a) > 0$ and $\text{Re}(c-a) > 0$. Note that, the series expansions for ${}_1F_1$ and ${}_2F_1$ given in (10) and (11) can be obtained by expanding $\exp(zt)$ and $(1-zt)^{-b}$, $|zt| < 1$, in (13) and (14) and integrating t. Substituting $z = 1$ in (14) and integrating, we obtain

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \text{Re}(c-a-b) > 0. \tag{15} $$

The hypergeometric functions ${}_1F_1(a; c; z)$ and ${}_2F_1(a, b; c; z)$ satisfy Kummer's and Euler's relations

$$ {}_1F_1(a; c; -z) = \exp(-z){}_1F_1(c-a; c; z) \tag{16} $$

and

$$ \begin{aligned} {}_2F_1(a, b; c; z) &= (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right) \\ &= (1-z)^{-b}\,{}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right) \\ &= (1-z)^{c-a-b}\,{}_2F_1(c-a, c-b; c; z). \end{aligned} \tag{17} $$

Appell's second hypergeometric function F_2 is defined by Prudnikov et al. [25, Eq. 7.2.4(44)],

$$F_2(a, b, b'; c, c'; w, z) = \frac{\Gamma(c)\Gamma(c')}{\Gamma(b)\Gamma(b')\Gamma(c-b)\Gamma(c'-b')} \int_0^1 \int_0^1 \frac{u^{b-1} v^{b'-1} (1-u)^{c-b-1} (1-v)^{c'-b'-1} \, du \, dv}{(1-wu-zv)^a},$$

where $\mathrm{Re}(b) > 0$, $\mathrm{Re}(b') > 0$, $\mathrm{Re}(c-b) > 0$ and $\mathrm{Re}(c'-b') > 0$. Replacing $(1-wu-zv)^{-a}$ by its equivalent integral, namely,

$$(1-wu-zv)^{-a} = \frac{1}{\Gamma(a)} \int_0^\infty \exp[-(1-wu-zv)t] t^{a-1} \, dt, \quad \mathrm{Re}(a) > 0,$$

and integrating u and v using (13), the above expression is re-written as

$$F_2(a, b, b'; c, c'; w, z) = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-t) t^{a-1} {}_1F_1(b; c; wt) {}_1F_1(b'; c'; zt) \, dt. \tag{18}$$

Further, expanding ${}_1F_1(b; c; wt)$ and ${}_1F_1(b'; c'; zt)$ in series form and integrating t, the series expansion of F_2 is obtained as

$$\begin{aligned} F_2(a, b, b'; c, c'; w, z) &= \sum_{r,s=0}^\infty \frac{(a)_{r+s} (b)_r (b')_s}{(c)_r (c')_s} \frac{w^r z^s}{r!\, s!} \\ &= \sum_{r=0}^\infty \frac{(a)_r (b)_r}{(c)_r} \frac{w^r}{r!} {}_2F_1(a+r, b'; c'; z) \\ &= \sum_{s=0}^\infty \frac{(a)_s (b')_s}{(c')_s} \frac{z^s}{s!} {}_2F_1(a+s, b; c; w), \end{aligned} \tag{19}$$

where $|w| + |z| < 1$. Further, for $c = c' = a$, F_2 reduces to a Gauss hypergeometric function. That is

$$F_2(a, b, b'; a, a; w, z) = (1-w)^{-b} (1-z)^{-b'} {}_2F_1\left(b, b'; a; \frac{wz}{(1-w)(1-z)}\right). \tag{20}$$

From the definition, it is easy to see that

$$\begin{aligned} F_2(a, b, b'; c, c'; w, z) &= {}_2F_1(a, b'; c'; z), \quad \text{if } b = 0, \\ &= {}_2F_1(a, b; c; w), \quad \text{if } b' = 0. \end{aligned} \tag{21}$$

In continuation we present a few special cases of Appell's second hypergeometric function F_2 given in [26]:

$$F_2(a+1, 1, 1; 2, 2; w, z) = \frac{1}{a(a-1)wz} [1 - (1-w)^{1-a} - (1-z)^{1-a} + (1-w-z)^{1-a}], \tag{22}$$

$$F_2(1, 1, 1; 2, 2; w, z) = \frac{1}{wz} [(1-w-z)\ln(1-w-z) - (1-w)\ln(1-w) - (1-z)\ln(1-z)], \tag{23}$$

and

$$F_2\left(1, \frac{1}{2}, 1; 1, 2; w, z\right) = \frac{2}{z} \ln\left(\frac{1+\sqrt{1-w}}{\sqrt{1-z} + \sqrt{1-w-z}}\right). \tag{24}$$

Note that the special cases of F_2 given above are expressed in terms of elementary functions. Recently, Murley and Saad [27] have listed more than 400 special cases of F_2.

For properties and further results on these functions the reader is referred to Luke [21], Bailey [28], and Srivastava and Karlsson [29].

3. Generalized Dirichlet type 1 distribution

In statistical distribution theory it is well known that if X_1, X_2 and X_3 are independent, $X_i \sim \mathrm{Ga}(\nu_i)$, $i = 1, 2, 3$, then $(X_1, X_2)/X_3 \sim \mathrm{D2}(\nu_1, \nu_2; \nu_3)$, $(X_1, X_2)/(X_1 + X_2 + X_3) \sim \mathrm{D1}(\nu_1, \nu_2; \nu_3)$ and $X_1 + X_2 + X_3 \sim \mathrm{Ga}(\nu_1 + \nu_2 + \nu_3)$. In this section and subsequent section we derive similar results when X_1 and X_2 are independent confluent hypergeometric function kind 1 variables and X_3 is a gamma variable. Note that if $X \sim \mathrm{CH}(\nu, \alpha, \beta, \text{kind } 1)$, then using (16), the p.d.f. of X can also be expressed as

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu)}{\Gamma(\nu)\Gamma(\beta)\Gamma(\alpha-\nu)} x^{\nu-1} \exp(-x) {}_1F_1(\beta-\alpha; \beta; x), \quad x > 0. \tag{25}$$

Theorem 3.1. *Let X_1, X_2 and X_3 be independent, $X_i \sim CH(\nu_i, \alpha_i, \beta_i, \text{kind } 1), i = 1, 2$ and $X_3 \sim Ga(\nu_3)$. Then, the p.d.f. of $(Z_1, Z_2) = (X_1, X_2)/(X_1 + X_2 + X_3)$ is given by*

$$K \frac{z_1^{\nu_1-1} z_2^{\nu_2-1} (1 - z_1 - z_2)^{\nu_3-1}}{B(\nu_1, \nu_2, \nu_3)} F_2 (\nu_1 + \nu_2 + \nu_3, \beta_1 - \alpha_1, \beta_2 - \alpha_2; \beta_1, \beta_2; z_1, z_2),$$
$$z_1 > 0,\ z_2 > 0,\ z_1 + z_2 < 1, \tag{26}$$

and the p.d.f. of $S = X_1 + X_2 + X_3$ is given by

$$K \frac{s^{\nu_1+\nu_2+\nu_3-1} \exp(-s)}{\Gamma(\nu_1 + \nu_2 + \nu_3)} \sum_{r=0}^{\infty} \frac{(\beta_2 - \alpha_2)_r (\nu_2)_r s^r}{(\beta_2)_r (\nu_1 + \nu_2 + \nu_3)_r\, r!} {}_2F_2 (\nu_1, \beta_1 - \alpha_1; \nu_1 + \nu_2 + \nu_3 + r, \beta_1; s), \quad s > 0,$$

where

$$K = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1 - \nu_1)\Gamma(\beta_2 - \nu_2)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1 - \nu_1)\Gamma(\alpha_2 - \nu_2)}.$$

