A novel cell-coupling leading to nonlocal interactions

You may also be interested in:

Electrical implementation of a complete synchronization dynamic system
C Gonçalves and L G Neto

Locking-time and Information Capacity in CML with Statistical Periodicity
R M Szmoski, R F Pereira, F A S Ferrari et al.

Characterization of hyperchaotic states in the parameter-space of a modified Lorenz system
Marcos J Correia and Paulo C Rech

A minimalist model of calcium-voltage coupling in GnRH cells
Martin Rennie, Rossanna Chan, David Schneider et al.

Scaling Laws in the Transient Dynamics of Firefly-like Oscillators
N Rubido, C Cabeza, G M Ramírez Ávila et al.

Self-organization and pattern formation in coupled Lorenz oscillators under a discrete symmetric transformation
A Carrillo and B A Rodríguez

Neural networks for emulation variational method for data assimilation in nonlinear dynamics
Helaine Cristina Morais Furtado, Haroldo Fraga de Campos Velho and Elbert E N Macau

Dynamic valid models for the conservative Hénon-Heiles system
S B Bastos and E M A M Mendes

On the effect of a parallel resistor in the Chua's circuit
Flavio Prebianca, Holokx A Albuquerque and Rero M Rubinger

A novel cell-coupling leading to nonlocal interactions

David Ciro[†] and Boris A. Rodríguez[‡]

[†,‡] Instituto de Física
Universidad de Antioquia
Medellín, A.A. 1226 Colombia

E-mail: [†]davidcirotaborda@pegasus.udea.edu.co
           [‡]banghelo@fisica.udea.edu.co

Abstract. It is shown that nonlocal interactions and phenomena can be achieved through local considerations, in which the departure of some scalar field from an harmonic one in a point changes as a function of the field itself in such point. After discretizing the equation of motion, it is shown that the shape of the nonlocal interaction function depends deeply on the choice of boundary conditions. As a physical implementation, the found interactions describes the evolution for the inductively coupled nonlinear networks. A qualitative analysis suggests that under such interactions the system self-organizes quite naturally, finally this is evidenced through the numerical solution of the equations of motion in the case of local cubic nonlinearities for two different boundary conditions.

1. Introduction

In the investigation of spatiotemporal behavior in arrays of coupled nonlinear oscillators, it has been very common to use local interaction functions. This leads to equations with diffusive structure that produce naturally the phenomena of propagation and the emergence of Turing patterns. The locality of the equations arises from the nearest neighbors interaction which is used for modeling coupled nonlinear networks (CNNs) [1]. This causes the dynamic of the system to be almost insensitive to the choice of boundary conditions, except at the borders; such locality can be achieved through the resistively coupling of cells [2] in electronic implementations. In this paper we show, that there is a straightforward way to induce nonlocal interactions through local considerations, but at variance with local systems, the interaction functions and the behavior of the system will be highly dependent on boundary conditions. Nonlocality can be a key idea in the electronic implementation of neural networks with no need of a large number of connections. However, a deeper knowledge of the implications on different kinds of local considerations in the resulting nonlocal system must be developed. For electronic implementations we show that inductive coupling of cells leads to such nonlocality.

The paper is organized as follows: In section 2 it is shown how to find nonlocal interactions through local considerations from a mathematical point of view; later in section 3, we generalize the interactions found in section 2, and show how those depends on boundary conditions; in section 4 we present a physical realization of such interactions through electronic devices and introduce the local dynamical systems; finally, section 5 is devoted to study self-organization under the found nonlocal interactions and numerical results are also presented.

1

2. A Source for Nonlocal Behavior

To begin with, we consider the scalar field $\xi(\boldsymbol{r}, t)$, defined in some domain $\Omega \in M$, where M is an Euclidean Space. From this, let's define the *harmonicity* field $\Upsilon(\boldsymbol{r}, t) = \nabla^2 \xi(\boldsymbol{r}, t)$ that says how far is ξ from being an harmonic function in the point $\boldsymbol{r}$ at the moment t, this also tell us about the "concentration" of the scalar field $\xi(\boldsymbol{r}, t)$ in each point and time. Now we introduce the following rule, or dynamical system

