A numerical method for solving Caputo's and Riemann-Liouville's fractional differential equiations which includes multi-order factional derivatives and variable coefficients

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A numerical method for solving Caputo's and Riemann-Liouville's fractional differential equations which includes multi-order fractional derivatives and variable coefficients

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ABSTRACT

In this paper, a numerical method is developed to obtain a solution of Caputo's and Riemann-Liouville's Fractional Differential Equations (CFDE and RLFDE). Scientific literature review shows that some numerical methods solve CFDE and there is only one paper that numerically solves RLFDE. Nevertheless, their solution is limited or the Fractional Differential Equation (FDE) to be solved is not in the most general form. To be best of the author's knowledge, the proposed method is presented as the first method that numerically solves RLFDE which includes multi-order fractional derivatives and variable coefficients. The method converts the RLFDE or CFDE to be solved into an algebraic equation. Each Riemann-Liouville's or Caputo's Fractional Derivative (RLFD and CFD), derived from the RLFDE or CFDE respectively, is conveniently written as a set of substitution functions and an integral equation. The algebraic equation, the sets of substitution functions and the integral equations are discretized; and then solved using arrays. Some examples are provided for comparing the obtained numerical results with the results of other papers (when available) and exact solutions. It is demonstrated that the method is accurate and easy to implement, being presented as a powerful tool to solve not only FDE but also a wide range of differential and integral equations.

1. Introduction

Numerical solution of FDE is a concerning topic for many researchers around the world, and one of the reasons for that is that FDE might provide a better explanation to many natural physical phenomena and facilitate the description of dynamic systems [1,2]. For instance, FDE can be used to characterize complex systems related to memory and hereditary properties such as viscoelastic deformation, anomalous diffusion, signal processing, and stock market [3]. A modern challenge of engineering and science is to obtain more realistic models and improve processes [2,4], and as a consequence, mathematical

Abbreviations: (RLFI), Riemann-Liouville's Fractional Integral; (RLSE), Riemann-Liouville's Simplified Equation; (CRLFI), Complementary-order Riemann-Liouville's Fractional Integral; (CSE), Caputo's Simplified Equation.

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models tend to be more complex and of difficult implementation. However, models should be simple to facilitate the implementation of engineering applications in micro-controllers or computers. In general terms, efforts are focused on finding a precise solution and obtaining the reduction of the computing effort with simpler algorithms [1,5].

Even though there is a great number of forms for FDE, CFDE and RLFDE are the most promising ones because they show many advantages for describing and simulating physical processes and dynamic systems [3,6]. Most researchers are focused on the solution of CFDE because RLFDE requires fractional order initial conditions, and there is not a consensus yet concerning their physical meaning [7]. On the other hand, because some researchers have been reporting interesting properties of RLFDE or have overcome some limitations [8], RLFDE mathematical models have caught the attention in areas such as chaotic, polymers, biological tissue, self-similar protein dynamics among others [9–11]. Concerning the solution of RLFDE, it was found in the literature review that only one paper focuses on obtaining a numerical solution for RLFDE [12] which is limited to solve FDE in the form of $D^{v}f(x) = \beta f(x) + u(x)$ with $\beta > 0$ and a restricted fractional order ($0 < v < 1$). The extensive use of RLFDE for modeling complex systems has the following limitations: 1) there is not a generalized method for the solution, and 2) there is not a consensus of the meaning of fractional initial conditions or there is not a method to convert integer initial conditions in fractional initial conditions. This paper proposes a method to overcome the first limitation. The main contribution is, therefore, a numerical method for the solution of RLFDE with multi-order fractional derivatives and variable coefficients as (1).

$$c_m(x)D^{v_m}f(x) + \cdots + c_j(x)D^{v_j}f(x) + \cdots + c_1(x)D^{v_1}f(x) + c_0(x)f(x) = u(x) \tag{1}$$

Where $j = 1, 2, \ldots, m$ and m is the number Riemann-Liouville fractional derivatives in (1).

With the fractional initial conditions; $I^{r_1}f(0), \ldots, I^{r_j}f(0), \ldots, I^{r_m}f(0)$; $\alpha^{1-r_j}(0), \ldots, \alpha^{k_j-r_j}(0), \ldots, \alpha^{n_j-r_j}(0)$.

The method also solves CFDE as (2) more efficiently than the methods reported in scientific literature [2,5,6,13–19].

$$c_m(x)^cD^{v_m}f(x) + \cdots + c_j(x)^cD^{v_j}f(x) + \cdots + c_1(x)^cD^{v_1}f(x) + c_0(x)f(x) = u(x) \tag{2}$$

Where $j = 1, 2, \ldots, m$ and m is the number of Caputo fractional derivatives into (2). With the integer initial conditions; $f(0)$, $f^{(1)}(0), \ldots, f^{(k_j)}(0), \ldots, f^{(n_m-1)}(0)$.

The proposed method simplifies the fractional derivatives and the FDE into simple algebraic equations, and then solves them using simple array operations. In order to show the accuracy and simplicity of the method, some examples are provided where the method is used to solve linear and nonlinear FDE (RLFDE and CFDE) which includes multi-order derivatives and variable coefficients.