Proof. Using independence, the joint p.d.f. of X_1, X_2 and X_3 is given by

$$K_1 x_1^{\nu_1-1} x_2^{\nu_2-1} x_3^{\nu_3-1} \exp[-(x_1 + x_2 + x_3)] {}_1F_1 (\beta_1 - \alpha_1; \beta_1; x_1)\, {}_1F_1 (\beta_2 - \alpha_2; \beta_2; x_2) \tag{27}$$

where $x_1 > 0, x_2 > 0, x_3 > 0$ and

$$K_1 = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1 - \nu_1)\Gamma(\beta_2 - \nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1 - \nu_1)\Gamma(\alpha_2 - \nu_2)}. \tag{28}$$

Making the transformation $Z_1 = X_1/(X_1 + X_2 + X_3), Z_2 = X_2/(X_1 + X_2 + X_3)$ and $S = X_1 + X_2 + X_3$ with the Jacobian $J(x_1, x_2, x_3 \to z_1, z_2, s) = s^2$ in (27), we obtain the joint p.d.f. of Z_1, Z_2 and S as

$$K_1 z_1^{\nu_1-1} z_2^{\nu_2-1} (1 - z_1 - z_2)^{\nu_3-1} s^{\nu_1+\nu_2+\nu_3-1} \exp(-s) {}_1F_1 (\beta_1 - \alpha_1; \beta_1; z_1 s)\, {}_1F_1 (\beta_2 - \alpha_2; \beta_2; z_2 s), \tag{29}$$

where $z_1 > 0, z_2 > 0, z_1 + z_2 < 1$ and $s > 0$. Now, integration of s by using (18) yields the density of (Z_1, Z_2). The marginal density of S is obtained by integrating z_1 and z_2 in (29). □

For $\beta_i = \alpha_i, i = 1, 2$, the density (26) slides to a Dirichlet density with parameters ν_1, ν_2 and ν_3. In Bayesian analysis, the Dirichlet distribution is used as a conjugate prior distribution for the parameters of a multinomial distribution. However, the Dirichlet family is not sufficiently rich in scope to represent many important distributional assumptions, because the Dirichlet distribution has few number of parameters. We, in Theorem 3.1, have provided a generalization of the Dirichlet distribution with added number of parameters.

Several special cases of the density (26) can be obtained for special values of parameters by simplifying F_2 using (22)–(24). For $\nu_1 = \nu_2 = 1/2, \alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$, the density in (26) simplifies to

$$\frac{(1 - z_1 - z_2)^{\nu_3-1}}{4\pi(\nu_3 - 1) z_1^{3/2} z_2^{3/2}} [1 - (1 - z_1)^{1-\nu_3} - (1 - z_2)^{1-\nu_3} + (1 - z_1 - z_2)^{1-\nu_3}].$$

For $\nu_1 = \nu_2 = \nu_3 = 1/3, \alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$, the density (26) slides to

$$\frac{4}{9\Gamma^3(1/3)} (1 - z_1 - z_2)^{-2/3} (z_1 z_2)^{-5/3} [(1 - z_1 - z_2) \ln(1 - z_1 - z_2) - (1 - z_1) \ln(1 - z_1) - (1 - z_2) \ln(1 - z_2)].$$

For $\nu_1 = \nu_2 = \nu_3 = 1/3, \alpha_1 = 1/2, \alpha_2 = 1$ and $\beta_1 = 1$ and $\beta_2 = 2$, the density (26) reduces to

$$\frac{2\sqrt{\pi}\,\Gamma(5/3)}{\Gamma(1/3)^3 \Gamma(1/6)} z_1^{-2/3} z_2^{-5/3} (1 - z_1 - z_2)^{-2/3} \ln\left(\frac{1 + \sqrt{1 - z_1}}{\sqrt{1 - z_2} + \sqrt{1 - z_1 - z_2}}\right).$$

Corollary 3.1.1. *Let X_1, X_2 and X_3 be independent random variables, $X_i \sim CH(\nu_i, \alpha_i, \nu_1 + \nu_2 + \nu_3, \text{kind } 1), i = 1, 2$, and $X_3 \sim \text{Ga}(\nu_3)$. Then, the p.d.f. of (Z_1, Z_2) is given by*

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\nu_2 + \nu_3)\Gamma(\nu_1 + \nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_1 + \nu_2 + \nu_3)\Gamma(\alpha_1 - \nu_1)\Gamma(\alpha_2 - \nu_2)} \frac{z_1^{\nu_1-1} z_2^{\nu_2-1} (1 - z_1 - z_2)^{\alpha_1+\alpha_2-\nu_1-\nu_2-1}}{(1 - z_1)^{\alpha_2} (1 - z_2)^{\alpha_1}}$$
$$\times {}_2F_1 \left(\alpha_1, \alpha_2; \nu_1 + \nu_2 + \nu_3; \frac{z_1 z_2}{(1 - z_1)(1 - z_2)}\right), \quad z_1 > 0,\ z_2 > 0,\ z_1 + z_2 < 1.$$

Proof. Substituting $\beta_1 = \beta_2 = \nu_1 + \nu_2 + \nu_3$ in (26) and applying the results given in (20) and (17), we obtain the desired results. □

Corollary 3.1.2. *Let X_1, X_2 and X_3 be independent random variables, $X_1 \sim CH(\nu_1, \alpha_1, \beta_1, \text{kind 1})$, and $X_i \sim Ga(\nu_i)$, $i = 2, 3$. Then, the p.d.f. of (Z_1, Z_2) is given by*

$$\frac{\Gamma(\alpha_1)\Gamma(\beta_1-\nu_1)\Gamma(\nu_1+\nu_2+\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\beta_1)\Gamma(\alpha_1-\nu_1)} z_1^{\nu_1-1} z_2^{\nu_2-1}(1-z_1-z_2)^{\nu_3-1} {}_2F_1(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1; \beta_1; z_1),$$

where $z_1 > 0, z_2 > 0$ and $z_1 + z_2 < 1$.

Proof. Substituting $\beta_2 = \alpha_2$ in (26) and using the result given in (21), we obtain the required result. □

If we take $\beta_1 = \nu_1 + \nu_2 + \nu_3$ in the above corollary, then the p.d.f. of (Z_1, Z_2) is given by

$$\frac{\Gamma(\alpha_1)\Gamma(\nu_2+\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\alpha_1-\nu_1)} z_1^{\nu_1-1} z_2^{\nu_2-1}(1-z_1)^{\alpha_1-\nu_1-\nu_2-\nu_3}(1-z_1-z_2)^{\nu_3-1}, \quad z_1 > 0,\ z_2 > 0,\ z_1+z_2 < 1.$$

The above bivariate distribution was first derived by Connor and Mosimann [7].

