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Stability and boundedness of solutions of a kind of third-order delay differential equations*

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Abstract. This paper studies the stability and boundedness of solutions of certain nonlinear third-order delay differential equations. Sufficient conditions for the stability and boundedness of solutions for the equations considered are obtained by constructing a Lyapunov functional.

Mathematical subject classification: 34K20.

Key words: stability, boundedness, Lyapunov functional, differential equations of third-order with delay.

1 Introduction

This paper deals with the stability and boundedness of solution of the delay differential equation

$$\begin{aligned} x'''(t) + h(x'(t))x''(t) + g(x'(t-r(t))) + f(x(t-r(t))) \\ = p(t, x(t), x'(t), x(t-r(t)), x'(t-r(t)), x''(t)) \end{aligned} \tag{1.1}$$

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or its equivalent system

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= -h(y)z - g(y) - f(x) + \int_{t-r(t)}^{t} g'(y(s))z(s)ds \\
&\quad + \int_{t-r(t)}^{t} f'(x(s))y(s)ds + p(t, x, y, x(t-r(t)), y(t-r(t)), z),
\end{aligned}
\tag{1.2}
$$

where $0 \leq r(t) \leq \gamma, r'(t) \leq \beta$, $0 < \beta < 1$, β and γ are some positive constants, γ will be determined later, $f(x), g(y), h(y), p(t, x, y, x(t-r(t)),$ $y(t-r(t)), z)$ are continuous in their respective arguments. Besides, it is supposed that the derivatives $f'(x), g'(y)$ are continuous for all x, y with $f(0) = g(0) = 0$. In addition, it is also assumed that the functions $f(x(t-r(t)))$, $g(y(t-r(t)))$ and $p(t, x, y, x(t-r(t)), y(t-r(t)), z)$ satisfy a Lipschitz condition in $x, y, x(t-r(t)), y(t-r(t))$ and z; throughout the paper $x(t), y(t)$ and $z(t)$ are, respectively, abbreviated as x, y and z. Then the solution is unique. (See [5, pp. 14]).

In recent year, many books and papers dealt with the delay differential equation and obtained many good results, for example, [1, 2, 3, 18, 19, 21], etc. In many references, the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability and boundedness. (See [1-21]).

In particular, recently, Tunç [15], obtained sufficient conditions which ensure the stability and the boundedness of systems

$$x''' + a_1x'' + f_2(x'(t-r(t))) + a_3x = 0$$

and

$$x''' + a_1x'' + f_2(x'(t-r(t))) + a_3x = p(t, x, x', x(t-r(t)), x'(t-r(t)), x''),$$

where $r(t)$ is as defined above, a_1 and a_3 are some positive constants.

Our objective in this paper is to establish some sufficient conditions for the stability and for the boundedness of solutions of (1.1) in the cases $p \equiv 0$, $p \not\equiv 0$, respectively.

2 Stability

First, we will give the stability criteria for the general autonomous delay differential system. We consider

$$x' = f(x_t), \quad x_t = x(t+\theta), \quad -r \le \theta \le 0, \; t \ge 0, \tag{2.1}$$

where $f\colon C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r,0],\mathbb{R}^n)\colon \|\phi\| \le H\}$ and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| \le L(H_1)$ when $\|\phi\| \le H_1$.

Definition 2.1. An element $\psi \in C$ is in the ω-limit set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,\infty)$ and there is a sequence $\{t_n\}$, $t_n \longrightarrow \infty$, as $n \longrightarrow \infty$, with $\|x_{t_n}(\phi) - \psi\| \longrightarrow 0$ as $n \longrightarrow \infty$ where $x_{t_n}(\phi) = x(t_n+\theta,0,\phi)$ for $-r \le \theta \le 0$.

Definition 2.2 (See [17])**.** A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t,0,\phi)$, is defined on $[0,\infty)$, and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.