$$\frac{\partial}{\partial t} \Upsilon(\boldsymbol{r}, t) = \Im[\xi(x, t)], \tag{1}$$

where the *Harmonizing Function* $\Im(x)$ is a one-variable function that says how Υ changes in time depending on the value of ξ; i.e. in this dynamical system the "harmonicity" of some scalar field ξ changes as a function of the field itself. Note that equation (1) is local, in the sense that $\Upsilon(\boldsymbol{r}, t)$ depends only on value of field in a neighborhood (see fig. 1), and this is also true for $\Im[\xi(\boldsymbol{r}, t)]$.
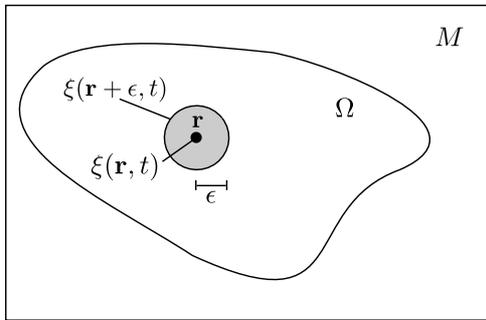

Figure 1. The spatial locality of eq. (1) comes from the fact that to define the *harmonicity* we need to know ξ in a arbitrarily small ϵ-ball around $\boldsymbol{r}$, and of course the quantity $\frac{\partial}{\partial t}\Upsilon$ is local in the time, because it does not involve future or past values of $\Upsilon(\boldsymbol{r}, t)$, only depends on the actual one.

Although eq. (1) may look simple, if we substitute the *harmonicity* function we get:

$$\frac{\partial}{\partial t} \nabla^2 \xi(\boldsymbol{r}, t) = \Im[\xi(\boldsymbol{r}, t)], \tag{2}$$

a third order partial differential equation, in which both time and space derivatives are to be applied to the same term. In the following we exhibit a procedure that renders such *local* equations as a nonlocal one.

2.1. Discretization and Standardization

We start considering the one-dimensional case, $\boldsymbol{r} = r$, where the ϵ-ball (fig. 1) becomes an interval and after discretizing the differential operators, we get:

$$\nabla^2 \xi(\boldsymbol{r}, t) = \frac{\partial^2}{\partial r^2} \xi(r, t) = \lim_{\epsilon \to 0} \frac{x_{n+1}(t) - 2x_n(t) + x_{n-1}(t)}{\epsilon^2}, \tag{3}$$

where the substitution $\xi(r, t) = \xi(n\epsilon, t) = x_n(t)$, was used. Taking this to eq. (2), and defining $f = \epsilon^2 \Im$ we get the discrete dynamical equations associated to the continuous field equation (2):

$$\frac{d}{dt}(x_{n+1} - 2x_n + x_{n-1}) = f(x_n). \tag{4}$$

Observe that eq. (4) does not have the usual structure for a dynamical system, i.e. $\dot{x} = f(x, t)$, but we can do some algebra to give it such form. In eq. (4) there are N variables x_n, then collecting all terms in vectors and defining $\boldsymbol{x} = (x_1, x_2, ..., x_N)^T$ and $\boldsymbol{F}(\boldsymbol{x}) = -(f(x_1), f(x_2), ..., f(x_N))^T$, we can write eq. (4) in vector form like:

$$\frac{d}{dt}\mathbb{A}\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{x}), \tag{5}$$

where the Coupling Matrix $\mathbb{A}$, is the matrix containing the weights of the corresponding neighbors for every independent equation. In the following sections the structure of $\mathbb{A}$ will be made precise depending on the boundary conditions. Now suppose for a moment that the coupling matrix can be inverted, and as it is time independent we have

$$\dot{\boldsymbol{x}} = \mathbb{A}^{-1}\boldsymbol{F}(\boldsymbol{x}). \tag{6}$$

This last expression has the usual form of a Dynamical System, so we will name it *The Standard Form* of eq. (5). If $\mathbb{A}$ can not be inverted, it is also possible to write the Standard Form by removing the *zero frequency* modes in the system [3].