Theorem 3.2. *If the p.d.f. of (Z_1, Z_2) is given by (26), then the marginal density of Z_1 is obtained as*

$$K\frac{z_1^{\nu_1-1}(1-z_1)^{\nu_2+\nu_3-1}}{B(\nu_1, \nu_2+\nu_3)} \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_1-\alpha_1)_r}{(\beta_1)_r\, r!} z_1^r \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \nu_2, \beta_2-\alpha_2; \beta_2, \nu_2+\nu_3; 1-z_1), \quad 0 < z_1 < 1.$$

Proof. The marginal density of Z_1 is derived as

$$\frac{K}{B(\nu_1, \nu_2, \nu_3)} z_1^{\nu_1-1} \int_0^{1-z_1} z_2^{\nu_2-1}(1-z_1-z_2)^{\nu_3-1} F_2(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; z_1, z_2)\, dz_2. \tag{30}$$

Substituting $x = z_2/(1-z_1)$, expanding F_2 in series form and integrating x, the above integral is evaluated as

$$\frac{\Gamma(\nu_2)\Gamma(\nu_3)}{\Gamma(\nu_2+\nu_3)}(1-z_1)^{\nu_2+\nu_3-1} \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_1-\alpha_1)_r}{(\beta_1)_r\, r!} z_1^r \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \nu_2, \beta_2-\alpha_2; \beta_2, \nu_2+\nu_3; 1-z_1). \tag{31}$$

Finally, substituting (31) in (30) and simplifying, we get the desired result. □

Theorem 3.3. *If the p.d.f. of (Z_1, Z_2) is given by (26), then the density of $Z = Z_1 + Z_2$ is given by*

$$K\frac{z^{\nu_1+\nu_2-1}(1-z)^{\nu_3-1}}{B(\nu_1+\nu_2, \nu_3)} \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_1-\alpha_1)_r(\nu_1)_r}{(\beta_1)_r(\nu_1+\nu_2)_r\, r!} z^r \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \nu_2, \beta_2-\alpha_2; \beta_2, \nu_1+\nu_2+r; z), \quad 0 < z < 1.$$

Proof. By using the convolution formula, the density of Z is given by

$$\begin{aligned} &K\frac{(1-z)^{\nu_3-1}}{B(\nu_1, \nu_2, \nu_3)} \int_0^z z_1^{\nu_1-1}(z-z_1)^{\nu_2-1} F_2(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; z_1, z-z_1)\, dz_1 \\ &= K\frac{z^{\nu_1+\nu_2-1}(1-z)^{\nu_3-1}}{B(\nu_1, \nu_2, \nu_3)} \int_0^1 v^{\nu_1-1}(1-v)^{\nu_2-1} F_2(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; zv, z(1-v))\, dv. \end{aligned} \tag{32}$$

Now, expanding F_2 in series form and integrating v, the above integral is evaluated as

$$\frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_2)} \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_1-\alpha_1)_r(\nu_1)_r}{(\beta_1)_r(\nu_1+\nu_2)_r\, r!} z^r {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_2-\alpha_2, \nu_2; \beta_2, \nu_1+\nu_2+r; z). \tag{33}$$

Finally, substituting (33) in (32) and simplifying, we get the desired result. □

If the p.d.f. of (Z_1, Z_2) is given by (26), then the (h, t)-th joint moment of (Z_1, Z_2) is derived as

$$\begin{aligned} E(Z_1^h Z_2^t) &= \frac{K}{B(\nu_1,\nu_2,\nu_3)} \int_0^1 \int_0^{1-z_1} z_1^{\nu_1+h-1} z_2^{\nu_2+t-1} (1-z_1-z_2)^{\nu_3-1} \\ &\quad \times F_2\left(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; z_1, z_2\right) \,\mathrm{d}z_2\,\mathrm{d}z_1 \\ &= \frac{K}{B(\nu_1,\nu_2,\nu_3)} \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_2-\alpha_2)_r}{(\beta_2)_r r!} \frac{\Gamma(\nu_2+t+r)\Gamma(\nu_3)\Gamma(\nu_1+h)}{\Gamma(\nu_1+\nu_2+\nu_3+h+t+r)} \\ &\quad \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_1-\alpha_1, \nu_1+h; \beta_1, \nu_1+\nu_2+\nu_3+h+t+r; 1), \end{aligned}$$

where $\beta_1 \geq \alpha_1 > \nu_1 > 0$, $\beta_2 \geq \alpha_2 > \nu_2 > 0$ and $\nu_3 > 0$. The final result has been obtained by expanding F_2 in series form, substituting $v = z_2/(1-z_1)$ and integrating with respect to z_1 and v by applying (12).

Substituting appropriately for h and t in $E(Z_1^h Z_2^t)$, we obtain expressions for $E(Z_1Z_2)$, $E(Z_1)$, $E(Z_2)$, $E(Z_1^2)$ and $E(Z_2^2)$ as

$$\begin{aligned} E(Z_1Z_2) &= K \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_2-\alpha_2)_r}{(\beta_2)_r r!} \frac{B(\nu_1+1, \nu_2+r+1, \nu_3)}{B(\nu_1,\nu_2,\nu_3)} \\ &\quad \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_1-\alpha_1, \nu_1+1; \beta_1, \nu_1+\nu_2+\nu_3+r+2; 1), \end{aligned} \tag{34}$$

$$\begin{aligned} E(Z_1) &= K \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_2-\alpha_2)_r}{(\beta_2)_r r!} \frac{B(\nu_1+1, \nu_2+r, \nu_3)}{B(\nu_1,\nu_2,\nu_3)} \\ &\quad \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_1-\alpha_1, \nu_1+1; \beta_1, \nu_1+\nu_2+\nu_3+r+1; 1), \end{aligned} \tag{35}$$

$$\begin{aligned} E(Z_2) &= K \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_2-\alpha_2)_r}{(\beta_2)_r r!} \frac{B(\nu_1, \nu_2+r+1, \nu_3)}{B(\nu_1,\nu_2,\nu_3)} \\ &\quad \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_1-\alpha_1, \nu_1; \beta_1, \nu_1+\nu_2+\nu_3+r+1; 1), \end{aligned} \tag{36}$$

$$\begin{aligned} E(Z_1^2) &= K \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_2-\alpha_2)_r}{(\beta_2)_r r!} \frac{B(\nu_1+2, \nu_2+r, \nu_3)}{B(\nu_1,\nu_2,\nu_3)} \\ &\quad \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_1-\alpha_1, \nu_1+2; \beta_1, \nu_1+\nu_2+\nu_3+r+2; 1) \end{aligned} \tag{37}$$

and

$$\begin{aligned} E(Z_2^2) &= K \sum_{r=0}^{\infty} \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_2-\alpha_2)_r}{(\beta_2)_r r!} \frac{B(\nu_1, \nu_2+r+2, \nu_3)}{B(\nu_1,\nu_2,\nu_3)} \\ &\quad \times {}_3F_2(\nu_1+\nu_2+\nu_3+r, \beta_1-\alpha_1, \nu_1; \beta_1, \nu_1+\nu_2+\nu_3+r+2; 1). \end{aligned} \tag{38}$$

Further, the correlation between Z_1 and Z_2 is defined as

$$\rho_{Z_1,Z_2} = \frac{\mathrm{Cov}(Z_1,Z_2)}{\sqrt{\mathrm{Var}(Z_1)\mathrm{Var}(Z_2)}}, \tag{39}$$

where $\mathrm{Cov}(Z_1, Z_2) = E(Z_1Z_2) - E(Z_1)E(Z_2)$, $\mathrm{Var}(Z_i) = E(Z_i^2) - E(Z_i)^2$, $i = 1, 2$, and $E(Z_1Z_2)$, $E(Z_1)$, $E(Z_2)$, $E(Z_1^2)$ and $E(Z_2^2)$ are given by (34)–(38), respectively.