This paper is organized as follows: Section 2 introduces some basic definitions and properties. Section 3 presents the proposed method supported by Theorem 3.1 and Corollary 3.1.1; and Theorem 3.2 and Corollary 3.2.1 which are presented as important contributions of this paper. Theorem 3.1 and Corollary 3.1.1 are necessary for transforming RLFDE and CFDE to algebraic equations, while Theorem 3.2 and Corollary 3.2.1 allow finding the additional initial conditions necessary for applying the method. Section 4 includes numerical examples such as: Example 4.1 solves a linear RLFDE including the explanation step by step of the proposed method, also the absolute error for different fractional orders is analyzed; Example 4.2 solves a RLFDE with multi-order fractional derivatives and variable coefficients, Example 4.3 solves a multi-order CFDE including a comparison in terms of precision and simulation time, and Example 4.4 solves a CFDE with multi-order fractional derivatives and variable coefficients. Section 5 concludes and highlights the contributions of this paper.

2. Basic definitions and properties

Operators and properties used in this paper are provided by [6,20,21], these are as follows.

Riemann-Liouville Fractional Integral (RLFI) of order $r (r \geq 0)$ is defined as:

$$I^{r}f(x) = \frac{1}{\Gamma(r)}\int_{o}^{x}(x-\tau)^{r-1}\ f(\tau)d\tau \tag{3}$$

Riemann-Liouville Fractional Derivative (RLFD) of order $v (v \geq 0)$ is defined as:

$$D^{v}f(x) = D^{n}I^{r}f(x) = \frac{1}{\Gamma(r)}\frac{d^{n}}{dx^{n}}\int_{o}^{x}(x-\tau)^{r-1}\ f(\tau)d\tau \tag{4}$$

Where $r = n - v$ and $n = \lceil v \rceil$

Caputo Fractional Derivative (CFD) of order $v (v \geq 0)$ is defined as:

$$^{c}D^{v}f(x) = I^{r}D^{n}f(x) = \frac{1}{\Gamma(r)}\int_{o}^{x}(x-\tau)^{r-1}\frac{d^{n}\ f(\tau)}{d\tau^{n}}d\tau \tag{5}$$

Where $r = n - v$ and $n = \lceil v \rceil$

Abelian semigroup property for RLFI of orders $v_1 (v_1 \geq 0)$ and $v_2 (v_2 \geq 0)$ is defined as:

$$I^{r_1}I^{r_2}f(x) = I^{r_2}I^{r_1}f(x) = I^{r_1+r_2}f(x) \tag{6}$$

3. Proposed method

The proposed method allows solving an v_j-order FDE where $n_j - 1 < v_j < n_j$ and $n = \lceil v \rceil$ in the sense of Riemann-Liouville (1) or Caputo (2). The method is composed of three steps:

1. Order reduction of fractional derivatives: this step allows converting each fractional derivative into an integral equation and a set of first-order differential equations which conform a system equation.
2. Transformation to recurrence equations: the system equation of step one is transformed into a system of recurrence equations, transforming a continuous problem in a discrete problem.
3. Search of additional initial conditions: this step allows finding the additional initial conditions necessary for solving a FDE using the set of discrete equations found in the previous step.

The following sections explain the steps of the proposed method.

3.1. Step 1: Order reduction of fractional derivatives

The method requires an order reduction of each derivative into the FDE to be solved; the order reduction allows converting each fractional derivative into an integral equation and a set of first-order differential equations that are valid for fractional derivatives in Riemann-Liouville's or Caputo's sense. The resultant set of first-order differential equations introduces all initial conditions to the method. Eq. (1) and (2) are composed of m fractional derivatives. The order reduction is shown in the following sections for an arbitrary v_j-order fractional derivative in the sense of Riemann-Liouville (Section 3.1.1) and Caputo (Section 3.1.2) where $r_j = n_j - v_j,\ n_j - 1 < v_j < n_j$ and $n_j = \lceil v_j \rceil$.

3.1.1. Riemann-Liouville order reduction

Definition of RLFD (4) shows that $D^{v_j} f(x) = \frac{d^{n_j}}{dx^{n_j}}(I^{r_j} f(x))$ where $I^{r_j} f(x)$ corresponds to RLFI (3), supposing that $G^j(x) = I^{r_j} f(x)$ then (3) is conveniently written as follows to simplify notation.

$$G^j(x) = I^{r_j} f(x) = \frac{1}{\Gamma(r_j)} \int_o^x (x-\tau)^{r_j - 1} f(\tau) d\tau \tag{7}$$

Each RLFD in (1) can be written as a first-order Differential Equation System (DES), applying (7) over (4) then a RLFD is written as $D^v f(x) = \frac{d^{n_j}}{dx^{n_j}} G^j(x)$ where $\frac{d^{n_j}}{dx^{n_j}} G^j(x)$ can be simplified to a first-order Differential Equations System (DES) given by (8).

$$\begin{aligned}
\alpha^{1-r_j}(x) &= \frac{dG^j(x)}{dx},\\
\alpha^{2-r_j}(x) &= \frac{d\alpha^{1-r_j}(x)}{dx} = \frac{d^2 G^j(x)}{dx^2},\\
&\vdots\\
\alpha^{k_j - r_j}(x) &= \frac{d\alpha^{k_j - 1 - r_j}(x)}{dx} = \frac{d^{k_j} G^j(x)}{dx^{k_j}},\\
&\vdots\\
\alpha^{n_j - r_j}(x) &= \frac{d\alpha^{n_j - 1 - r_j}(x)}{dx} = \frac{d^{n_j} G^j(x)}{dx^{n_j}}
\end{aligned} \tag{8}$$