2.2. Nonlocal Interaction
Looking the structure of eq. (4), we can see that the Coupling Matrix is a *tridiagonal Toeplitz matrix* [4] in most cases; but this does not means that $\mathbb{A}^{-1}$ is also tridiagonal. In the following sections it will be shown that the inverse of the Coupling Matrix is a dense matrix, so that the product $\mathbb{A}^{-1}\boldsymbol{F}(\boldsymbol{x})$ involves all inputs of the vector $\boldsymbol{F}(\boldsymbol{x})$, this means that the n'th element in equation (6) is:

$$\dot{x}_n = \sum_{m=1}^{N} c_{n,m} f(x_m), \tag{7}$$

where the $c_{n,m}$ are generally different from *zero* and represent the *discrete interaction function*. Therefore we have a nonlocal interaction, given that the evolution of the n'th variable involves the actual values of all the other elements in the network. This appears to be a simple consequence of the proposed coupling, but is remarkable that through *local considerations* and even *local equations* we obtained nonlocal interactions.

3. The Nonlocal Interaction Functions
In order to find the Standard Form equation, we will use the usual procedure for decomposing systems of linear second order coupled differential equations in its *normal oscillating modes* [5]. $\mathbb{A}$ is always an *hermitic matrix*, then defining

$$\mathbb{B} = (\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_N), \tag{8}$$

as the matrix whose columns are the eigenvectors $\{\boldsymbol{a}_n\}$ of $\mathbb{A}$ and after premultiplying by $\mathbb{B}^T$, eq. (5) can be written as:

$$\mathbb{A}_D \mathbb{B}^T \dot{\boldsymbol{x}} = \mathbb{B}^T \boldsymbol{F}(\boldsymbol{x}), \tag{9}$$

where $\mathbb{A}_D$ is the diagonal form of $\mathbb{A}$. Then we can suppress in eq. (9) those rows corresponding to *zero* eigenvalues, and define the *non-square matrix* $\tilde{\mathbb{B}}$ as the obtained after suppressing from $\mathbb{B}$ the eigenvectors related to *null* eigenvalues. We finally get:

$$\dot{\boldsymbol{x}} = (\tilde{\mathbb{B}}\tilde{\mathbb{A}}_D^{-1}\tilde{\mathbb{B}}^T)\boldsymbol{F}(\boldsymbol{x}) = \mathbb{C}\boldsymbol{F}(\boldsymbol{x}), \tag{10}$$

in which $\tilde{\mathbb{A}}_D^{-1}$ is the diagonal matrix containing the inverse of no-null eigenvalues. This is the needed result, given that it does not carry with singularity problems produced by a *non-invertible* Coupling Matrix. Note also that eq. (10) is well defined, and is a generalization of eq. (6), where $\mathbb{C}$ is the interaction function matrix.
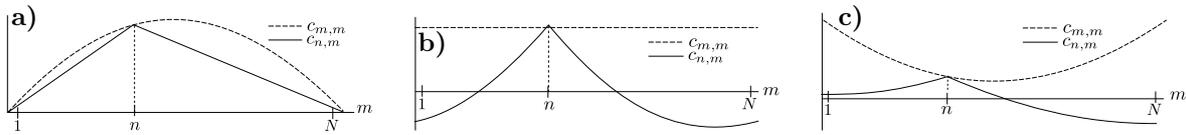
Figure 2. Interaction function matrix elements $c_{n,m}$ for different boundary conditions. a) For fixed boundaries: $c_{n,m} = \min(n,m) - \frac{mn}{N+1}$, which has a maximum in the position of the affected cell. The plot of $c_{m,m}$ shows that the strength of the interactions vanishes to the borders, so the dynamics are richest in the center. b) For periodic boundaries: $c_{n,m} = |n-m|\frac{|n-m|-N}{2N} + \frac{N^2-1}{12N}$, that again is maximum in the affected cell. Now $c_{m,m}$ shows that the strength in this case does not change, as is expected in a periodic system. c) For free borders: $c_{n,m} = \frac{n^2+m^2-n-m}{2N} - \max(n,m) + \frac{(2N+1)(N+1)}{6N}$, that depends strongly on the cell's neighbors, but now the interaction strength grows in the borders.