Note that the expressions for $E(Z_1Z_2)$, $E(Z_1)$, $E(Z_2)$, $E(Z_1^2)$ and $E(Z_2^2)$ contain the generalized hypergeometric function ${}_3F_2(a, b, c; a_1, b_1; 1)$, which can be calculated using MATHEMATICA. The software Mathematica includes this function in the form *HypergeometricPFQ*$[\{a, b, c\}, \{a_1, b_1\}, 1]$. Table 1 gives the correlations between Z_1 and Z_2 for different values of $\alpha_1, \alpha_2, \beta_1, \beta_2, \nu_1, \nu_2$ and ν_3. For $\alpha_2 = \beta_2$, the density of (Z_1, Z_2) is given in Corollary 3.1.2 and in this case the correlation coefficients for different values of α_1, β_1, ν_1, ν_2 and ν_3 are given by rows in bold. Since the bivariate beta distribution proposed by Connor and Mosimann [7] is a special case of Corollary 3.1.2 when $\beta_1 = \nu_1+\nu_2+\nu_3$, the correlation coefficients in this case are given in boxes among rows in bold. It can be observed that the correlation between Z_1 and Z_2 is always negative due to the condition $z_1 + z_2 < 1$. Further, for selected values of the parameters, it is possible to find correlations close to 0 or -1. For example, for $\alpha_1 = \alpha_2 = \beta_2 = 1.5$, $\beta_1 = 2$, $\nu_1 = \nu_2 = 1$ and $\nu_3 = 0.01$ the correlation is -0.994, whereas for $\alpha_1 = \beta_2 = 3$, $\alpha_2 = 2.5$, $\beta_1 = 3.5$, $\nu_1 = \nu_2 = 0.5$ and $\nu_3 = 10$ the correlation is -0.079. Note that when ν_3 is big the correlation between variables is close to zero.

Fig. 1 gives graphs of the correlation coefficient ρ_{Z_1,Z_2} as a function of α_2 for $\alpha_1 = 2.5, 3, 3.5, 4$ and $\beta_1 = \nu_1+\nu_2+\nu_3 = 4$, $\beta_2 = 3$, $\nu_1 = 2$, $\nu_2 = 1$, $\nu_3 = 1$. For $\alpha_2 = 3$ we get correlation coefficient for Connor and Mosimann [7] case for $\alpha_1 = 2.5$, $\alpha_1 = 3$, $\alpha_1 = 3.5$, and $\alpha_1 = 4$.

Fig. 2 gives graphs of the correlation coefficient ρ_{Z_1,Z_2} for the Connor and Mosimann [7] case as a function of α_1 for (i) $\beta_1 = 5$, $\nu_1 = 0.5$, $\nu_2 = 0.5$, $\nu_3 = 4$, (ii) $\beta_1 = 2.5$, $\nu_1 = 0.5$, $\nu_2 = 1$, $\nu_3 = 1$, (iii) $\beta_1 = 4$, $\nu_1 = 1$, $\nu_2 = 1$, $\nu_3 = 2$, (iv) $\beta_1 = 4$, $\nu_1 = 1$, $\nu_2 = 2$, $\nu_3 = 1$, (v) $\beta_1 = 3$, $\nu_1 = 1$, $\nu_2 = 1$, $\nu_3 = 1$, (vi) $\beta_1 = 5$, and $\nu_1 = 1$, $\nu_2 = 1$, $\nu_3 = 3$.

So far we have studied the exact distribution of (Z_1, Z_2) and several of its properties. In the following theorem, we derive limiting density of $\nu_3(Z_1, Z_2)$, when ν_3 tends to infinity, following the procedure similar to that of Tiao and Guttman [30].

Table 1
Correlations for different values of the parameters.

α_1	α_2	β_1	β_2	ν_1	ν_2	ν_3								
						0.01	0.1	0.5	1	2	3	4	5	10
2	**2**	**2**	**2**	**1**	**1**	**−0.99**	**−0.909**	**−0.667**	**−0.5**	**−0.333**	**−0.25**	**−0.2**	**−0.167**	**−0.091**
1	**1**	**2**	**1**	**0.5**	**0.5**	**−0.988**	**−0.897**	**−0.67**	**−0.542**	**−0.429**	**−0.374**	**−0.341**	**−0.318**	**−0.26**
1.5	**1.5**	**2**	**1.5**	**1**	**1**	**−0.994**	**−0.948**	**−0.802**	**−0.692**	**−0.574**	**−0.509**	**−0.467**	**−0.437**	**−0.36**
1.5	**1.5**	**2.5**	**1.5**	**1**	**1**	**−0.995**	**−0.956**	**−0.83**	**−0.735**	**−0.628**	**−0.568**	**−0.529**	**−0.5**	**−0.422**
1.5	1.5	2	2.5	1	1	−0.892	−0.879	−0.826	−0.771	−0.688	−0.626	−0.577	−0.538	−0.415
3	2.5	3.5	3	0.5	0.5	−0.984	−0.858	−0.557	−0.397	−0.259	−0.197	−0.16	−0.135	−0.079
3	2.5	3.5	3	1	1	−0.993	−0.936	−0.754	−0.615	−0.46	−0.373	−0.317	−0.277	−0.176
3	**2.5**	**4**	**2.5**	**1**	**1**	**−0.992**	**−0.929**	**−0.73**	**−0.583**	**−0.426**	**−0.342**	**−0.289**	**−0.251**	**−0.159**
4	3.5	5	4.5	0.5	0.5	−0.984	−0.863	−0.569	−0.408	−0.269	−0.204	−0.166	−0.141	−0.081
4	3.5	5	4.5	1.5	1.5	−0.996	−0.966	−0.854	−0.751	−0.611	−0.52	−0.455	−0.405	−0.269
4	**4.5**	**5**	**4.5**	**0.5**	**0.5**	**−0.982**	**−0.847**	**−0.532**	**−0.368**	**−0.232**	**−0.172**	**−0.137**	**−0.114**	**−0.064**
4	**4.5**	**5**	**4.5**	**1.5**	**1.5**	**−0.995**	**−0.951**	**−0.800**	**−0.672**	**−0.515**	**−0.423**	**−0.361**	**−0.316**	**−0.201**
4	**4.5**	**8**	**4.5**	**1.5**	**1.5**	**−0.996**	**−0.965**	**−0.854**	**−0.755**	**−0.626**	**−0.544**	**−0.485**	**−0.441**	**−0.314**

Theorem 3.4. *If the p.d.f. of (Z_1, Z_2) is given by (26), then when ν_3 tends to infinity, the limiting joint density of the random variables $T_i = \nu_3 Z_i$, $i = 1, 2$, is such that*

$$\lim_{\nu_3\to\infty} f(t_1,t_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\nu_1)\Gamma(\beta_2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1-\nu_1)\Gamma(\alpha_2-\nu_2)} t_1^{\nu_1-1} t_2^{\nu_2-1}\exp(-t_1-t_2) \times {}_1F_1(\beta_1-\alpha_1;\beta_1;t_1)\,{}_1F_1(\beta_2-\alpha_2;\beta_2;t_2), \quad (40)$$

where f is the joint density function of the random variables $T_i = \nu_3 Z_i$, $i = 1, 2$, $t_1 > 0$ and $t_2 > 0$.

