Spatial Epidemic Patterns Recognition Using Computer Algebra

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Abstract. An exploration in Symbolic Computational bio-surveillance is showed. The main obtained results are that the geometry of the habitat determines the critical parameters via the zeroes of the Bessel functions and the explicit forms of the static and non-static spatial epidemic patterns.

Keywords: spatial epidemic patterns, pan-endemic state, pan-epidemic state, critical parameter, velocity of propagation, endemic boundary, special functions.

1 Introduction

A very interesting issue, which remains practically unexplored, it is concerned with the spatial propagation of the diseases inside habitats with boundaries and the patterns or spatial profiles that the disease displays. In the present work we consider the question about the spatial patterns that the infectious diseases generate, for the case of a circular habitat with endemic boundary, which is initially free of infection. The present work is an exploration in the land of the Symbolic Computational Spatial Bio-Surveillance (SCSBS).

2 Methods

We consider here a circular habitat with a radio *a*. The model is as follows [1-3]

$$\frac{\partial}{\partial t}\mathrm{X}(r,t) = \frac{\eta\left(\left(\frac{\partial}{\partial r}\mathrm{X}(r,t)\right) + r\left(\frac{\partial^2}{\partial r^2}\mathrm{X}(r,t)\right)\right)}{r} - \beta\,\mathrm{X}(r,t)\,\mathrm{Y}(r,t) \tag{1}$$

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$$\frac{\partial}{\partial t}\mathrm{Y}(r,t)=\frac{\eta\left(\left(\frac{\partial}{\partial r}\mathrm{Y}(r,t)\right)+r\left(\frac{\partial^2}{\partial r^2}\mathrm{Y}(r,t)\right)\right)}{r}+\beta\,\mathrm{X}(r,t)\,\mathrm{Y}(r,t)-\gamma\,\mathrm{Y}(r,t) \quad (2)$$

$$\frac{\partial}{\partial t}\mathrm{Z}(r,t)=\frac{\eta\left(\left(\frac{\partial}{\partial r}\mathrm{Z}(r,t)\right)+r\left(\frac{\partial^2}{\partial r^2}\mathrm{Z}(r,t)\right)\right)}{r}+\gamma\,\mathrm{Y}(r,t) \quad (3)$$

Where X(r,t) is the susceptible density, Y(r,t) is the infective density, Z(r,t) is the density of removed, β is the infectiousness, γ the recovery constant and η is the diffusivity.

2.1 Static Spatial Epidemic Patterns

The following patterns for infected and removed people are obtained if (6) is satisfied.

$\mathrm{Y}(r)=\dfrac{Y_b\,J_0\left(\sqrt{\dfrac{\beta N-\gamma}{\eta}}\,r\right)}{J_0\left(\sqrt{\dfrac{\beta N-\gamma}{\eta}}\,a\right)}$ (4)	$\mathrm{Z}(r)=Z_b+\dfrac{\gamma\,Y_b\left(\dfrac{J_0\left(\sqrt{\dfrac{\beta N-\gamma}{\eta}}\,r\right)}{J_0\left(\sqrt{\dfrac{\beta N-\gamma}{\eta}}\,a\right)}-1\right)}{\beta N-\gamma}$ (5)

The total number of infected individuals is given by (7).

$(1<R_0)<1+\dfrac{5.784025\,\eta}{\gamma\,a^2}$ (6)	$Y=\dfrac{2\,\pi\,Y_b\,a\,\sqrt{\eta}\,J_1\left(\dfrac{\sqrt{\beta N-\gamma}\,a}{\sqrt{\eta}}\right)}{J_0\left(\sqrt{\dfrac{\beta N-\gamma}{\eta}}\,a\right)\sqrt{\beta N-\gamma}}$ (7)

The Fig.1. illustrates the pan-endemic threshold condition (6).

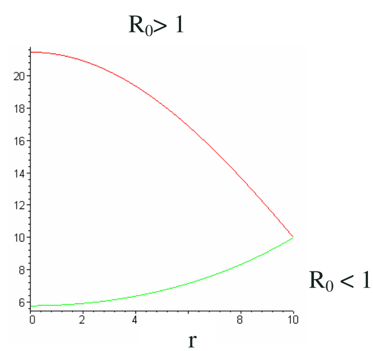


Fig. 1. Two static epidemic profiles of infected people according with the equation (4) are showed. This is an illustration of the pan-endemic threshold condition (6). The curve of the top corresponds to the super-critical case and the curve of the bottom corresponds to the sub-critical case. In the super- critical case the infection is more protuberant at the center of the habitat r = 0 but in the sub-critical case the infection remains confined to the endemic boundary r=*a*.

