Rational solutions of the full Kostant-Toda equation

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Abstract

In this paper a commuting hierarchy of flows including the full Kostant-Toda equation is studied. A Mulase's approach which uses the so-called Borel-Gauss decomposition leads to explicit rational solutions of the full Kostant-Toda equation. Finally, we investigate how the solution blows up in finite time in the case of matrices of order 2.

Keywords: Kostant-Toda equation, discrete KP hierarchy, Borel-Gauss decomposition.

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1 Introduction

In [3] Ercolani, Flaschka and Singer introduced an extension of the Toda equation which can be written in the Lax form

$$\frac{dL}{dt} = [L, P] \tag{1}$$

where

$$L = \sum_{i=1}^{n-1} E_{i,i+1} + \sum_{1 \leq j \leq i \leq n} b_{ij} E_{i,j} \tag{2}$$

and

$$P = 2 \sum_{1 \leq j < i \leq n} b_{ij} E_{i,j} \tag{3}$$

We have denoted by $E_{p,q}$ the matrix with 1 in the (p,q) entry and 0 in all other entries. The equation (1) with L and P expressed by (2) and (3) is called the "full Kostant-Toda" equation. Kodama and Ye [5] have given an explicit formula for the solution to the initial value problem. The Kodama-Ye solution is obtained by a beautiful generalization of the orthogonalization procedure of Szegö.

Also, they studied the behaviors of the solutions and showed that the solutions, found by them, are of two types, having either sorting property or blowing up to infinity in finite time.

Bloch and Gekhtman has shown how the "full Kostant-Toda" equation may be viewed as a gradient flow [2]. In that paper the relationship of this equation to double bracket equations was discussed.

On the other hand, in the work of Adler and van Moerbeke [1] one knows that t-perturbed weights lead to moments, polynomials and a matrix evolving according to the semi-infinite discrete KP-hierarchy. The relation of the Riemann-Hilbert method to obtain asymptotics for orthogonal polynomials to a semi-infinite LU factorization of the moment matrix was extensively studied by these authors. Also Felipe and Ongay [4] adapted the algebraic approach due to Mulase [7], [8] for the study of the KP hierarchy to the discrete KP hierarchy. In particular, this approach allowed them to consider on an almost equal footing the cases of semi-infinite and bi-infinite matrices.

In this work we show that "full Kostant-Toda" appear in a finite counterpart of the discrete KP hierarchy, and that the results of [4] remain valid in this context almost verbatim, then a large class of solutions is obtained. Despite the fact that our method is quite different from that used by Kodama and Ye, in the cases of 2 by 2 matrices the solutions turn out to be of the same type. We conjecture that all our solutions are of the type found by Kodama and Ye.

2   Algebraic approach for the finite discrete KP hierarchy

The goals of this section are:

1) To introduce a natural commuting hierarchy of flows including the "full Kostant-Toda" equation as the first element of this hierarchy. We make three comments of this hierarchy: first, it can be defined by a Lax type operator (matrix) with respect to the shift matrix and its transpose; second, the Lax matrix introduced admits a dressing matrix in terms of which the hierarchy can be rewritten and third, the existence of a Sato-Wilson matrix. We mention that the situation is similar to the Sato theory and his dressing technique (pseudodifferential theory).

2) To study the integrability in the sense of Frobenius for the hierarchy introduced. The key point in our method is the so-called Gauss-Borel decomposition

3) To ilustrate the result obtained in the first part of this section by taking an explicit form of the matrix L of order 2.

Now, let us observe how the "full Kostant-Toda" is included into the Sato's framework. This being, to the best of our knowledge, the first time that it is introduced to a finite system of nonlinear ordinary differential equations.

Let Λ be the n by n matrix with ones on the first upper diagonal and zero in the remaining entries

$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

and Λ^T its transpose. The matrix Λ is the shift operator of coordinates for vectors in R^n. One can check that the components of Λ^k are zero except $k - th$ upper diagonal where it has ones. Note that $\Lambda^n = \Lambda^{T^n}$ is the zero matrix. Let L be the matrix

$$L = \Lambda + D_0 + \sum_{k=1}^{n-1} D_i (\Lambda^T)^i \tag{4}$$

where D_i are diagonal matrices. We assume that the entries of L are functions depending on parameters $t = (t_1, ..., t_{n-1})$.

Definition 2.1. *The **finite discrete KP hierarchy** is the Lax system*

$$\frac{\partial L}{\partial t_k} = \left[L_{\geq}^k, L \right] \qquad k = 1, ..., n - 1 \tag{5}$$

where $L_{\geq}$ is the upper triangular part of the matrix L. Note that the "full Kostant-Toda" equation is the first of this hierarchy after a change of scale.