Proof. Transforming $T_1 = \nu_3 Z_1$ and $T_2 = \nu_3 Z_2$ with the Jacobian $J(z_1, z_2 \to t_1, t_2) = 1/\nu_3^2$ in (26), and expanding F_2 in series form, we obtain the joint p.d.f. of T_1 and T_2 as

$$f(t_1,t_2) = \frac{K}{\Gamma(\nu_1)\Gamma(\nu_2)}\sum_{r,s=0}^{\infty}\frac{(\beta_1-\alpha_1)_r(\beta_2-\alpha_2)_s}{(\beta_1)_r(\beta_2)_s r!s!} t_1^{\nu_1+r-1} t_2^{\nu_2+s-1} \times \frac{\Gamma(\nu_1+\nu_2+\nu_3+r+s)}{\Gamma(\nu_3)\nu_3^{\nu_1+\nu_2+r+s}}\left(1-\frac{t_1+t_2}{\nu_3}\right)^{\nu_3-1}. \quad (41)$$

Now, taking $\nu_3 \to \infty$ in (41) and using the results

$$\lim_{\nu_3\to\infty}\frac{\Gamma(\nu_1+\nu_2+\nu_3+r+s)}{\Gamma(\nu_3)\nu_3^{\nu_1+\nu_2+r+s}} = 1$$

and

$$\lim_{\nu_3\to\infty}\left(1-\frac{t_1+t_2}{\nu_3}\right)^{\nu_3-1} = \exp(-t_1-t_2)$$

the limiting p.d.f. is obtained as

$$\lim_{\nu_3\to\infty} f(t_1,t_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\nu_1)\Gamma(\beta_2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1-\nu_1)\Gamma(\alpha_2-\nu_2)}\exp(-t_1-t_2) \times t_1^{\nu_1-1}t_2^{\nu_2-1}\,{}_1F_1(\beta_1-\nu_1;\beta_1;t_1)\,{}_1F_1(\beta_2-\nu_2;\beta_2;t_2).$$

The above expression is called the joint limiting p.d.f. of (T_1, T_2). □

Note that in the above theorem the limiting density of (T_1, T_2), when ν_3 tends to infinity, is the product of the densities of two independent random variables, each of them has confluent hypergeometric function type 1 distribution.

4. Generalized Dirichlet type 2 distribution

Theorem 4.1. *Let X_1, X_2 and X_3 be independent, $X_i \sim CH(\nu_i, \alpha_i, \beta_i, \text{kind } 1)$, $i = 1, 2$ and $X_3 \sim \text{Ga}(\nu_3)$. Then, the p.d.f. of $(Y_1, Y_2) = (X_1/X_3, X_2/X_3)$ is given by*

$$K\frac{y_1^{\nu_1-1}y_2^{\nu_2-1}(1+y_1+y_2)^{-(\nu_1+\nu_2+\nu_3)}}{B(\nu_1,\nu_2,\nu_3)}F_2\left(\nu_1+\nu_2+\nu_3,\beta_1-\alpha_1,\beta_2-\alpha_2;\beta_1,\beta_2;\frac{y_1}{1+y_1+y_2},\frac{y_2}{1+y_1+y_2}\right), \quad (42)$$

where $y_1 > 0$ and $y_2 > 0$.

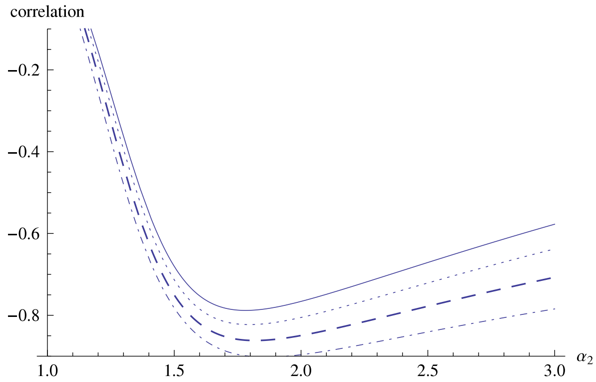

Fig. 1. Plots of the correlation coefficient ρ_{Z_1,Z_2} as a function of α_2 for $\alpha_1 = 2.5, 3, 3.5, 4$ and $\beta_1 = 4$, $\beta_2 = 3$, $\nu_1 = 2$, $\nu_2 = 1$, $\nu_3 = 1$. The four curves are: the curve of dots and dashes $\alpha_1 = 2.5$, the curve of dashes $\alpha_1 = 3$, the curve of dots $\alpha_1 = 3.5$, the solid curve $\alpha_1 = 4$.