3.1. Dependence of Interaction Functions on Boundary Conditions

Contrary to the case of local interactions, here the choice of boundary conditions have deep consequences in the shape of the interacting functions and the resulting behavior. The boundary conditions, in some cases can be included completely on the Coupling Matrix, but in other ones auxiliary conditions on the eigenvectors are needed (see Table 1). Due to the discretization of the Laplace operator, our Coupling Matrix is:

$$\mathbb{A} = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1(0) \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -1(0) & 0 & 0 & \cdots & 2 \end{pmatrix}, \tag{11}$$

where the top-right and down-left corner inputs "$-1(0)$" are related to boundary conditions like "periodic (fixed, free)" respectively. In this case, it is easy to show that the eigenvectors are *discrete harmonic functions* [7] for *all* boundary conditions, and that the eigenvalues have the structure:

$$a_m = 4\sin^2\left(\frac{mk}{2}\right) , \ m \in \{1, 2, \cdots, N\}, \tag{12}$$

with k depending on the boundary conditions.

Table 1. Dependence of k on the boundary conditions.

	Fixed	Free	Periodic
k :	$\frac{\pi}{N+1}$	$\frac{\pi}{N}$	$\frac{2\pi}{N}$
Conditions:	$x_0 = 0$	$x_0 = x_1$	$x_0 = x_N$
	$x_{N+1} = 0$	$x_N = x_{N+1}$	$x_{N+1} = x_1$

After solving the eigenvalue problem, and suppressing all trivial modes, we perform the matrix products in (10) to find the coefficients $c_{n,m}$ in eq. (7), obtaining as can be seen in figure 2 quite different interaction functions [7].

Those are long-range interactions, and their strength does not necessarily decreases for far cells, so that this kind of systems are strongly coupled and can not be reduced through local approximations to simpler systems. There is something interesting about this coupling, namely for free and periodic borders the interaction becomes negative at the far cells, likewise in the diffusive local interactions where is negative in the first neighbors and positive in the affected cells.

3.2. The Continuum Limit

Now that we have successfully inverted the equation (5) we are in position to solve the problem numerically, provided the system is already in standard form. But first, it is worth making some observations about the continuum limit of such dynamical system. Let's write eq. (7) in the special case of the periodic boundaries:

$$\dot{x}_n = \sum_{m=1}^{N} \left(|n - m| \frac{|n - m| - N}{2N} + \frac{N^2 - 1}{12N} \right) f(x_n), \tag{13}$$

using $x_n(t) = \xi(n\epsilon, t)$, $r = n\epsilon, r' = m\epsilon$, and $f(x_n) = \epsilon^2 \Im[\xi(r, t)]$ we have:

$$\frac{\partial}{\partial t} \xi(r, t) = \int_0^L \Im[\xi(r', t)] \left(\frac{L}{12} - |r - r'| \frac{L - |r - r'|}{2L} \right) dr', \tag{14}$$

where we made $N \to \infty$, $\epsilon N = L$, and the limit $\epsilon \to 0$ was taken. This is an integro-differential equation, that puts in evidence the non-local character of this interaction. Generally, for any boundary condition the problem may be transformed into:

$$\frac{\partial}{\partial t} \xi(r, t) = \int_0^L K(r, r') \Im[\xi(r', t)] dr', \tag{15}$$

with $K(r, r')$ a well defined kernel emerging from the found discrete non-local interactions. Note that this last equation is the formal solution of eq. (2), but now is clear that the scalar field $\xi(r, t)$ changes in relation to the current state of the field in every point of the domain Ω, in this case the interval $(0, L)$. This procedure is equivalent to applying the inverse operator of ∇^2 in eq. (2), and this will be always possible for any local consideration by the presented analysis. Something important to say is that coupling matrices similar to $\mathbb{A}$ with no vanishing values near the main diagonal will generally be dense, then local considerations similar to those used for defining eq. (1) will produce nonlocal interactions. Observe that this procedure give us consistent interaction functions depending on boundary conditions and the coupling matrix, then is not necessary to introduce by hand boundary conditions in the numerical simulations.

4. Devising nonlocality

Here we want to show that equations with the structure in (2) can be easily achieved through the *Inductive Coupling* of electronic nonlinear cells. In figure 3 nonlinear Current Controlled Voltage Sources (CCVS elements) [6], represented by rectangular shapes, are connected in a one-dimensional array containing inductors as the inter-cell coupling element.