Where $k_j =\ 1,\ 2,\ \ldots,\ n_j$ and $n_j = \lceil v_j \rceil$

The first-order DES (8) represents the n_j functions and are defined in this paper as the substitution functions in the Riemann-Liouville's sense. After the application of the order reduction to the m Riemann-Liouville's fractional derivatives in (1), m RLFIs are obtained as (7) and m sets of first-order differential equations as (8). Eq. (9) is obtained when the substitution functions (8) are replaced for all the m Riemann-Liouville's fractional derivatives into (1) and is defined in this paper as Riemann-Liouville's Simplified Equation (RLSE).

$$c_m(x)\alpha^{n_m - r_m} + \ldots + c_j(x)\alpha^{n_j - r_j} + \ldots + c_1(x)\alpha^{n_1 - r_1} + c_0(x) f(x) = u(x) \tag{9}$$

Proposition 3.1. *Let (1) be a Riemann-Liouville's FDE where f(x) and $G^j(x)$ are continuous functions in the interval [a, x) and all their integer derivatives are continuous in the interval [a, x), then (1) can be represented as an equation system using (9), m RLFIs as (7) and m sets of substitution functions as (8).*

For example, let $10D^{2.45} f(x) - 7D^{1.8} f(x) + 9f(x) = 13$ be a RLFDE as (1), for $D^{2.45}f(x)$ $v_2 = 2.45,\ n_2 = 3,\ r_2 = 0.55$ and for $D^{1.8}f(x)$ $v_1 = 1.8,\ n_1 = 2,\ r_1 = 0.2$. Applying Proposition 3.1, the following RLSE is obtained:

$$10\alpha^{2.45}(x) - 7\alpha^{1.8}(x) + 9f(x) = 13$$

Two RLFIs as (7) are shown as follows:

$$G^2(x) = \frac{1}{\Gamma(0.55)}\int_o^x (x-\tau)^{0.55-1} f(\tau)d\tau \qquad G^1(x) = \frac{1}{\Gamma(0.2)}\int_o^x (x-\tau)^{0.2-1} f(\tau)d\tau$$

Two sets of substitution functions as (8) are obtained:

$$\begin{aligned} \alpha^{0.45}(x) &= \frac{dG^2(x)}{dx} & \alpha^{0.8}(x) &= \frac{dG^1(x)}{dx} \\ \alpha^{1.45}(x) &= \frac{d\alpha^{0.45}(x)}{dx} & \alpha^{1.8}(x) &= \frac{d\alpha^{0.8}(x)}{dx} \\ \alpha^{2.45}(x) &= \frac{d\alpha^{1.45}(x)}{dx} \end{aligned}$$

3.1.2. Caputo order reduction

It is supposed that $f(x)$ and its derivates are continuous in a solution interval $[a, x)$, (5) can be represented as a RLFI given by (10) and a first-order DES given by (11).

$$G^j(x) = {}^cD^{\nu} f(x) = \frac{1}{\Gamma(r_j)}\int_o^x (x-\tau)^{r_j-1}\beta^{n_j}(\tau)d\tau \tag{10}$$

$$\begin{aligned} \beta^1(x) &= \frac{df(x)}{dx}, \\ \beta^2(x) &= \frac{d\beta^1(x)}{dx} = \frac{d^2 f(x)}{dx^2}, \\ &\vdots \\ \beta^{k_j}(x) &= \frac{d\beta^{k_j-1}(x)}{dx} = \frac{d^{k_j} f(x)}{dx^{k_j}}, \\ &\vdots \\ \beta^{n_j}(x) &= \frac{d\beta^{n_j-1}(x)}{dx} = \frac{d^{n_j} f(x)}{dx^{n_j}} \end{aligned} \tag{11}$$

Where $k_j = 1, 2, \ldots, n_j$ and $n_j = \lceil \nu_j \rceil$

Eq. (11) represent the n_j functions and are defined in this paper as the substitution functions in the Caputo's sense, after applying the order reduction to the m Caputo's fractional derivatives in (2), m RLFIs are obtained as (10) and m sets of first-order differential equations as (11). Eq. (12) is obtained when the substitution functions (11) are replaced into (2) and is defined in this paper as Caputo's Simplified Equation (CSE).

$$c_m(x)G^m(x) + \cdots + c_j(x)G^j(x) + \cdots + c_1(x)G^1(x) + c_0(x)f(x) = u(x) \tag{12}$$

Proposition 3.2. *Let (2) be a Caputo's FDE where $f(x)$ and $G^j(x)$ are continuous functions in the interval $[a, x)$ and all integer-derivates of $f(x)$ are continuous in the interval $[a, x)$, then (2) can be represented as an equation system using (12), m RLFIs as (10) and m sets of substitution functions as (11).*

For example, let $10{}^cD^{2.45}f(x) - 7{}^cD^{1.8}f(x) + 9f(x) = 13$ be a CFDE as (2), for ${}^cD^{2.45}f(x)$ $\nu_2 = 2.45,\ n_2 = 3,\ r_2 = 0.55$ and for ${}^cD^{1.8}f(x)$ $\nu_1 = 1.8,\ n_1 = 2,\ r_1 = 0.2$. Applying Proposition 3.2, the following CSE is obtained:

$$10G^2(x) - 7G^1(x) + 9f(x) = 13$$

The following RLFIs as (10) are obtained:

$$G^2(x) = \frac{1}{\Gamma(0.55)}\int_o^x (x-\tau)^{0.55-1}\beta^3(\tau)d\tau \qquad G^1(x) = \frac{1}{\Gamma(0.2)}\int_o^x (x-\tau)^{0.2-1}\beta^1(\tau)d\tau$$