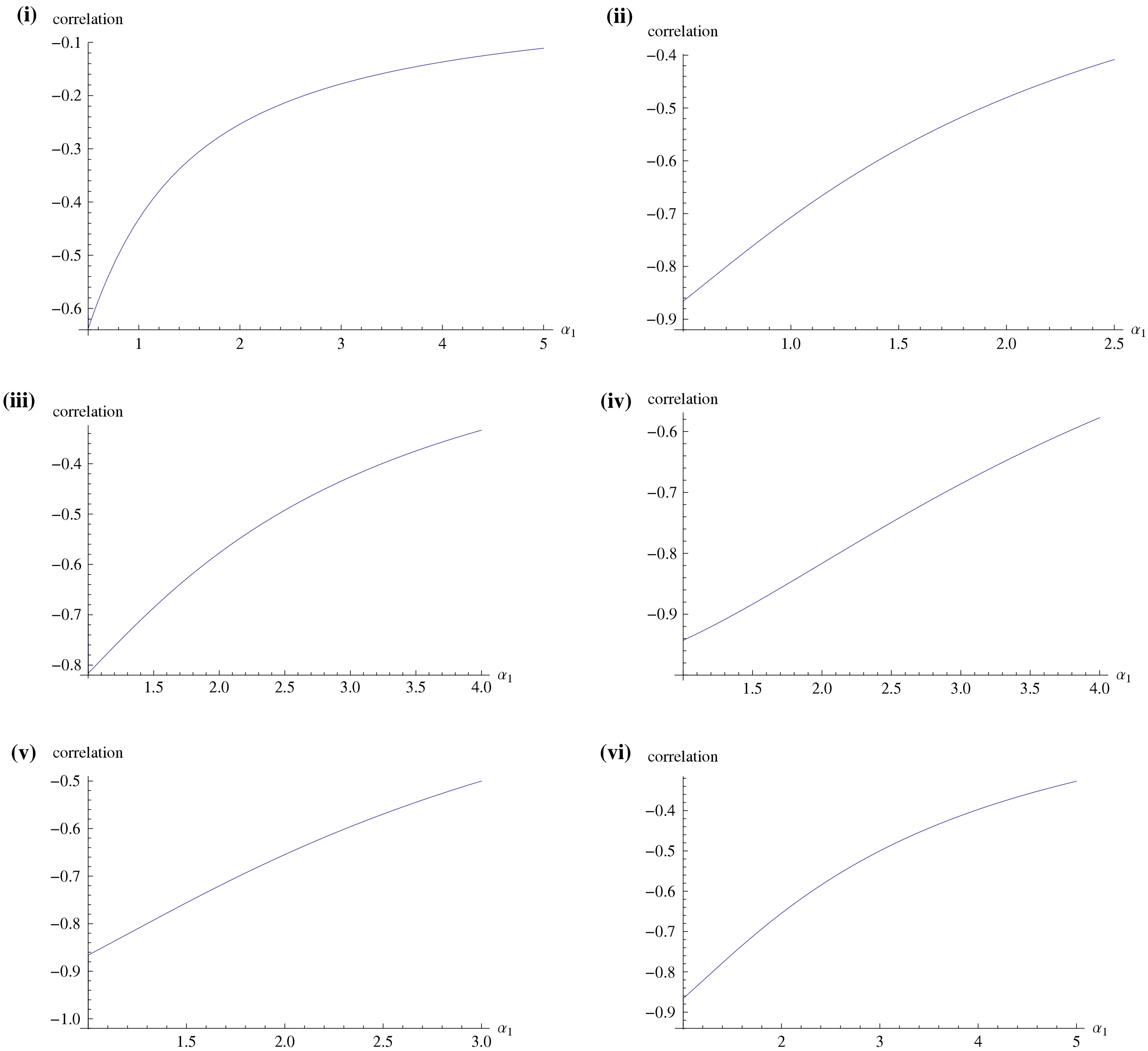

Fig. 2. Plots of the correlation coefficient ρ_{Z_1,Z_2} as a function of α_1. The six graphs are (i) $\beta_1 = 5, \alpha_2 = \beta_2, \nu_1 = 0.5, \nu_2 = 0.5, \nu_3 = 4$, (ii) $\beta_1 = 2.5, \alpha_2 = \beta_2, \nu_1 = 0.5, \nu_2 = 1, \nu_3 = 1$, (iii) $\beta_1 = 4, \alpha_2 = \beta_2, \nu_1 = 1, \nu_2 = 1, \nu_3 = 2$, (iv) $\beta_1 = 4, \alpha_2 = \beta_2, \nu_1 = 1, \nu_2 = 2, \nu_3 = 1$, (v) $\beta_1 = 3, \alpha_2 = \beta_2, \nu_1 = 1, \nu_2 = 1, \nu_3 = 1$, (vi) $\beta_1 = 5, \alpha_2 = \beta_2, \nu_1 = 1, \nu_2 = 1, \nu_3 = 3$.

Proof. Transforming $Y_1 = X_1/X_3$, $Y_2 = X_2/X_3$, $X_3 = X_3$ with the Jacobian $J(x_1, x_2, x_3 \to y_1, y_2, x_3) = x_3^2$ in (27), we obtain the joint p.d.f. of Y_1, Y_2 and X_3 as

$$K_1 y_1^{\nu_1-1} y_2^{\nu_2-1} x_3^{\nu_1+\nu_2+\nu_3-1} \exp[-(1+y_1+y_2)x_3]\,{}_1F_1(\beta_1-\alpha_1;\beta_1;y_1x_3) \\ \times {}_1F_1(\beta_2-\alpha_2;\beta_2;y_2x_3), \quad y_1>0,\ y_2>0,\ x_3>0, \tag{43}$$

where K_1 is defined in (28). To find the marginal p.d.f. of (Y_1, Y_2), we integrate (43) with respect to x_3 to get

$$K_1 y_1^{\nu_1-1} y_2^{\nu_2-1} \int_0^\infty x_3^{\nu_1+\nu_2+\nu_3-1} \exp[-(1+y_1+y_2)x_3]\,{}_1F_1(\beta_1-\alpha_1;\beta_1;y_1x_3)\,{}_1F_1(\beta_2-\alpha_2;\beta_2;y_2x_3)\,\mathrm{d}x_3. \tag{44}$$

Now, applying (18) and substituting for K_1, we obtain the desired result. □

For $\nu_1 = \nu_2 = 1/2$, $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$, the density in (42) reduces to

$$\frac{(y_1y_2)^{-3/2}}{4\pi(\nu_3-1)}[1+(1+y_1+y_2)^{1-\nu_3}-(1+y_1)^{1-\nu_3}-(1+y_2)^{1-\nu_3}].$$

If $\nu_1 = \nu_2 = \nu_3 = 1/3$, $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 2$, then the density in (42) simplifies to

$$\frac{4}{9\Gamma^3(1/3)}(y_1y_2)^{-5/3}[(1+y_1+y_2)\ln(1+y_1+y_2)-(1+y_1)\ln(1+y_1)-(1+y_2)\ln(1+y_2)]$$

and finally for $\nu_1 = \nu_2 = \nu_3 = 1/3$, $\alpha_1 = 1/2$, $\alpha_2 = 1$ and $\beta_1 = 1$ and $\beta_2 = 2$, it reduces to

$$\frac{4\sqrt{\pi}\,\Gamma(2/3)}{3\Gamma^3(1/3)\Gamma(1/6)} y_1^{-2/3} y_2^{-5/3} \ln\left(\frac{\sqrt{1+y_1+y_2}+\sqrt{1+y_2}}{1+\sqrt{1+y_1}}\right).$$

Corollary 4.1.1. *Let X_1, X_2 and X_3 be independent, $X_i \sim CH(\nu_i, \alpha_i, \nu_1+\nu_2+\nu_3, \text{kind } 1)$, $i = 1, 2$ and $X_3 \sim \mathrm{Ga}(\nu_3)$. Then, the p.d.f. of $(Y_1, Y_2) = (X_1/X_3, X_2/X_3)$ is given by*

$$\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\nu_2+\nu_3)\Gamma(\nu_1+\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\nu_1+\nu_2+\nu_3)\Gamma(\alpha_1-\nu_1)\Gamma(\alpha_2-\nu_2)} \frac{y_1^{\nu_1-1}y_2^{\nu_2-1}}{(1+y_1)^{\alpha_1}(1+y_2)^{\alpha_2}} \\ \times {}_2F_1\left(\alpha_1, \alpha_2; \nu_1+\nu_2+\nu_3; \frac{y_1y_2}{(1+y_1)(1+y_2)}\right), \quad y_1>0,\ y_2>0.$$

Proof. In (42), substitute $\beta_1 = \beta_2 = \nu_1+\nu_2+\nu_3$ and simplify the resulting expression using (20) and (17). □

Corollary 4.1.2. *Let X_1, X_2 and X_3 be independent, $X_1 \sim \mathrm{CH}(\nu_1, \alpha_1, \beta_1, \text{kind } 1)$, and $X_i \sim \mathrm{Ga}(\nu_i)$, $i = 2, 3$. Then, the p.d.f. of $(Y_1, Y_2) = (X_1/X_3, X_2/X_3)$ is given by*

$$\frac{\Gamma(\alpha_1)\Gamma(\beta_1-\nu_1)\Gamma(\nu_1+\nu_2+\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\beta_1)\Gamma(\alpha_1-\nu_1)} \frac{y_1^{\nu_1-1}y_2^{\nu_2-1}}{(1+y_1+y_2)^{\nu_1+\nu_2+\nu_3}} \\ \times {}_2F_1\left(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1; \beta_1; \frac{y_1}{1+y_1+y_2}\right), \quad y_1>0,\ y_2>0.$$