Applying the first Kirchhoff law in the nodes A and B, and the second to the n'th loop we obtain the equation:

$$\frac{d}{dt}(i_{n+1} - 2i_n + i_{n-1}) = L^{-1}v(i_n) \tag{16}$$

where $v_n = v(i_n)$ is the n'th potential drop produced by the n'th current i_n. This equation is identical to eq. (4). Therefore for this structure we have nonlocal interactions, or in other words
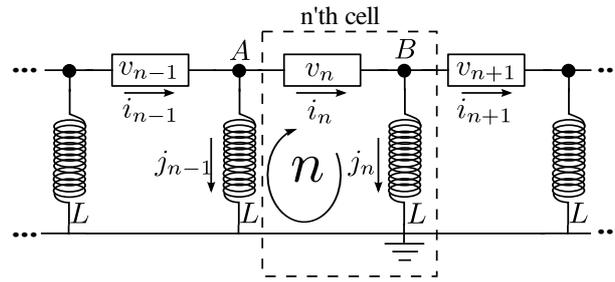
Figure 3. Inductive Coupling of Nonlinear Cells.

each cell interact with all others *directly* without the extensive connectivity of a neural network. Then, for each cell we can write:

$$\frac{d}{dt}i_n = -L^{-1}\sum_{m=0}^{N} c_{n,m}v_m \tag{17}$$

4.1. The Case of Dynamic CCVS Elements

Consider now that in the figure 3 the CCVS elements are replaced with dynamic elements, i.e. the rectangular shape has its own dynamics and the current entering to the element is just one of the dynamical variables that evolves in time. The most simple case would be

$$\dot{v}_n = F(v_n, i_n). \tag{18}$$

We can think of v_n as a *local variable*, given that its evolution only depends on the cell's state; and i_n as a global variable, because its evolution depends on the current value of *all* local variables.

In figure 4 we show such replacement, where a Chua's diode [8] has been used to introduce a "cubic" nonlinearity in the local function $F(v,i)$. Nondimensionalizing the current ($i_n \rightarrow y_n$), voltage ($v_n \rightarrow x_n$) and making $g(v_n) \rightarrow f(x_n)$

$$\dot{x}_n = f(x_n) + \epsilon y_n, \tag{19}$$

$$\dot{y}_n = -\delta \sum_{m=0}^{N} c_{n,m}x_m, \tag{20}$$

With

$$f(x_n) = \begin{cases} -mx - 1 & , & x < -1 \\ mx & , & -1 \leq x \leq 1 \\ -mx + 1 & , & x > 1 \end{cases} \tag{21}$$

This structure allows us to consider the global variable y_n as a small perturbation on the *local system* ,eq. (19), given that ϵ can be made as small as we want.

Locally we have three fixed points, two stable ones (x_-, x_+), and one unstable (x_0) disposed has in the figure 5. Observe that the effect of the global perturbation is to *move* the cubic shape up or down. This means that the basins of attraction of the stable fixed points may change under such influence, even, if y_n is big enough ($|\epsilon y_n| > h$) it may produce *local* saddle-node bifurcations, so that just one stable fixed point survives.
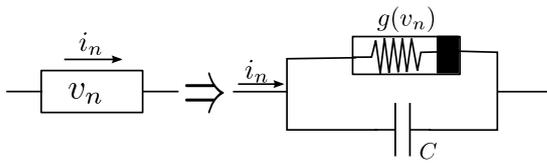
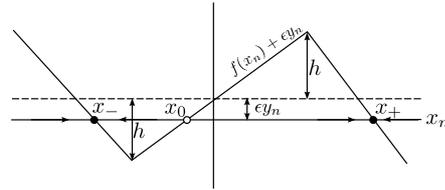**Figure 4.** Replacement for the CCVS elements.

Figure 5. Nonlinear local function for the n't cell.

5. Self-organization Under Nonlocal Interactions

Now that we have introduced the basic structure of our non-local system (and its electronic realization), and convert it to the standard form, it is possible to solve numerically some problems. As we have said, different boundary conditions produces different nonlocal interactions, then we need to consider separately each one.