Lemma 2.1 (See [13])**.** *If $\phi \in C_H$ is such that the solution $x_t(\theta)$ of* (2.1) *with $x_0(\phi) = \phi$ is defined on* $[0,\infty)$ *and $\|x_t(\phi)\| \le H_1 < H$ for $t \in$* $[0,\infty)$*, then $\Omega(\phi)$ is a nonempty, compact, invariant set and*

$$\operatorname{dist}(x_t(\phi), \Omega(\phi)) \longrightarrow 0, \quad \text{as} \quad t \longrightarrow \infty$$

Lemma 2.2 (See [13])**.** *Let $V(\phi)\colon C_H \longrightarrow \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(0) = 0$ and such that*

(i) $W_1(|\phi(0)|) \le V(\phi) \le W_2(\|\phi\|)$ *where $W_1(r)$, $W_2(r)$ are wedges.*

(ii) $V'_{(2.1)}(\phi) \le 0$*, for $\phi \le C_H$.*

The the zero solution of (2.1) *is uniformly stable. If we define $Z = \{\phi \in C_H\colon V'_{(2.1)}(\phi) = 0\}$, then the zero solution of* (2.1) *is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.*

The following will be our main stability result for (1.1).

Theorem 2.1. *Consider system* (1.2) *with*

$$p(t,x,y,x(t-r(t)),y(t-r(t)),z)\equiv 0,\ f(x),\ f'(x),\ g(y),\ g'(y),\ h(y)$$

continuous in their respective arguments. Suppose further that

(i) *for some* $a>0, \epsilon_0>0, h(y)\geq a+\epsilon_0$ *for all* y;

(ii) *for some* $b>0, \frac{g(y)}{y}\geq b$ *for all* $y\neq 0$;

(iii) *for some* $c_0, \frac{f(x)}{x}\geq c_0$ *for all* $x\neq 0$;

(iv) *for some* $c>0, f'(x)\leq c$ *for all* x, *where* $ab-c>0$;

(v) *for some constants* $L, M, |f'(x)|\leq L, |g'(y)|\leq M$, *for all* x, y.

Then the zero solution of (1.2) *is asymptotically stable, provided that*

$$\gamma<\min\left\{\frac{2\epsilon_0(1-\beta)}{(L+M)(1-\beta)+(1+a)M},\frac{2(ab-c)(1-\beta)}{a[(L+M)(1-\beta)+(1+a)L]}\right\}.$$

Proof. Using the equivalent system form (1.2), our main tool is the following Lyapunov functional $V(x_t,y_t,z_t)$ defined as

$$\begin{aligned}V(x_t,y_t,z_t) &= \int_0^x f(\xi)d\xi+\int_0^y vh(v)dv\\ &\quad+a^{-1}\int_0^y g(u)du+\frac{1}{2}a^{-1}z^2+yz+a^{-1}f(x)y \qquad (2.2)\\ &\quad+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds,\end{aligned}$$

where λ and δ are positive constants which will be determined later.

The Lyapunov functional $V=V(x_t,y_t,z_t)$ defined in (2.2) can be arranged in the form

$$\begin{aligned}V(x_t,y_t,z_t) &= \frac{1}{2a}(ay+z)^2+\frac{1}{2ab}(f(x)+by)^2+\int_0^y[h(v)-a]vdv\\ &\quad+\frac{1}{a}\int_0^y\left[\frac{g(v)}{v}-b\right]vdv+\frac{1}{ab}\int_0^x[ab-f'(s)]f(s)ds\\ &\quad+\lambda\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds+\delta\int_{-r(t)}^0\int_{t+s}^t z^2(\theta)d\theta ds.\end{aligned}$$

On using (i), (ii), (iii) and (iv) of Theorem (2.1), we obtain

$$\begin{aligned} V(x_t, y_t, z_t) \ \geq \ & \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}(f(x)+by)^2 \\ & + \frac{1}{2}\epsilon_0 y^2 + \frac{c_0}{2ab}(ab-c)x^2 + \lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds \\ & + \delta \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds. \end{aligned}$$