Further, if $\beta_1 = \nu_1+\nu_2+\nu_3$, then the p.d.f. of $(Y_1, Y_2) = (X_1/X_3, X_2/X_3)$ simplifies to

$$\frac{\Gamma(\alpha_1)\Gamma(\nu_2+\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\alpha_1-\nu_1)} \frac{y_1^{\nu_1-1}y_2^{\nu_2-1}(1+y_2)^{\alpha_1-(\nu_1+\nu_2+\nu_3)}}{(1+y_1+y_2)^{\alpha_1}}, \quad y_1>0,\ y_2>0. \tag{45}$$

It may be remarked here that the special case of the generalized Dirichlet type 2 density (42) given in the above corollary coincides with the bivariate case of the generalized Dirichlet type 2 density given in Thomas and Jacob [31, Eq. 2]. The bivariate density of Thomas and Jacob [31] has the form

$$\frac{\Gamma(a_1+a_2+a_3+b_1+b_2)\Gamma(a_2+a_3+b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_2+a_3+b_1+b_2)\Gamma(a_3+b_2)} \frac{v_1^{a_1-1}v_2^{a_2-1}(1+v_2)^{b_1}}{(1+v_1+v_2)^{a_1+a_2+a_3+b_1+b_2}}, \tag{46}$$

where $v_1 > 0$ and $v_2 > 0$. By comparing, one can easily observe that the densities in (45) and (46) are identical for $\nu_1 = a_1$, $\nu_2 = a_2$, $\nu_3 = a_3+b_2$ and $\alpha_1 = a_1+a_2+a_3+b_1+b_2$.

Theorem 4.2. *If the p.d.f. of* (Y_1, Y_2) *is given by* (42)*, then the marginal density of* Y_1 *is given by*

$$\frac{\Gamma(\alpha_1)\Gamma(\beta_1-\nu_1)\Gamma(\nu_1+\nu_3)}{\Gamma(\nu_1)\Gamma(\nu_3)\Gamma(\beta_1)\Gamma(\alpha_1-\nu_1)} \frac{y_1^{\nu_1-1}}{(1+y_1)^{\nu_1+\nu_3}} {}_2F_1\left(\beta_1-\alpha_1, \nu_1+\nu_3; \beta_1; \frac{y_1}{1+y_1}\right), \quad y_1 > 0.$$

Proof. The marginal density of Y_1 is given by

$$\frac{K}{B(\nu_1,\nu_2,\nu_3)} y_1^{\nu_1-1} \int_0^\infty \frac{y_2^{\nu_2-1}}{(1+y_1+y_2)^{\nu_1+\nu_2+\nu_3}} \times F_2\left(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; \frac{y_1}{1+y_1+y_2}, \frac{y_2}{1+y_1+y_2}\right) \mathrm{d}y_2. \tag{47}$$

Substituting $x = y_2/(1+y_1)$, expanding F_2 in series form and integrating x, the above integral is evaluated as

$$\begin{aligned}&\frac{\Gamma(\nu_2)\Gamma(\nu_1+\nu_3)}{\Gamma(\nu_1+\nu_2+\nu_3)}(1+y_1)^{-(\nu_1+\nu_3)} {}_2F_1(\beta_2-\alpha_2, \nu_2; \beta_2; 1)\, {}_2F_1\left(\beta_1-\alpha_1, \nu_1+\nu_3; \beta_1; \frac{y_1}{1+y_1}\right)\\ &= \frac{\Gamma(\nu_2)\Gamma(\nu_1+\nu_3)\Gamma(\beta_2)\Gamma(\alpha_2-\nu_2)}{\Gamma(\nu_1+\nu_2+\nu_3)\Gamma(\alpha_2)\Gamma(\beta_2-\nu_2)}(1+y_1)^{-(\nu_1+\nu_3)} {}_2F_1\left(\beta_1-\alpha_1, \nu_1+\nu_3; \beta_1; \frac{y_1}{1+y_1}\right),\end{aligned} \tag{48}$$

where the last line has been obtained by using (15). Finally, substituting (48) in (47) and simplifying, we get the desired result. □

Theorem 4.3. *If the p.d.f. of* (Y_1, Y_2) *is given by* (42)*, then the density of* $S = Y_1 + Y_2$ *is given by*

$$\frac{K}{B(\nu_1+\nu_2,\nu_3)} \frac{s^{\nu_1+\nu_2-1}}{(1+s)^{\nu_1+\nu_2+\nu_3}} \sum_{r=0}^\infty \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_1-\alpha_1)_r(\nu_1)_r}{(\beta_1)_r(\nu_1+\nu_2)_r\, r!} \frac{s^r}{(1+s)^r} \times {}_3F_2\left(\nu_1+\nu_2+\nu_3+r, \nu_2, \beta_2-\alpha_2; \beta_2, \nu_1+\nu_2+r; \frac{s}{1+s}\right), \quad s > 0. \tag{49}$$

Proof. By using the convolution formula, the density of S is given by

$$\begin{aligned}&\frac{K}{B(\nu_1,\nu_2,\nu_3)} \int_0^s \frac{y_1^{\nu_1-1}(s-y_1)^{\nu_2-1}}{(1+s)^{\nu_1+\nu_2+\nu_3}} F_2\left(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; \frac{y_1}{1+s}, \frac{s-y_1}{1+s}\right) \mathrm{d}y_1\\ &= \frac{K}{B(\nu_1,\nu_2,\nu_3)} \frac{s^{\nu_1+\nu_2-1}}{(1+s)^{\nu_1+\nu_2+\nu_3}}\\ &\quad\times \int_0^1 z^{\nu_1-1}(1-z)^{\nu_2-1} F_2\left(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; \frac{sz}{1+s}, \frac{s(1-z)}{1+s}\right) \mathrm{d}z.\end{aligned} \tag{50}$$

Now, expanding F_2 in series form and integrating z, the above integral is evaluated as

$$\begin{aligned}&\int_0^1 z^{\nu_1-1}(1-z)^{\nu_2-1} F_2\left(\nu_1+\nu_2+\nu_3, \beta_1-\alpha_1, \beta_2-\alpha_2; \beta_1, \beta_2; \frac{sz}{1+s}, \frac{s(1-z)}{1+s}\right) \mathrm{d}z\\ &= \frac{\Gamma(\nu_1)\Gamma(\nu_2)}{\Gamma(\nu_1+\nu_2)} \sum_{r=0}^\infty \frac{(\nu_1+\nu_2+\nu_3)_r(\beta_1-\alpha_1)_r(\nu_1)_r}{(\beta_1)_r(\nu_1+\nu_2)_r\, r!} \frac{s^r}{(1+s)^r}\\ &\quad\times {}_3F_2\left(\nu_1+\nu_2+\nu_3+r, \nu_2, \beta_2-\alpha_2; \beta_2, \nu_1+\nu_2+r; \frac{s}{1+s}\right).\end{aligned} \tag{51}$$