The substitution functions as (11) are:

$$\begin{aligned} \beta^1(x) &= \frac{df(x)}{dx} & \beta^1(x) &= \frac{df(x)}{dx} \\ \beta^2(x) &= \frac{d\beta^1(x)}{dx} & \beta^2(x) &= \frac{d\beta^1(x)}{dx} \\ \beta^3(x) &= \frac{d\beta^2(x)}{dx} \end{aligned}$$

3.2. Step 2: Transformation to recurrence equations

The proposed numerical method requires discretization of equations. The discretization begins supposing that a line is a proper adjustment to a segment of curve between two points (x_a, y_a) and (x_b, y_b) when these points are so close, the distance $h = |x_b - x_a|$ corresponds to the step of the method.

Proposition 3.3. *Let* (x_{l+1}, y_{l+1}) *and* (x_l, y_l) *be two points in a curve* $y(x)$ *such that* $h = |x_{l+1} - x_l|$ *and* $h \to 0$ *then* $y(x) \approx (\frac{y_{l+1}-y_l}{h})(x - x_l) + y_l$ *in the interval* $[x_l, x_{l+1}]$.

3.2.1. Recurrence equations for RLFDE

The discrete form for the substitution functions in the Riemann-Liouville's sense is obtained applying Proposition 3.3 over the system (8), then the discrete form of (8) is (13) and is shown as follows:

$$
\begin{aligned}
\alpha_{l+1}^{1-r_j} &= \frac{G_{l+1}^j - G_l^j}{h},\\
\alpha_{l+1}^{2-r_j} &= \frac{\alpha_{l+1}^{1-r_j} - \alpha_l^{1-r_j}}{h},\\
&\vdots\\
\alpha_{l+1}^{k_j-r_j} &= \frac{\alpha_{l+1}^{k_j-1-r_j} - \alpha_l^{k_j-1-r_j}}{h},\\
&\vdots\\
\alpha_{l+1}^{n_j-r_j} &= \frac{\alpha_{l+1}^{n_j-1-r_j} - \alpha_l^{n_j-1-r_j}}{h}
\end{aligned}
\tag{13}
$$

Where $k_j = 1, 2, \ldots, n_j$ and $n_j = \lceil v_j \rceil$

Applying Proposition 3.3 over (9), the discrete form of RLSE is obtained:

$$
c_{m,l}\alpha_{l+1}^{n_m-r_m} + \cdots + c_{j,l}\alpha_{l+1}^{n_j-r_j} + \cdots + c_{1,l}\alpha_{l+1}^{n_1-r_1} + c_{0,l}f_{l+1} = u_l \tag{14}
$$

3.2.2. Recurrence equations for CFDE

The substitution functions for fractional derivatives in the sense of Caputo have the same discrete form than Riemann-Liouville's substitution functions. Applying Proposition 3.3 over the system (11), the discrete form of the substitution functions in the sense of Caputo (15) is obtained:

$$
\begin{aligned}
\beta_{l+1}^{1} &= \frac{f_{l+1} - f_l}{h},\\
\beta_{l+1}^{2} &= \frac{\beta_{l+1}^{1} - \beta_l^{1}}{h},\\
&\vdots\\
\beta_{l+1}^{k_j} &= \frac{\beta_{l+1}^{k_j-1} - \beta_l^{k_j-1}}{h},\\
&\vdots\\
\beta_{l+1}^{n_j} &= \frac{\beta_{l+1}^{n_j-1} - \beta_l^{n_j-1}}{h}
\end{aligned}
\tag{15}
$$

Where $k_j = 1, 2, \ldots, n_j$ and $n_j = \lceil v_j \rceil$

Applying Proposition 3.3 over (12), the discrete form of CSE is obtained. Eq. (16) corresponds to the discrete form of CSE and is shown as follows:

$$
c_{m,l}G_{l+1}^{m} + \ldots + c_{j,l}G_{l+1}^{j} + \ldots + c_{1,l}G_{l+1}^{1} + c_{0,l}f_{l+1} = u_l \tag{16}
$$

3.2.3. Proposition of Complementary-order RLFI

The Complementary-order RLFI (I^{α}) (CRLFI) is defined to facilitate the demonstration of Theorem 3.1. CRLFI of a RLFI ($I^{\beta}f(x)$) is proposed in this paper as a RLFI such that $I^{\alpha}(I^{\beta}f(x))$ and $\alpha + \beta = 1$, being used Abelian semigroup property (6) to find a complementary form of RLFI. RLFIs (7) and (10) are equal to $G^j(x) = I^{r_j} f(x)$ and $G^j(x) = I^{r_j}\beta^{n_j}(x)$ respectively, so CRLFIs of (7) and (10) are $I^{1-r_j}G^j(x) = I^{1-r_j}I^{r_j}f(x)$ and $I^{1-r_j}G^j(x) = I^{1-r_j}I^{r_j}\beta^{n_j}(x)$ respectively. Applying Abelian semigroup property over RLFIs (7) and (10), $I^1 f(x) = I^{1-r_j}G^j(x)$ and $I^1\beta^{n_j} = I^{1-r_j}G^j(x)$ are respectively obtained.