5.1. The evolution for a Periodic System

First of all let's consider the system with periodic boundaries. In this case, the Coupling Matrix $\mathbb{A}$ is singular because there exist a *zero* eigenvalue. Such eigenvalue is related to an eigenvector in which all entries have the same value ($N^{-\frac{1}{2}}$, after normalization), i.e. an homogeneous state. This means that in this case $\tilde{\mathbb{B}}$ is the $(N-1) \times N$ matrix $(\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_{N-1})$. with $\boldsymbol{a}_N$ the homogeneous vector. Now, if we assume that the system is in a global equilibrium state we have:

$$\mathbb{C}\boldsymbol{x} = \boldsymbol{0} \Rightarrow \tilde{\mathbb{B}}\mathbb{A}_{\mathbb{D}}^{-1}\tilde{\mathbb{B}}^T\boldsymbol{x} = \boldsymbol{0}, \tag{22}$$

which can be reduced to:

$$p_1 a_1^{-1}\boldsymbol{a}_1 + p_2 a_2^{-1}\boldsymbol{a}_2 + \cdots + p_{N-1} a_{N-1}^{-1}\boldsymbol{a}_{N-1} = \boldsymbol{0}, \tag{23}$$

where a_n and $\boldsymbol{a}_n$ are the no-null eigenvalues and nonsingular eigenvectors of $\mathbb{A}$ respectively, and $p_n = \boldsymbol{x} \cdot \boldsymbol{a}_n$ is the projection of the *local* variables vector over the eigenvector $\boldsymbol{a}_n$. Due to the orthogonality of the set $\{\boldsymbol{a}_n\}$, it is clear that this equation is satisfied only if $p_n = 0, \forall n < N$, but this only happens if $\boldsymbol{x}$ is proportional to the singular eigenvector $\boldsymbol{a}_N$. The last means that for periodic boundaries only *homogeneous* states can be the equilibrium states. The stability of such states is determined by the stability of the local system, whose fixed points satisfies $\epsilon y_n + f(x_n) = 0$, but due to the homogeneity: $x_n = x^0, \forall n$, which means $y_n = -f(x_0), \forall n$. This can only be true if $f(x^0) = 0$, because we have to accomplish $\sum_{n=0}^{N} y_n = 0$ due to the application of the second Kirchhoff's law to the global loop containing all currents.

In conclusion, there are three global attractors in this system, in which all internal variables have the same value x_-, x_+ or x_0, and all currents are null; the stability for such attractors are the same as in the local system, and of course the homogeneous asymptotic state (see figures 6 and 7) is sensitive to initial conditions.

5.2. Phenomena for Fixed Boundaries

For fixed boundaries $\mathbb{A}$ is fully invertible, then there are no trivial modes or null eigenvalues, however, we can make some qualitative observations after joining equations (19) and (20).

$$\ddot{x}_n - f'(x_n)\dot{x}_n + \epsilon\delta\sum_{m=1}^{N} c_{n,m}x_m = 0, \tag{24}$$
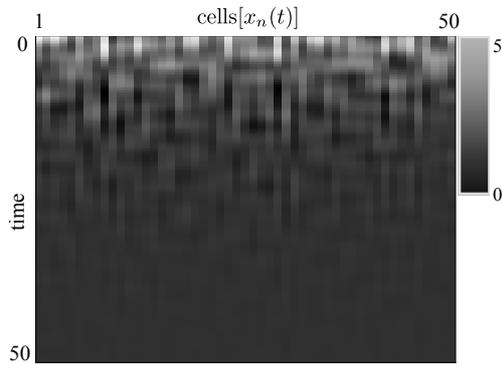
Figure 6. Evolution from random initial condition ($x_\pm = \pm 2, x_0 = 0$).
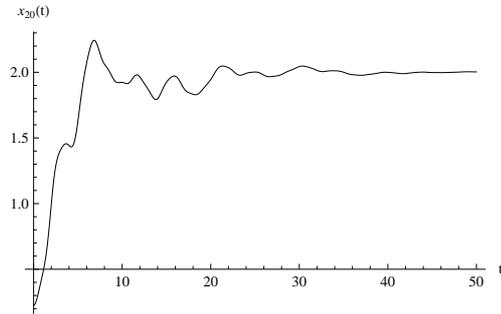