Finally, substituting (51) in (50) and simplifying, we get the desired result. □

Further, using the synthetic representation $(Y_1, Y_2) = (X_1/X_3, X_2/X_3)$, where X_1, X_2 and X_3 are independent, $X_i \sim \mathrm{CH}(\nu_i, \alpha_i, \beta_i, \text{kind } 1)$, $i = 1, 2$ and $X_3 \sim \mathrm{Ga}(\nu_3)$, we derive

$$E(Y_1^h Y_2^t) = E(X_1^h)E(X_2^t)E(X_3^{-(h+t)}).$$

Now, evaluating $E(X_1^h)$, $E(X_2^t)$ and $E(X_3^{-(h+t)})$, one obtains

$$E(Y_1^h Y_2^t) = \frac{\Gamma(\beta_1-\nu_1)\Gamma(\beta_2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\nu_3)\Gamma(\alpha_1-\nu_1)\Gamma(\alpha_2-\nu_2)} \times \frac{\Gamma(\nu_1+h)\Gamma(\nu_2+t)\Gamma(\nu_3-h-t)\Gamma(\alpha_1-\nu_1-h)\Gamma(\alpha_2-\nu_2-t)}{\Gamma(\beta_1-\nu_1-h)\Gamma(\beta_2-\nu_2-t)},$$

where $\beta_1 \ge \alpha_1$, $\beta_2 \ge \alpha_2$, $-\nu_1 < \mathrm{Re}(h) < \min\{\alpha_1-\nu_1, \beta_1-\nu_1\}$, $-\nu_2 < \mathrm{Re}(t) < \min\{\alpha_2-\nu_2, \beta_2-\nu_2\}$ and $\mathrm{Re}(h+t) < \nu_3$.

Table 2
Correlations between Y_1 and Y_2.

α_1	α_2	β_1	β_2	ν_1	ν_2	ν_3								
						2.5	3	5	10	15	20	30	50	100
3.5	3.5	4.5	4.5	1	1	0.204	0.161	0.088	0.041	0.027	0.02	0.013	0.008	0.004
4	4	5	5	1.5	1.5	0.25	0.2	0.111	0.053	0.034	0.026	0.017	0.010	0.005
7	8	9	10	3	4	0.548	0.477	0.313	0.169	0.115	0.088	0.059	0.036	0.018
7	8	9	10	4	5	0.443	0.374	0.230	0.117	0.079	0.059	0.04	0.024	0.012
15	**10**	**20**	**10**	**5**	**5**	**0.747**	**0.688**	**0.525**	**0.33**	**0.240**	**0.189**	**0.133**	**0.083**	**0.043**
15	**10**	**30**	**10**	**5**	**5**	**0.731**	**0.671**	**0.505**	**0.313**	**0.227**	**0.178**	**0.124**	**0.078**	**0.04**
10	**20**	**15**	**20**	**5**	**5**	**0.686**	**0.622**	**0.455**	**0.273**	**0.195**	**0.152**	**0.105**	**0.065**	**0.034**
10	**20**	**25**	**20**	**5**	**5**	**0.659**	**0.593**	**0.426**	**0.252**	**0.179**	**0.139**	**0.096**	**0.059**	**0.030**
20	**30**	**30**	**30**	**5**	**5**	**0.753**	**0.695**	**0.533**	**0.337**	**0.246**	**0.194**	**0.136**	**0.085**	**0.044**
20	22	22	24	5	5	0.760	0.704	0.543	0.346	0.254	0.200	0.141	0.088	0.046
20	22	22	24	15	15	0.814	0.767	0.623	0.425	0.322	0.26	0.187	0.120	0.063
30	**25**	**40**	**25**	**5**	**5**	**0.763**	**0.707**	**0.546**	**0.349**	**0.256**	**0.202**	**0.142**	**0.09**	**0.046**

Substituting appropriately for h and t in $E(Y_1^h Y_2^t)$, we obtain expressions for $E(Y_1Y_2)$, $E(Y_1)$, $E(Y_2)$, $E(Y_1^2)$ and $E(Y_2^2)$ as

$$E(Y_1Y_2) = \frac{\nu_1\nu_2(\beta_1-\nu_1-1)(\beta_2-\nu_2-1)}{(\nu_3-1)(\nu_3-2)(\alpha_1-\nu_1-1)(\alpha_2-\nu_2-1)}, \tag{52}$$

$$E(Y_i) = \frac{\nu_i(\beta_i-\nu_i-1)}{(\nu_3-1)(\alpha_i-\nu_i-1)}, \quad i=1,2, \tag{53}$$

$$E(Y_i^2) = \frac{\nu_i(\nu_i+1)(\beta_i-\nu_i-1)(\beta_i-\nu_i-2)}{(\nu_3-1)(\nu_3-2)(\alpha_i-\nu_i-1)(\alpha_i-\nu_i-2)}, \quad i=1,2. \tag{54}$$

Further, the correlation between Y_1 and Y_2 is defined as

$$\rho_{Y_1,Y_2} = \frac{\mathrm{Cov}(Y_1,Y_2)}{\sqrt{\mathrm{Var}(Y_1)\mathrm{Var}(Y_2)}},$$

where $\mathrm{Cov}(Y_1,Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$, $\mathrm{Var}(Y_i) = E(Y_i^2) - E(Y_i)^2$, $i=1,2$, and $E(Y_1Y_2)$, $E(Y_i)$, and $E(Y_i^2)$ are given in (52)–(54), respectively.

Table 2 gives the correlation coefficients between Y_1 and Y_2 for different values of $\alpha_1, \alpha_2, \beta_1, \beta_2, \nu_1, \nu_2$ and ν_3. For $\alpha_2 = \beta_2$, the density of (Y_1, Y_2) is given in Corollary 4.1.2 and in this case the correlation coefficients for different values of $\alpha_1, \beta_1, \nu_1, \nu_2$ and ν_3 are given by rows in bold. Further, for the bivariate beta distribution given by the density (45), the correlation coefficients are given in boxes among rows in bold. Note that, the bivariate density considered by Thomas and Jacob [31] is identical to the density (45) after re-parametrization. One can observe that for selected values of the parameters it is possible to find correlations close to 0 or 1. Note that when ν_3 is big the correlation between variables is small.

Finally, in the next result we give the limiting form of the joint density of $V_1 = \nu_3 Y_1$ and $V_2 = \nu_3 Y_2$ when ν_3 tends to infinity.

Theorem 4.4. *If the density of (Y_1, Y_2) is given by (42), then the joint density in limit of the variables $V_i = \nu_3 Y_i$, $i = 1, 2$, when ν_3 tends to infinite, is given by*

$$\lim_{\nu_3\to\infty} g(v_1,v_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1-\nu_1)\Gamma(\beta_2-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\alpha_1-\nu_1)\Gamma(\alpha_2-\nu_2)} \exp(-v_1-v_2)$$
$$\times v_1^{\nu_1-1} v_2^{\nu_2-1} {}_1F_1(\beta_1-\alpha_1;\beta_1;v_1) {}_1F_1(\beta_2-\alpha_2;\beta_2;v_2), \quad v_1>0,\ v_2>0,$$

where g is the joint density of the variables $V_1 = \nu_3 Y_1$ and $V_2 = \nu_3 Y_2$.

Proof. Similar to the proof of Theorem 3.4. □

Note that the density in limit of (V_1, V_2), when ν_3 tends to infinity, is the product of the densities of two independent random variables each distributed as confluent hypergeometric function type 1.

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