From now we will use the notation $L_>$, $L_<$ and $L_{\leq}$ for the strictly upper, strictly lower and lower triangular parts of the matrix L. Let L be a matrix defined by (4). If there exists the matrix S of the form

$$S = I + S_1 \Lambda^T + S_2 (\Lambda^T)^2 + ... + S_{n-1} (\Lambda^T)^{n-1} \tag{6}$$

such that

$$L = S \Lambda S^{-1} \tag{7}$$

where S_i are diagonal matrices, we call L of (7) as **Lax matrix**.
The operator S is called **dressing matrix** and it is unique, up to right multiplication by an invertible matrix, taking the form of (6) that commutes with Λ.
If there is a dressing matrix such that

$$\frac{\partial S}{\partial t_k} = -L_{<}^k S \qquad k = 1, ..., n - 1 \tag{8}$$

where $L = S \Lambda S^{-1}$ then L satisfies (5). Conversely, if L is a Lax matrix which is a solution of (5) then exists a dressing matrix S of L which is a solution of (8), operator S is called the Sato-Wilson matrix. It was shown in [4] that from (5) it follows the following equations

$$\frac{\partial L_{\geq}^i}{\partial t_j} - \frac{\partial L_{\geq}^j}{\partial t_i} = \left[L_{\geq}^i, L_{\geq}^j \right] \qquad i \neq j, \qquad i, j = 1, ..., n - 1. \tag{9}$$

Let us observe that for a given order, it is always possible to find matrices L of the type (4), which are not Lax matrices. In fact, next we put the Lax matrices as functions of their dressing operators for $n = 2$ and $n = 3$.

$$n = 2 \qquad\qquad\qquad n = 3$$

$$S^{-1} \qquad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}$$

$$S \qquad \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ab - c & -b & 1 \end{pmatrix}$$

$$L = S\Lambda S^{-1} \quad \begin{pmatrix} a & 1 \\ -a^2 & -a \end{pmatrix} \quad \begin{pmatrix} a & 1 & 0 \\ -a^2 + c & -a + b & 1 \\ (a^2 - c)b - ac & (ab - c) - b^2 & -b \end{pmatrix}$$

For $L \in \Lambda + B_-$ (B_- is the algebra of lower triangular matrices), there exists $U \in N_-$ (N_- is subgroup of lower triangular matrices where all of diagonal components are 1) such that $U^{-1}LU = \Lambda_p$ where $\Lambda_p = \Lambda - \sum_{k=1}^{n} p_{n-k} E_{k,1}$. This is an application of the sweeping out method. In general case, Kostant studied about this normalization in [6].

Theorem 2.2. *Suppose that $L \in \Lambda + B_-$ is normalized such as $U^{-1}LU = \Lambda_p$ by $U \in N_-$, then L is not a Lax matrix unless $p_{n-1} = p_{n-2} = \cdots = p_1 = p_0 = 0$, that is $\Lambda_p = \Lambda$.*

Proof. Suppose that L is normalized such as $U^{-1}LU = \Lambda_p$, that is $L = U\Lambda_p U^{-1}$, where $U \in N_-$. By easy calculation one has

$$|\lambda - L| = |\lambda - \Lambda_p| = \lambda^n - p_{n-1}\lambda^{n-1} - \cdots - p_0.$$

Since L is a Lax matrix, the any eigenvalues of L are 0. This implies $p_{n-1} = p_{n-2} = \cdots = p_1 = p_0 = 0$. $\qquad \square$

Remark 2.3. *In general, the eigenvalues of matrices L of the form (4) that are solutions of the finite discrete KP hierarchy are the celebrated preserving quantities. If L is a Lax matrix then*

$$|\lambda I - L| = |\lambda I - S\Lambda S^{-1}| = |\lambda I - \Lambda| = \lambda^n,$$

thus the eigenvalues are all 0.

We say that a matrix M admits a Gauss-Borel descomposition if M can be written in the form

$$M = M_\leq M_\geq \qquad\qquad (10)$$

where $M_\leq$ has ones on the principal diagonal and $M_\geq$ has non-zero elements on the principal diagonal. Descomposition (11) is equivalent to

$$M = (I + G_< G_0^{-1})(G_0 + G_>)$$

where G_0 is a diagonal matrix with non-zero elements. It can be proved that the Gauss-Borel descomposition is unique. A necessary and sufficient condition in order to M admits the Gauss-Borel decomposition is that $M_k \neq 0$, $k = 1, ..., n$ where M_k is the determinant of the principal order k submatrix [4].