Since the integrals

$$\lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds \quad \text{and} \quad \delta \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds$$

are non-negative,

$$V(x_t, y_t, z_t) \geq \frac{1}{2a}(ay+z)^2 + \frac{1}{2ab}(f(x)+by)^2 + \frac{1}{2}\epsilon_0 y^2 + \frac{c_0}{2ab}(ab-c)x^2.$$

Thus, we can find a positive constant D_1, small enough such that

$$V(x_t, y_t, z_t) \geq D_1(x^2+y^2+z^2). \tag{2.3}$$

Next, our target is to show that $V(x_t, y_t, z_t)$ satisfies the conditions of Lemma 2.2. First, by (1.2) and (2.2), we obtain

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) \ = \ & -\frac{1}{a}(h(y)-a)z^2 - \frac{1}{a}\left(a\frac{g(y)}{y} - f'(x)\right)y^2 \\ & + \left(y + \frac{1}{a}z\right)\left\{\int_{t-r(t)}^{t} g'(y(s))z(s)ds \right. \\ & \left. + \int_{t-r(t)}^{t} f'(x(s))y(s)ds\right\} + \lambda y^2 r(t) + \delta z^2 r(t) \\ & - \lambda(1-r'(t))\int_{t-r(t)}^{t} y^2(\theta)d\theta - \delta(1-r'(t)) \\ & \times \int_{t-r(t)}^{t} z^2(\theta)d\theta \end{aligned}$$

By (v) and using $2uv \leq u^2 + v^2$, we obtain

$$\begin{aligned}
\frac{d}{dt}V(x_t, y_t, z_t) \ \leq \ & -\frac{1}{a}(h(y) - a)z^2 - \frac{1}{a}\left(a\frac{g(y)}{y} - f'(x)\right)y^2 \\
& + \frac{1}{2a}(L + M + 2a\delta)z^2 r(t) + \frac{1}{2}(L + M + 2a\lambda)y^2 r(t) \\
& + \frac{L}{2a}\left[1 + a - 2\frac{a\lambda}{L}(1 - \beta)\right]\int_{t-r(t)}^{t} y^2(s)ds \\
& + \frac{M}{2a}\left[1 + a - 2\frac{a\delta}{M}(1 - \beta)\right]\int_{t-r(t)}^{t} z^2(s)ds,
\end{aligned}$$

since $r'(t) \leq \beta$, $0 < \beta < 1$.

If we choose $\lambda = \frac{(1+a)L}{2a(1-\beta)} > 0$, and $\delta = \frac{(1+a)M}{2a(1-\beta)} > 0$, and using (i), (ii), (iv) and $r(t) \leq \gamma$, we obtain

$$\begin{aligned}
\frac{d}{dt}V(x_t, y_t, z_t) \ \leq \ & -\frac{1}{2a}\left\{2\epsilon_0 - \gamma\left(\frac{(L + M)(1 - \beta) + (1 + a)M}{(1 - \beta)}\right)\right\}z^2 \\
& -\frac{1}{2a}\left\{2(ab - c) - \gamma\left(\frac{a[(L + M)(1 - \beta) + (1 + a)L]}{(1 - \beta)}\right)\right\}y^2,
\end{aligned}$$

choosing

$$\gamma < \min\left[\frac{2\epsilon_0(1 - \beta)}{(L + M)(1 - \beta) + (1 + a)M}, \frac{2(1 - \beta)(ab - c)}{a[(L + M)(1 - \beta) + (1 + a)L]}\right],$$

we have

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq -K(y^2 + z^2) \ \text{ for some } \ K > 0. \tag{2.4}$$

Finally, it follows that $\frac{d}{dt}V(x_t, y_t, z_t) \equiv 0$ if and only if $y_t = z_t = 0$, $\frac{d}{dt}V(\phi) < 0$ for $\phi \neq 0$ and $V(\phi) \geq u(|\phi(0)|) \geq 0$. Thus, in view of (2.3), (2.4) and the last discussion, it is seen that all the conditions of Lemma 2.2 are satisfied. This shows that the trivial solution of Eq. (1.1) is asymptotically stable. Hence the proof of Theorem 2.1 is complete.