Proposition 3.4. *Let (7) and (10) be RLFIs where* $f(x)$ *and* β^{n_j} *are continuous and integrable functions, CRLFIs of (7) and (10) are* $F(x) = I^{1-r_j}G^j(x)$ *and* $\beta^{n_j-1} = I^{1-r_j}G^j(x)$ *respectively where* $F(x) = I^1 f(x) = \int f(x)dx$.

3.2.4. Recurrence equations for CRLFI

Applying Proposition 3.4 over (3), the CRLFI of (3) is obtained. Eqs. (7) and (10) correspond to RLFIs, and then discrete form of CRLFIs (3), (7) and (10) can be obtained using Theorem 3.1.

Theorem 3.1. *Let (3) be the RLFI of f(x) and supposing that f(x) is a continuous function in the interval* $[0, b]$ *then the discrete form of CRLFI of (3) is given by* $hf_{l+1} + F_l = \frac{h^{1-r_j}}{\Gamma(3-r_j)}(G^j_{l+1} - G^j_l) + \frac{h^{1-r_j}}{\Gamma(2-r_j)}G^j_l + \frac{1}{\Gamma(3-r_j)}\sum_{i=0}^{l-1} k_i + \frac{1}{\Gamma(2-r_j)}\sum_{i=0}^{l-1} m_i$ *where* $k_i = \frac{(G^j_{i+1}-G^j_i)}{h}[(\rho+h)^{2-r_j} - \rho^{2-r_j}]$, $m_i = G^j_i(\rho+h)^{1-r_j} - G^j_{i+1}\rho^{1-r_j}$, *and* $\rho = (l-i)h$.

Proof. Applying Proposition 3.4 over (3), (17) is obtained as follows:

$$F(x) = \frac{1}{\Gamma(1-r_j)}\int_0^x f(x-\tau)^{-r_j} G^j(\tau)d\tau \tag{17}$$

Let identity (18) be

$$H(\tau) = \frac{1}{\Gamma(1-r_j)}(x-\tau)^{-r_j} G^j(\tau) \tag{18}$$

$$F(x) = \int_0^x H(\tau)d\tau \tag{19}$$

Considering that solution interval of (17) is $[0,\ x_{l+1}]$ where $x_{l+1} = (l+1)h$ and supposing that $x_1, x_2, \ldots, x_i, x_{i+1}, .., x_l \in [0,\ x_{l+1}]$

$$F(x) = \int_0^{x_1} H(\tau)d\tau + \int_{x_1}^{x_2} H(\tau)d\tau + \ldots + \int_{x_i}^{x_{i+1}} H(\tau)d\tau + \ldots + \int_{x_l}^{x_{l+1}} H(\tau)d\tau = \sum_{i=0}^{l}\int_{x_i}^{x_{i+1}} H(\tau)d\tau \tag{20}$$

Replacing (18) in (20)

$$F(x) = \sum_{i=0}^{l}\frac{1}{\Gamma(1-r_j)}\int_{x_i}^{x_{i+1}}(x_{l+1}-\tau)^{-r_j} G^j(\tau)d\tau \tag{21}$$

Applying Proposition 3.3 over functions $F(x)$ and $G^j(\tau)$ in (21)

$$F_{l+1} = \sum_{i=0}^{l}\frac{1}{\Gamma(1-r_j)}\int_{x_i}^{x_{i+1}}(x_{l+1}-\tau)^{-r_j}\left[\frac{(G^j_{i+1}-G^j_i)}{\Delta\tau}(\tau-\tau_i) + G^j_i\right]d\tau \tag{22}$$

Separating the integral in (22) and supposing the identity $b = \frac{G^j_{i+1}-G^j_i}{\Delta\tau}$

$$F_{l+1} = \sum_{i=0}^{l}\left[b\int_{x_i}^{x_{i+1}}\frac{\tau(x_{l+1}-\tau)^{-r_j}}{\Gamma(1-r_j)}d\tau - b\int_{x_i}^{x_{i+1}}\frac{\tau_i(x_{l+1}-\tau)^{-r_j}}{\Gamma(1-r_j)}d\tau + \int_{x_l}^{x_{l+1}}\frac{G_i(x_{l+1}-\tau)^{-r_j}}{\Gamma(1-r_j)}d\tau\right] \tag{23}$$

Solving integrals in (23)

$$\begin{aligned} F_{l+1} = \sum_{i=0}^{l}\Bigg[&\frac{-bx_{i+1}(x_{l+1}-x_{i+1})^{1-r_j}}{\Gamma(2-r_j)} + \frac{bx_i(x_{l+1}-x_i)^{1-r_j}}{\Gamma(2-r_j)} - \frac{b(x_{l+1}-x_{i+1})^{2-r_j}}{\Gamma(3-r_j)} + \frac{b(x_{l+1}-x_i)^{2-r_j}}{\Gamma(3-r_j)} \\ &+ \frac{b\tau_i(x_{l+1}-x_{i+1})^{1-r_j}}{\Gamma(2-r_j)} - \frac{b\tau_i(x_{l+1}-x_i)^{1-r_j}}{\Gamma(2-r_j)} - \frac{G^j_i(x_{l+1}-x_{i+1})^{1-r_j}}{\Gamma(2-r_j)} + \frac{G^j_i(x_{l+1}-x_i)^{1-r_j}}{\Gamma(2-r_j)}\Bigg] \end{aligned} \tag{24}$$