2.2 Dynamic Spatial Epidemic Patterns

The equation (8) can be derived [1], for the case of a circular habitat with endemic boundary when the infective profile Y(r,t) is restricted by the boundary condition Y(a,t)=Y_b and it is subjected to the initial condition Y(r,0)=0. The initial condition indicates that initially the habitat is free of infection but the boundary condition represents the case when the disease is permanently introduced from the endemic boundary. This boundary condition of the Dirichlet kind corresponds to an emergent infectious disease but a boundary condition of the Newman kind is a model of a biological attack. Here only the Dirichlet condition is studied.

$$Y(r,t) = \frac{Y_b J_0\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, r\right)}{J_0\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, a\right)} + \left(\sum_{n=1}^{\infty}\left(\frac{2 Y_b J_0\left(\frac{\alpha_n r}{a}\right) e^{\left(\frac{(\beta N a^2-\gamma a^2-\alpha_n^2\eta)t}{a^2}\right)}\alpha_n \eta}{J_1(\alpha_n)(\beta N a^2-\gamma a^2-\alpha_n^2\eta)}\right)\right) \quad (8)$$

The corresponding critical parameter is quantized as (9) [1] , where n is an positive integer such $n \geq 1$, and α_n are the zeros of the Bessel function J_0. The fundamental critical parameter corresponds to n=1 with α_1=2.405 and it is given like (10) [1].

$R_{0,n} = \frac{\beta N}{\gamma + \frac{\alpha_n^2 \eta}{a^2}}$ (9)	$R_{0,1} = \frac{\beta N}{\gamma + \frac{5.784025\,\eta}{a^2}}$ (10)

For hence the pan-epidemic state is established in the circular habitat with endemic boundary only when $R_{0,1} > 1$.

The total number of infectious individuals for every time in the habitat is computed as

$$Y(t) = 2\pi\left(\frac{Y_b\sqrt{\eta}\, a\, J_1\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, a\right)}{J_0\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, a\right)\sqrt{\beta N-\gamma}} + \left(\sum_{n=1}^{\infty}\left(\frac{2 Y_b a^2 e^{\left(\frac{(\beta N a^2-\gamma a^2-\alpha_n^2\eta)t}{a^2}\right)}\eta}{\beta N a^2-\gamma a^2-\alpha_n^2\eta}\right)\right)\right) \quad (11)$$

and the incidence is given by

$$\frac{d}{dt}Y(t) = 2\pi\left(\sum_{n=1}^{\infty}\left(2 Y_b e^{\left(\frac{(\beta N a^2-\gamma a^2-\alpha_n^2\eta)t}{a^2}\right)}\eta\right)\right) \quad (12)$$

The velocity of propagation of the disease in the circular habitat with endemic boundary and when the initial focus of infection is located justly in such endemic boundary, can be defined by (13) where r(t) is the instantaneous center of mass for the profile of infectious people at time t, Y(r,t); and it is given by (14).

$$v(t) = -\left(\frac{d}{dt} r(t)\right)$$ (13)	$$r(t) = \frac{\int_0^a Y(r,t)\, r^2\, dr}{\int_0^a Y(r,t)\, r\, dr}$$ (14)

The Fig. 2 gives an illustration of the equation (8). The Fig. 3 is depicted according with the equations (13) and (14).

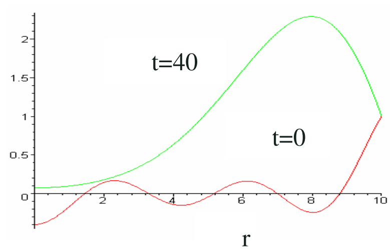


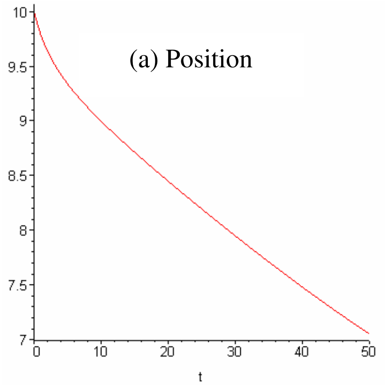
Fig. 2. Two spatial epidemic profiles of infected people for two different times, according with the equation (8) are showed. The curve at the top is for *t*=40 and the curve at the bottom is for *t*=0. Initially the infection is only concentrated at the endemic boundary but for *t* =40 the infection is yet present in the interior of the habitat and the more affected zone is not now the boundary.