Remark 2.1. If $h(x') = a$ in (1.1), then Theorem 2.1 reduces to Theorem 1 of [13] and a result of [1].

Remark 2.2. If $h(x') = a$, $f(x(t - r(t))) = cx(t)$ in (1.1), then Theorem 2.1 reduces to Theorem 2 of [15].

Example 1.1. Consider the third order nonlinear delay differential equation

$$\begin{aligned} x'''(t) + \left[x'^2(t) + x'(t) + 2\right] x''(t) + 4x'(t - r(t)) \\ + \sin x'(t - r(t)) + \frac{x(t - r(t))}{1 + x^2(t - r(t))} = 0 \end{aligned} \tag{2.5}$$

or its equivalent system form

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= -\left[y^2 + y + 2\right]z(t) - [4y + \sin y] - \frac{x}{1 + x^2} \\ &\quad + \int_{t-r(t)}^{t} (4 + \cos y(s))z(s)ds + \int_{t-r(t)}^{t} \frac{1 - x^2(s)}{(1 + x^2(s))^2} y(s)ds \end{aligned} \tag{2.6}$$

where we suppose that $0 \leq r(t) \leq \gamma$, $r'(t) \leq \beta$, β and γ are positive constants, γ will be determined later, $t \in [0, \infty)$. It is obvious that

$$3 \leq 4 + \frac{\sin y}{y} \ \text{ for all } \ y, \ (y \neq 0), \qquad 1 < y^2 + y + 2 \ \text{ for all } \ y.$$

Our main tool is the Lyapunov functional

$$\begin{aligned} V(x_t, y_t, z_t) &= \frac{1}{2}(y + z)^2 + \frac{1}{6}\left(\frac{x}{1 + x^2} + 3y\right)^2 \\ &\quad + \int_0^y \left[(v^2 + v + 2) - 1\right] v dv + \int_0^y \left(1 + \frac{\sin v}{v}\right) v dv \\ &\quad + \frac{1}{3}\int_0^x \left(3 - \frac{1 - \xi}{(1 + \xi^2)^2}\right) \frac{\xi}{1 + \xi^2} d\xi \\ &\quad + \lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds + \delta \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds \end{aligned} \tag{2.7}$$

where λ and δ are some positive constants which will be determined later.

It is clear that the functional $V(x_t, y_t, z_t)$ is positive definite. Hence it is evident from the terms contained in (2.7), that there exist sufficiently small positive constant δ_i, $(i = 1, 2, 3)$ such that

$$V(x_t, y_t, z_t) \geq \delta_1 x^2 + \delta_2 y^2 + \delta_3 z^2 + \lambda \int_{-r(t)}^{0} \int_{t+s}^{t} y^2(\theta) d\theta ds$$

$$+\delta \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta) d\theta ds \geq \delta_1 x^2 + \delta_2 y^2 + \delta_3 z^2 \geq \delta_4 (x^2 + y^2 + z^2)$$

where $\delta_4 = \min\{\delta_1, \delta_2, \delta_3\}$.

Now, the time derivative of the functional $V(x_t, y_t, z_t)$ in (2.7) with respect to the system (2.6) can be calculated as follows:

$$\begin{aligned} \frac{d}{dt} V(x_t, y_t, z_t) = & -\left[(1 + y + y^2) - \delta r(t)\right] z^2 \\ & -\left[\left(4 + \frac{\sin y}{y}\right) - \frac{1 - x^2}{(1 + x^2)^2} - \lambda r(t)\right] y^2 \\ & + (y + z)\left[\int_{t-r(t)}^{t} (4 + \cos y(s)) z(s) ds \right. \\ & \left. + \int_{t-r(t)}^{t} \frac{1 - x^2(s)}{(1 + x^2(s))^2} y(s) ds\right] - \lambda (1 - r'(t)) \\ & \times \int_{t-r(t)}^{t} y^2(s) ds - \delta (1 - r'(t)) \int_{t-r(t)}^{t} z^2(\theta) d\theta. \end{aligned} \tag{2.8}$$