Supposing $\tau = \tau_{i+1}$ where $\tau_{i+1} = \tau_0 + (i+1)\Delta\tau,\ \tau_0 = 0,\ \Delta\tau = h$ then $\tau_{i+1} = x_{i+1}$. Replacing $x_{i+1} = (i+1)\Delta\tau$ in (24) and simplifying

$$F_{l+1} = \sum_{i=0}^{l}\left[\frac{-bh(lh-ih)^{1-r_j}}{\Gamma(2-r_j)} - \frac{b(lh-ih)^{2-r_j}}{\Gamma(3-r_j)} + \frac{b(lh-ih+h)^{2-r_j}}{\Gamma(3-r_j)} - \frac{G^j_i(lh-ih)^{1-r_j}}{\Gamma(2-r_j)} + \frac{G^j_i(lh-ih+h)^{1-r_j}}{\Gamma(2-r_j)}\right] \tag{25}$$

Applying the identities $\rho = lh - ih,\ b = \frac{G^j_{i+1}-G^j_i}{\Delta\tau}$ in (25) and simplifying

$$F_{l+1} = \sum_{i=0}^{l}\left[\frac{-G^j_{i+1}\rho^{1-r_j}}{\Gamma(2-r_j)} - \frac{(G^j_{i+1}-G^j_i)}{h}\frac{\rho^{2-r_j}}{\Gamma(3-r_j)} + \frac{(G^j_{i+1}-G^j_i)}{h}\frac{(\rho+h)^{2-r_j}}{\Gamma(3-r_j)} + \frac{G^j_i(\rho+h)^{1-r_j}}{\Gamma(2-r_j)}\right] \tag{26}$$

Extracting the terms associated to $l+1$ and grouping the common terms:

$$F_{l+1} = \frac{h^{1-r_j}}{\Gamma(3-r_j)}\left(G^j_{l+1} - G^j_l\right) + \frac{h^{1-r_j}}{\Gamma(2-r_j)}G^j_l + \frac{1}{\Gamma(3-r_j)}\sum_{i=0}^{l-1}\frac{(G^j_{i+1}-G^j_i)}{h}\left[(\rho+h)^{2-r_j} - \rho^{2-r_j}\right] + \frac{1}{\Gamma(2-r_j)}\sum_{i=0}^{i-1} G^j_i\left[(\rho+h)^{1-r_j} - \rho^{1-r_j}\right] \tag{27}$$

Applying Proposition 3.3 over the identity $f(x) = \frac{dF(x)}{dx}$ and organizing it:

$$F_{l+1} = F_l + hf_{l+1} \tag{28}$$

Replacing (28) in (27) and supposing that $k_i = \frac{(G^j_{i+1}-G^j_i)}{h}[(\rho+h)^{2-r_j} - \rho^{2-r_j}]$, $m_i = G^j_i(\rho+h)^{1-r_j} - G^j_{i+1}\rho^{1-r_j}$

$$F_l + hf_{l+1} = \frac{h^{1-r_j}}{\Gamma(3-r_j)}\left(G^j_{l+1} - G^j_l\right) + \frac{h^{1-r_j}}{\Gamma(2-r_j)}G^j_l + \frac{1}{\Gamma(3-r_j)}\sum_{i=0}^{l-1} k_i + \frac{1}{\Gamma(2-r_j)}\sum_{i=0}^{l-1} m_i \tag{29}$$

□

Corollary 3.1.1. *Let (5) be the CFD of f(x) and supposing that f(x), $\beta^1(x)$, ..., $\beta^n(x)$ are continuous functions in the interval* $[0, b]$*, then the discrete form of (5) is given by* $G_{l+1} = \frac{h^{r_j}}{\Gamma(2+r_j)}(\beta^{n_j}_{l+1} - \beta^{n_j}_l) + \frac{h^{r_j}}{\Gamma(1+r_j)}\beta^{n_j}_l + \frac{1}{\Gamma(2+r_j)}\sum_{i=0}^{l-1} k_i + \frac{1}{\Gamma(1+r_j)}\sum_{i=0}^{l-1} m_i$ *where* $k_i = \frac{(\beta^{n_j}_{i+1}-\beta^{n_j}_i)}{h}[(\rho+h)^{1+r_j} - \rho^{1+r_j}]$, $m_i = \beta^{n_j}_i(\rho+h)^{r_j} - \beta^{n_j}_{i+1}\rho^{r_j}$*, and* $\rho = (l-i)h$.

Proof. Applying Proposition 3.3 over (5) and the same procedure exposed in the proof of Theorem (3.1), (30) is obtained

$$G_{l+1} = \frac{h^{r_j}}{\Gamma(2+r_j)}\left(\beta^{n_j}_{l+1} - \beta^{n_j}_l\right) + \frac{h^{r_j}}{\Gamma(1+r_j)}\beta^{n_j}_l + \frac{1}{\Gamma(2+r_j)}\sum_{i=0}^{l-1} k_i + \frac{1}{\Gamma(1+r_j)}\sum_{i=0}^{l-1} m_i \tag{30}$$

□

3.3. Step 3: Search of initial conditions

The proposed method requires initial conditions for solving a FDE. The initial conditions required for Riemann-Liouville's Fractional derivatives are different to the initial condition required for Caputo's fractional derivatives. In the following section, the method for finding the initial conditions for RLFDE and CFDE is explained.