2.3 Parameter Estimation

Using the previously given equations jointly with real data it is possible to estimate the basic parameters and the critical parameters. We use the least square method for the optimal adjustment between observed and predicted static and non-stationary spatial epidemic patterns. Assuming that the real data are organized as the pattern y(r,t) and the predicted pattern has the form Y(r,t, p_1, p_2, …) where p_i are the parameters of the model, it is possible to construct the summation of squares of deviations given by (15).

$$S(r, t, p_i) = \sum_j \int_0^a r^2 \left(y(r, t_j) - Y(r, t_j, p_i)\right)^2 dr \quad , \qquad (15)$$

where t_j are the sampling times. and for hence the optimal adjustment is obtained when the basic parameters of the model are determined as the solutions of the following system of equations.



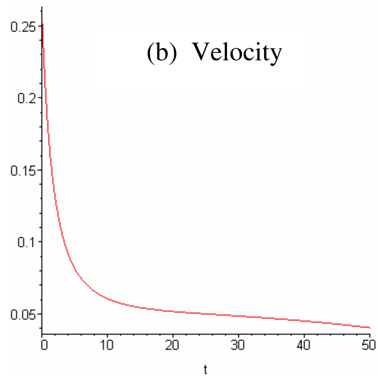


Fig. 3. (a) Curve for the trajectory of the infection according with (14). (b) Curve for the velocity of propagation according with (13) and (14). These curves were computed assuming that the fundamental pan-epidemic threshold condition $R_{0,1} > 1$ is satisfied.

3 Spatial Epidemic Patterns Recognition

Using the equation (15) for the case of the equations (4) and (8), it is possible to estimate the epidemic parameters, to obtain the critical parameters and to derive an approximation for the velocity of propagation. Such substitutions generate very complicated integrals which are difficult to hand at general and they will not be presented here by space reasons. Only the calculations for the equation (4) will be considered here. Specifically, assuming that the observed profile y(r) has the form $y(r) = AJ_0(\delta r)$, where A and δ are known parameters, the substitution of (4) in (15) gives an integral which can be symbolically computed using Maple and the final result is showed by the equation (16)

$$S(\beta,\gamma) = \frac{1}{3} a^3 A^2 \, F\left(\left[\frac{1}{2},\frac{3}{2}\right],\left[1,1,\frac{5}{2}\right],-a^2\delta^2\right) - \int_0^a \frac{r^2 A J_0(\delta r) Y_b J_0\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, r\right)}{J_0\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, a\right)} dr + \frac{1}{3} \frac{a^3 Y_b^2 \, F\left(\left[\frac{1}{2},\frac{3}{2}\right],\left[1,1,\frac{5}{2}\right],-\frac{(\beta N-\gamma)a^2}{\eta}\right)}{J_0\left(\sqrt{\frac{\beta N-\gamma}{\eta}}\, a\right)^2} \qquad (16)$$

where F is the hypergeometric function. The optimization of (16) gives the following results:

$A = \frac{Y_b}{J_0\left(\sqrt{\frac{\beta N - \gamma}{\eta}}\, a\right)}$	$\delta = \sqrt{\frac{\beta N - \gamma}{\eta}}$ $\beta = \frac{\gamma + \eta\, \delta^2}{N}$	$R_0 = 1 + \frac{\eta\, \delta^2}{\gamma}$ $\delta < \frac{2.405}{a}$

4 Conclusions

1. The critical parameter for the existence of the pan-endemic configuration depends only over epidemical parameters but it is limited by the geometry of the habitat by mean of the first zero of J_0.
2. The critical parameter for the existence of the pan-epidemic configuration is quantized in a similar way to a certain systems of quantum mechanics and it depends not only over the usual epidemical parameters but that also it depends on the geometry via the zeros of J_0.

References

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