Making use of the fact that

$$|4 + \cos y(s)| \leq 5, \quad \left|\frac{\sin y}{y}\right| \leq 1, \quad \left|\frac{1 - x^2}{(1 + x^2)^2}\right| \leq 1,$$

$$0 \leq r(t) \leq \gamma, \ r'(t) \leq \beta, \ 0 < \beta < 1$$

and the inequality $2|uv| \leq u^2 + v^2$, we obtain the following inequalities for all terms contained in the inequality (2.8), respectively:

$$-\left[(1 + y + y^2) - \delta r(t)\right] z^2 \leq -(1 - \delta\gamma) z^2;$$

$$-\left[\left(4 + \frac{\sin y}{y}\right) - \frac{1 - x^2}{(1 + x^2)^2} - \lambda r(t)\right] y^2 \leq -(2 - \lambda\gamma) y^2;$$

$$(y+z)\left\{\int_{t-r(t)}^{t}(4+\cos y(s))z(s)+\int_{t-r(t)}^{t}\frac{1-x^2(s)}{(1+x^2(s))^2}y(s)ds\right\}$$

$$\leq\left(\frac{5}{2}+\frac{1}{2}\right)r(t)y^2(t)+\left(\frac{5}{2}+\frac{1}{2}\right)r(t)z^2(t)$$

$$+\left(\frac{1}{2}+\frac{1}{2}\right)\int_{t-r(t)}^{t}y^2(s)ds+\left(\frac{5}{2}+\frac{5}{2}\right)\int_{t-r(t)}^{t}z^2(s)ds$$

$$\leq 3\gamma y^2(t)+3\gamma z^2(t)+\int_{t-r(t)}^{t}y^2(s)ds+5\int_{t-r(t)}^{t}z^2(s)ds$$

and

$$-\delta(1-r'(t))\int_{t-r(t)}^{t}z^2(s)ds\leq-\delta(1-\beta)\int_{t-r(t)}^{t}z^2(s)ds.$$

Gathering all these inequalities into (2.8), we have

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t) \;\leq\; & -(1-(\delta+3)\gamma)z^2-(2-(\lambda+3)\gamma)y^2\\ & -(\delta(1-\beta)-5)\int_{t-r(t)}^{t}z^2(s)ds\\ & -(\lambda(1-\beta)-1)\int_{t-r(t)}^{t}y^2(\theta)d\theta.\end{aligned}$$

Let us choose $\delta=\frac{5}{1-\beta}$ and $\lambda=\frac{1}{1-\beta}$. Then, it is easy to see that

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\left(1-\left(\frac{8-3\beta}{1-\beta}\right)\gamma\right)z^2-\left(2-\left(\frac{4-3\beta}{1-\beta}\right)\gamma\right)y^2. \quad (2.9)$$

Now, in view of (2.9), one can conclude for some positive constants ν and ρ that

$$\frac{d}{dt}V(x_t,y_t,z_t)\leq-\nu y^2-\rho z^2 \quad (2.10)$$

provided

$$\gamma<\min\left\{\frac{1-\beta}{8-3\beta},\frac{2(1-\beta)}{4-3\beta}\right\}.$$

It is also easy to see that $\frac{d}{dt}V(x_t,y_t,z_t)\equiv 0$ if and only if $z_t=y_t=0$, $\frac{d}{dt}V(\phi)<0$ for $\phi\neq 0$ and $V(\phi)\geq u(|\phi(0)|)\geq 0$. Thus, all the conditions of Lemma 2.2 are satisfied. This shows that the trivial solution of (2.5) is globally asymptotically stable.