3.3.1. Initial conditions for RLFDE

Each Riemann-Liouville's fractional derivative into a RLFDE requires a set of fractional initial conditions. Laplace's rule shows that a v_j-order RLFDE needs n_j initial conditions [22] where $n_j = \lceil v_j \rceil$. s initial conditions are needed for solving the RLFDE of (1), where $s = \sum_{j=1}^{m} n_m$ and $n_m = \lceil v_m \rceil$. Eqs. (13), (14) and (29) have in total $s+m+2$ unknowns, these are in total $s+m+1$ equations, then other initial condition is required. This initial condition is found using the s initial conditions given for solving (1), Eqs. (13), (14), (29) and the discrete form of RLFI.

Theorem 3.2. *Let* $G^1(0)$, ..., $G^j(0)$, ..., $G^m(0)$; $\alpha^{1-r_j}(0)$, ..., $\alpha^{k_j-r_j}(0)$, ..., $\alpha^{n_j-r_j}(0)$, ..., $\alpha^{n_m-r_m}(0)$ *be the s initial conditions for solving RLFDE (1), (13), (14),* $f_1 - \frac{h^{-r_j}}{\Gamma(2-r_j)}G^j_1 = -\frac{h^{-r_j}}{\Gamma(2-r_j)}G^j_0 + \frac{h^{-r_j}}{\Gamma(1-r_j)}G^j_0$*, and* $G^j_1 + \frac{h^{r_j}}{\Gamma(2+r_j)}(f_0 - f_1) - \frac{h^{r_j}}{\Gamma(1+r_j)}f_0 = 0$ *allow finding the unknown initial conditions* G^j_1 *that is necessary for solving (1).*

Proof. Applying Proposition 3.4 over (3) and deriving it yields to (31).

$$f(x) = \frac{1}{\Gamma(1-r_j)}\frac{d}{dx}\int_0^x (x-\tau)^{-r_j} G^j(\tau)d\tau \tag{31}$$

Applying Proposition 3.3 over (31) and considering only one step in the interval $[0, h]$, (32) is obtained

$$f_1 = \frac{1}{\Gamma(1-r_j)}\frac{d}{dx}\int_0^x (x-\tau)^{-r_j}\left[\frac{(G^j_1 - G^j_0)}{\Delta\tau}\tau + G^j_0\right]d\tau \tag{32}$$

Separating the integral in (32) and supposing the identity $b = \frac{G^j_1 - G^j_0}{\Delta\tau}$

$$f_1 = \frac{b}{\Gamma(1-r_j)}\frac{d}{dx}\int_0^x (x-\tau)^{-r_j}\tau d\tau + \frac{G^j_0}{\Gamma(1-r_j)}\frac{d}{dx}\int_0^x (x-\tau)^{-r_j}d\tau \tag{33}$$

Solving the integrals in (33) and applying the identity $b = \frac{G_1^j - G_0^j}{\Delta\tau}$

$$f_1 = \frac{d}{dx}\left[\frac{(G_1^j - G_0^j)}{\Delta\tau}\frac{x^{2-r_j}}{\Gamma(3-r_j)} + \frac{G_0^j x^{1-r_j}}{\Gamma(2-r_j)}\right] \tag{34}$$

Deriving (34), then applying one step $(x = h)$ in the interval $[0, h]$, (35) is obtained

$$f_1 - \frac{h^{-r_j}}{\Gamma(2-r_j)}G_1^j = -\frac{h^{-r_j}}{\Gamma(2-r_j)}G_0^j + \frac{h^{-r_j}}{\Gamma(1-r_j)}G_0^j \tag{35}$$

Now applying Proposition 3.3 over (3) and considering one step in the interval $[0, h]$, (36) is obtained

$$G_1^j = \frac{1}{\Gamma(r_j)}\int_0^x (x-\tau)^{r_j-1}\left[\frac{(f_1 - f_0)}{\Delta\tau}\tau + f_0\right]d\tau \tag{36}$$

Separating the integral in (32) and supposing the identity $b = \frac{G_1^j - G_0^j}{\Delta\tau}$

$$G_1^j = \frac{b}{\Gamma(r_j)}\int_0^x (x-\tau)^{r_j-1}\tau d\tau + \frac{f_0}{\Gamma(r_j)}\int_0^x (x-\tau)^{r_j-1}d\tau \tag{37}$$

Solving the integrals in (37), applying the identities $b = \frac{G_1^j - G_0^j}{\Delta\tau}$ and one step $(x = h)$ in the interval $[0, h]$, (38) is obtained

$$G_1^j + \frac{h^{r_j}}{\Gamma(2+r_j)}(f_0 - f_1) - \frac{h^{r_j}}{\Gamma(1+r_j)}f_0 = 0 \tag{38}$$

Eqs. (35) and (38) correspond to the equations proposed in Theorem 3.2, being G_0^j initial conditions. Now Eqs. (13), (14), (35) and (38) are in total $s+m+2$ and these have $s+m+2$ unknowns. □

3.3.2. Initial conditions for CFDE

Each Caputo's fractional derivative into a CFDE requires a set of integer initial conditions, Laplace's rule shows that a v_j-order CFDE needs n_j initial conditions [22] where $n_j = \lceil v_j \rceil$, the CFDE (2) needs n_m initial conditions where $n_m = \lceil v_m \rceil$, because $v_m \geq v_j$ with $j = 1, 2, \ldots, m$. Eqs. (15), (16) and (30) have in total $s+m+2$ unknown, but these equations are in total $s+m+1$ then another initial condition is required. For obtaining the required initial condition, it is necessary: (1) the s initial conditions given for solving (2), (2) Eqs. (15), (16), (30), and (3) the discrete form of RLFI.