Figure 7. Evolution of an internal variable ($x_{20}(t)$).

that can be written as:

$$\ddot{x}_n - f'(x_n)\dot{x}_n + c_{n,n}\omega_0^2[x_n - x_n^0(t)] = 0, \qquad (25)$$

with $\omega_0^2 = \epsilon\delta$ and $x_n^0(t) = -c_{n,n}^{-1}\sum_{m\neq n}c_{n,m}x_m$. Equation (25), says that x_n behaves like a damped oscillator whose equilibrium point moves like $x_n^0(t)$, and also whose damping ratio $f'(x_n)$ can take three different values (m_-, m_0 and m_+), depending on the x_n value. Note that the *new* equilibrium point contains the non-local effects.
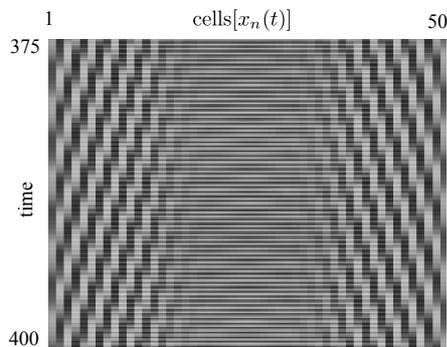

Figure 8. Evolution from random ($>$ 0) initial condition ($x_\pm = \pm 2, x_0 = 0$).
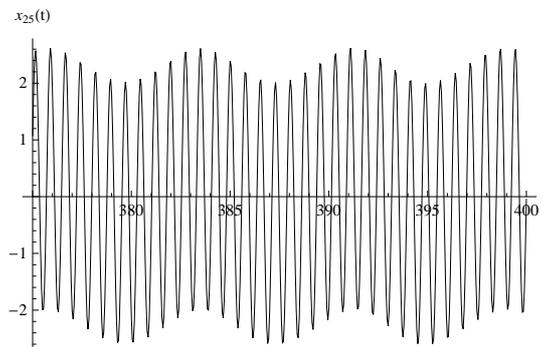

Figure 9. Evolution of the internal variable $x_{25}(t)$, for $t \in [375, 400]$.

Given the definition of $x_n^0(t)$, and due to the large number of cells N, it is possible to think that $x_n^0(t) \approx x_m^0(t), m \neq n$, so that the whole system affects each cell in a similar way. Now, as the whole network is acting as a forcing agent over the cells, this may lead to self-organization produced by the robustness of $x_n^0(t)$. In figure 8 we can see the asymptotic state for the system if it starts from random positive initial conditions. The time series for one variable ($x_{25}(t)$) shows that there are two embedded dynamics: one fast and the other slow. The fast dynamics frequency is close to the predicted natural one $\omega_n^2 = c_{25,25}\epsilon\delta$, then we can think that the slow one corresponds to the net influence of the whole system over the cell; such influence is similar for every cell, but differs in the scaling factor, that's why at the center slow oscillations happens with a smaller amplitude that in the borders (figure 10). The Fourier transform in figure 11 shows the natural larger frequency for $x_1(t)$, but also shows that the net influence of the network is not harmonic. After building a delay plot, we can see that the system may live inside a torus and is cuasiperiodic.
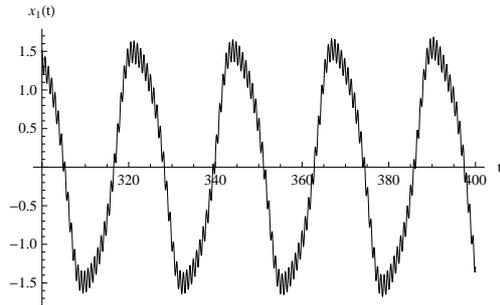
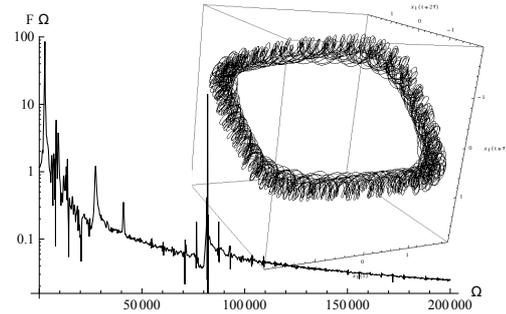**Figure 10.** Evolution for the internal variable $x_1(t)$, for $t \in [300, 400]$.