3 The boundedness of solutions

Now, we shall state and prove our main result on boundedness of (1.1) with $p(t, x(t), x'(t), x(t - r(t)), x'(t - r(t)), x''(t)) \neq 0$.

Theorem 3.1 *Let all the conditions of Theorem* 2.1 *be satisfied, in addition assume that there are positive constants* H *and* H_1 *such that the following conditions are satisfied for every* x, y *and* z *in*

$$\Omega := \left\{(x, y, z) \in \mathbb{R}^3 \colon |x| < H_1, |y| < H_1, |z| < H_1, H_1 < H\right\}.$$

(i) $|p(t, x(t), y(t), x(t - r(t)), y(t - r(t)), z(t))| \leq q(t)$,

where $\max q(t) < \infty$ *and* $q \in L^1(0, \infty)$ *the space of integrable Lebesgue functions.*

Then, there exists a finite positive constant K_1 *such that the solution* $x(t)$ *of* (1.1) *defined by the initial functions*

$$x(t) = \phi(t), \quad x'(t) = \phi'(t), \quad x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \leq K_1, \quad |x'(t)| \leq K_1, \quad |x''(t)| \leq K_1$$

for all $t \geq t_0$, *where* $\phi \in C^2([t_0 - r, t_0], \mathbb{R})$, *provided that*

$$\gamma < \min\left\{\frac{2\epsilon_0(1-\beta)}{(L+M)(1-\beta)+(1+a)M}, \frac{2(ab-c)(1-\beta)}{a[(L+M)(1-\beta)+(1+a)L]}\right\}.$$

Proof. As in Theorem 2.1, the proof of this theorem also depends on the scaler differentiable Lyapunov functional $V = V(x_t, y_t, z_t)$ defined in (2.2). Now, since $p(t, x(t), y(t), x(t - r(t)), y(t - r(t)), z(t)) \neq 0$,in view of (2.2), (1.2) and (2.4), it can be easily followed that the derivative of the functional $V(x_t, y_t, z_t)$ along (1.2) satisfies the following inequality,

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\leq -K(y^2 + z^2) + |y + a^{-1}z| \\ &\times |p(t, x(t), y(t), x(t - r(t)), y(t - r(t)), z(t))| \\ &\leq -K(y^2 + z^2) + |y + a^{-1}z|q(t). \end{aligned}$$

Hence it follows that

$$\begin{aligned}\frac{d}{dt}V(x_t, y_t, z_t) &\leq -K(y^2 + z^2) + D_2(|y| + |z|)q(t) \\ &\leq D_2(|y| + |z|)q(t)\end{aligned}$$

for a constant $D_2 > 0$, where $D_2 = \max\{1, a^{-1}\}$.

Making use of the inequalities $|y| < 1 + y^2$ and $|z| < 1 + z^2$, it is clear that

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq D_2(2 + y^2 + z^2)q(t).$$

By (2.3), we have

$$(x^2 + y^2 + z^2) \leq D_1^{-1} V(x_t, y_t, z_t)$$

hence

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq D_2(2 + D_1^{-1} V(x_t, y_t, z_t))q(t).$$

Now, integrating the last inequality from 0 to t, using the assumption $q \in L(0, \infty)$ and Gronwall-Reid-Bellman inequality, we obtain

$$\begin{aligned}V(x_t, y_t, z_t) &\leq V(x_0, y_0, z_0) + 2D_2A + D_2D_1^{-1}\int_0^t (V(x_s, y_s, z_s))q(s)ds \\ &\leq (V(x_0, y_0, z_0) + 2D_2A)\exp\left(D_2D_1^{-1}\int_0^t q(s)ds\right) \qquad (3.1) \\ &\leq (V(x_0, y_0, z_0) + 2D_2A)\exp\left(D_2D_1^{-1}A\right) = K_2 < \infty,\end{aligned}$$

where $K_2 > 0$ is a constant, $K_2 = (V(x_0, y_0, z_0) + 2D_2A)\exp\left(D_2D_1^{-1}A\right)$ and $A = \int_0^\infty q(s)ds$.