Particular we are interested in the space M^* of the matrices M depending on t, admitting a Gauss-Borel decomposition and satisfying $M_{\geq}(0) = I$. Note that we dont consider the condition $M_{\leq}(0) = I$ because we would like to exclude the trivial solution $L = \Lambda$.

Let Z be a 1-form associate to L given by

$$Z = \sum_{k=1}^{n-1} L_{\geq}^k dt_k. \tag{11}$$

If L satisfies (5) then Z satisfies

$$dZ = \frac{1}{2}[Z, Z] \tag{12}$$

which is equivalent to the Zakharov-Shabat equations (9).

Let Ω be the 1-form such as

$$\Omega = \sum_{k=1}^{n-1} \Lambda^k dt_k \tag{13}$$

which is a trivial solution of (12). The 1-form Ω is the heart of the Mulase technique in the discrete case, in fact each solution of (5) yields a solution of

$$dM = \Omega M \tag{14}$$

in M^* and conversely for a solution of (14) in M^*, we can build a matrix L that is a solution of (5). The solutions of (14) take the form

$$M = e^{\sum_{k=1}^{n-1} \Lambda^k t_k} M_0 \ .$$

Let us consider the Gauss-Borel factorization of M

$$M = S^{-1}Y \tag{15}$$

where S^{-1} is a lower triangular matrix with ones on the principal diagonal and Y is an upper triangular matrix with non-zero elements on the principal diagonal. For $M \in M^*$, we have that

$$M_0 = M(0) = S^{-1}(0)Y(0) = S^{-1}(0)I = S_0^{-1}$$

then

$$M = e^{\sum_{k=1}^{n-1} \Lambda^k t_k} M_0 = e^{\sum_{k=1}^{n-1} \Lambda^k t_k} S_0^{-1}$$

where S_0^{-1} is a matrix takes the form of (6).

3   Blows up to infinity of rational solutions

In order to obtain rational solutions that blows up to infinity, we present some auxiliar results.

Lemma 3.1. *The matrix*

$$M_p(t) = \begin{pmatrix} g(t) & e(t) \\ f(t)g(t) & f(t)e(t) + h(t) \end{pmatrix}$$

belongs to M^* if the conditions $g(0) = h(0) = 1$ and $e(0) = 0$ hold. We suppose that the function $g(t)$ is not identicaly zero.

Proof. The proof is straightforward.   □

Remark 3.2. *If in the previous Lemma we assume that $|M_p(t)| = 1$, then $h(t) = \frac{1}{g(t)}$.*

Proposition 3.3. *Let $L_p = S_p \Lambda S_p^{-1}$ such that $M_p(t) = S_p^{-1} \cdot Y = (M_p(t))_{\leq} \cdot ((M_p(t)))_{\geq}$, then in order to the Lax matrix L_p can be a solution of the finite discrete KP hierarchy is neccesary and sufficient that $f' + f^2 = 0$.*

Proof. As it was seen before

$$L_p = \begin{pmatrix} f & 1 \\ -f^2 & -f \end{pmatrix},$$

thus

$$\frac{\partial L_p}{\partial t} = \begin{pmatrix} f' & 0 \\ -2ff' & -f' \end{pmatrix},$$

on the other hand,

$$\left[(L_p)_{\geq}, L_p \right] = \begin{pmatrix} -f^2 & 0 \\ 2f^3 & f^2 \end{pmatrix},$$

so, the Proposition is proved.   □

Next, we construct an explicit solution for the full Kostant-Toda equation in the case of 2 by 2 matrices. As will be seen, the solution can blow up to infinity in finite time.
Let

$$M_p(t) = \begin{pmatrix} 1 + at & t \\ a & 1 \end{pmatrix}$$

this case correspond to take

$$f(t) = \frac{a}{1 + at}, \; g(t) = 1 + at, \; e(t) = t, \; h(t) = \frac{a}{1 + at},$$

thus

$$L_p(t) = \begin{pmatrix} \frac{a}{1+at} & 1 \\ -\frac{a^2}{(1+at)^2} & -\frac{a}{1+at} \end{pmatrix}. \tag{16}$$

To see that this $L_p(t)$ satisfies the equation, it is enough to observe that $f(t) = \frac{a}{1+at}$ is solution of the equation $f' + f^2 = 0$, but it is evident. In particular, (16) implies that if $a < 0$ then for $t = -\frac{1}{a}$ the solution blows up to infinity, it also implies that the solution has the sorting property if $a > 0$.

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