Figure 11. Fourier transform for $x_1(t)$, and delayed diagram.

In this system we also found stronger forms of self-organization, in which it evolves to a global limit cycle, as can be seen from figures 12 and 13. Such asymptotic state is reached through collisions of "structures" that propagates with constant speed until collide with another to form a new structure with a different speed. In every simulation it was observed that in the asymptotic state only one structure survives with null velocity. Obviously this behavior escapes from the scope of this paper.
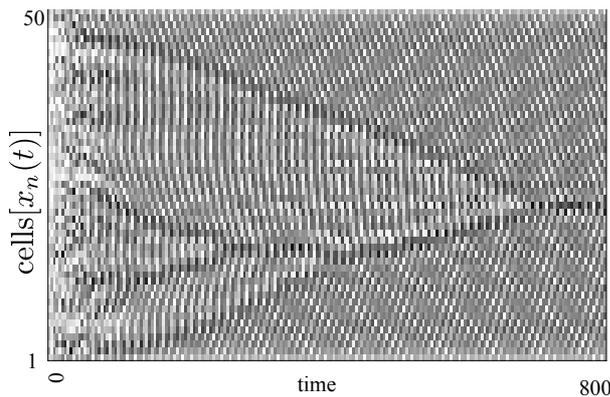

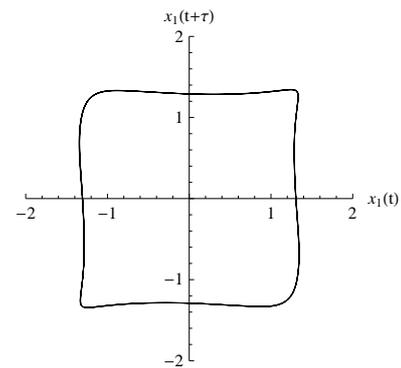**Figure 12.** Self-organization for random initial conditions

Figure 13. Delayed diagram for $x_1(t)$.

6. Conclusions

First of all, we demonstrated that through local considerations based in how close is a scalar field to be an harmonic one, it is possible to build evolution rules that explicitly involves the whole scalar field in the domain, i.e. a nonlocal interaction. By the introduction of physical implementations, we shown that it is possible, through *local connections*, to build a system endowed with nonlocal interactions. In other words, it is possible to build a highly connected system by using only local connections. When the connected cells contains dynamic elements, we shown that for periodic boundaries, for each stable fixed point in the local system exists a global attractor consisting of an homogeneous state. In the special case of cubic-like local nonlinearities, for random positive initial conditions, the system self-organizes through the robust behavior of the global system reaching an asymptotic cuasiperiodic state. But in the case of uniform random initial conditions a stronger type of self-organization by means of propagating structures leads the system to a global limit cycle.

References

[1] Chua L O and Yang L 1988 *IEEE Trans. Circuit Syst.* **35** 1257
[2] Muruganandam P, Murali K and Lakshmanan M 1999 *Int. J. Bif. Chaos* **9** 805
[3] Goldstein H 1980 *Classical Mechanics, Second Ed.* (Addison-Wesley)
[4] Bottcher A and Grudsky S 2000 *Toeplitz Matrices, Asymptotic Linear Algebra, and Functional Analysis* (Birkhauser)
[5] Guerrero A 2005 *Oscilaciones y Ondas* (Bogotá: Editorial Universidad Nacional)
[6] Chua L O, Desoer C A and Kuh E S *Linear and Nonlinear Circuits* (McGraw-Hill)
[7] Ciro D 2010 Chaos & Complexity in non-Linear Oscillator Networks: Nonlocal systems *Preprint* (Undergraduated Thesis), Instituto de Física, Universidad de Antioquia
[8] Chua LO 1993 *Proc. Int. Symp. NOLTA (Hawaii)* Part 2