Now, the inequalities (2.3) and (3.1) together yield that

$$x^2 + y^2 + z^2 \leq D_1^{-1} V(x_t, y_t, z_t) \leq K_3,$$

where $K_3 = K_2D_1^{-1}$. Thus, we conclude that

$$|x(t)| \leq K_3, \quad |y(t)| \leq K_3, \quad |z(t)| \leq K_3$$

for all $t \geq t_0$. That is

$$|x| \leq K_3, \quad |x'(t)| \leq K_3, \quad |x''(t)| \leq K_3$$

for all $t \geq t_0$.

The proof of the theorem is now complete.

Example 3.1. Consider the third order nonlinear delay differential equation

$$
\begin{aligned}
x'''(t) + [x'^2(t) + x'(t) + 2]\, x''(t) + 4x'(t - r(t)) \\
+ \sin x'(t - r(t)) + \frac{x(t - r(t))}{1 + x^2(t - r(t))} \qquad (3.2)
\end{aligned}
$$

$$
= \frac{2}{1 + t^2 + x^2(t) + x'^2(t) + x^2(t - r(t)) + x'^2(t - r(t)) + x''^2(t)}
$$

or its equivalent system form

$$
\begin{aligned}
x' &= y, \\
y' &= z, \\
z' &= -[y^2 + y + 2]z - [4y + \sin y] - \frac{x}{1 + x^2} \\
&\quad + \int_{t-r(t)}^{t} (4 + \cos y(s))z(s) + \int_{t-r(t)}^{t} \frac{1 - x(s)}{(1 + x^2(s))^2} y(s)ds \qquad (3.3) \\
&\quad + \frac{2}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2(t)}.
\end{aligned}
$$

Observe that

$$
\frac{2}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2} \le \frac{2}{1 + t^2} = q(t)
$$

for all $t \in \mathbb{R}^+, x, y, x(t - r(t)), y(t - r(t)), z$ and

$$
\int_0^\infty q(s)ds = \int_0^\infty \frac{2}{1 + s^2} ds = \pi < \infty, \quad \text{that is } q \in L^1(0, \infty).
$$

To show the boundedness of solutions we use as a main tool the Lyapunov functional (2.7). Now, in view of (2.10), the time derivative of the functional $V(x_t, y_t, z_t)$ with respect to the system (3.3) can be revised as follows:

$$
\frac{d}{dt} V(x_t, y_t, z_t) \le
$$

$$
-\nu y^2 - \rho z^2 + \frac{y + a^{-1}z}{1 + t^2 + x^2 + y^2 + x^2(t - r(t)) + y^2(t - r(t)) + z^2}.
$$

Making use of the fact

$$\frac{1}{1+t^2+x^2+y^2+x^2(t-r(t))+y^2(t-r(t))+z^2} \leq \frac{1}{1+t^2}$$

we get

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq -\nu y^2 - \rho z^2 + \frac{2|y+a^{-1}z|}{1+t^2}.$$

Hence it is obvious that

$$\begin{aligned} \frac{d}{dt}V(x_t, y_t, z_t) &\leq \frac{2|y+z|}{1+t^2} \leq \frac{2|y|+|z|}{1+t^2} \\ &\leq \frac{2(2+y^2+z^2)}{1+t^2} = \frac{4}{1+t^2} + \frac{2(y^2+z^2)}{1+t^2} \\ &\leq \frac{4}{1+t^2} + \frac{2D_1^{-1}}{1+t^2}V(x_t, y_t, z_t). \end{aligned} \tag{3.4}$$

Now, integrating (3.4) from 0 to t, using the fact $\frac{1}{1+t^2} \in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, it can be easily concluded the boundedness of all solutions of (3.2).

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