Corollary 3.2.1. *Let $f(0)$, $\beta^1(0), \ldots, \beta^{k_j}(0), \ldots, \beta^{n_m-1}(0)$ be the n_m initial conditions for solving CFDE (2); then (15), (16), $\beta_1^{n_m} - \frac{h^{-r_m}}{\Gamma(2-r_m)}G_1^m = 0$, and $G_1^m + \frac{h^{r_m}}{\Gamma(2+r_m)}(\beta_0^{n_m} - \beta_1^{n_m}) - \frac{h^{r_m}}{\Gamma(1+r_m)}\beta_0^{n_m} = 0$ allow finding the unknown initial conditions $\beta_0^{n_m}$ and $\beta_1^{n_m}$.*

Proof. Applying Proposition 3.4 over (10) and deriving it, (39) is obtained

$$\beta^{n_m}(x) = \frac{1}{\Gamma(1-r_m)}\frac{d}{dx}\int_0^x (x-\tau)^{-r_m}G^m(\tau)d\tau \tag{39}$$

Applying the same procedure exposed in the proof of Theorem 3.2, (40) is obtained

$$\beta_1^{n_m} - \frac{h^{-r_m}}{\Gamma(2-r_m)}G_1^m = -\frac{h^{-r_m}}{\Gamma(2-r_m)}G_0^m + \frac{h^{-r_m}}{\Gamma(1-r_m)}G_0^m \tag{40}$$

Considering that $G_0^m = 0$ for the definition of Caputo's derivative and applying it in (40), (41) is obtained

$$\beta_1^{n_m} - \frac{h^{-r_m}}{\Gamma(2-r_m)}G_1^m = 0 \tag{41}$$

Applying Proposition 3.3 over (10) and the same procedure exposed in the proof of Theorem 3.2, (42) is obtained

$$G_1^m + \frac{h^{r_m}}{\Gamma(2+r_m)}(\beta_0^{n_m} - \beta_1^{n_m}) - \frac{h^{r_m}}{\Gamma(1+r_m)}\beta_0^{n_m} = 0 \tag{42}$$

Eqs. (41) and (42) correspond to the equations proposed in the Corollary 3.2.1; being $f_0, \ldots, \beta_0^{n_m-1}$ the initial conditions. Now Eqs. (13), (14), (35) and (38) are in total $s+m+2$ and these have $s+m+2$ unknowns. □

4. Numerical examples

Example 4.1. Consider the linear RLFDE (43) exposed in [12]

$$D^{v_1}f(x) + f(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-v_1)}x^{3-v_1} + \frac{24}{\Gamma(5-v_1)}x^{4-v_1} \tag{43}$$

Where $0 < \nu_1 < 1$ and with the initial condition $I^{\nu_1 - 1} f(0) = 0$. The exact solution is $f(x) = x^4 - \frac{1}{2}x^3$. Considering $\nu_1 = 0.1,\ 0.25,\ 0.5,\ 0.9$ and step size $h = 0.1$ for solving (43). For this example: $0 < \nu_1 < 1$, $n_1 = 1$ and $r_1 = 1 - \nu_1$. Steps of the proposed method are explained below.

First step (order reduction): the resultant equations after the application of Proposition 3.1 are as follows:

$$\alpha^{1-r_1}(x) + f(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{\Gamma(4-\nu_1)}x^{3-\nu_1} + \frac{24}{\Gamma(5-\nu_1)}x^{4-\nu_1} \tag{44}$$

In this case, there is only one RLFD into (43) then one RLFIs is obtained

$$G^1(x) = \frac{1}{\Gamma(r_1)}\int_o^x (x-\tau)^{r_1-1} f(\tau)d\tau \tag{45}$$

In this case, $n_1 = 1$ and there is only one RLFD then one substitution function is obtained

$$\alpha^{1-r_1}(x) = \frac{dG^1(x)}{dx} \tag{46}$$

Second step (transform the previous equations to recurrence equations): applying Proposition 3.3 over Eqs. (44) and (46), the following equations are obtained:

$$\alpha_{l+1}^{1-r_1} + f_{l+1} = x_{l+1}^4 - \frac{1}{2}x_{l+1}^3 - \frac{3}{\Gamma(4-\nu_1)}x_{l+1}^{3-\nu_1} + \frac{24}{\Gamma(5-\nu_1)}x_{l+1}^{4-\nu_1} \tag{47}$$

$$\alpha_{l+1}^{1-r_1} = \frac{G_{l+1}^1 - G_l^1}{h} \tag{48}$$

Now, applying Theorem 3.1 over (45), (49) is obtained.

$$hf_{l+1} + F_l = \frac{h^{1-r_j}}{\Gamma(3-r_j)}(G_{l+1}^j - G_l^j) + \frac{h^{1-r_j}}{\Gamma(2-r_j)}G_l^j + \frac{1}{\Gamma(3-r_j)}\sum_{i=0}^{l-1} k_i + \frac{1}{\Gamma(2-r_j)}\sum_{i=0}^{l-1} m_i \tag{49}$$

Where $k_i = \frac{(G_{i+1}^j - G_i^j)}{h}[(\rho+h)^{2-r_j} - \rho^{2-r_j}]$, $m_i = G_i^j(\rho+h)^{1-r_j} - G_{i+1}^j \rho^{1-r_j}$, and $\rho = (l-i)h$. Eqs. (47) to (49) have 4 unknowns $(\alpha_{l+1}^{1-r_1}, f_{l+1}, G_{l+1}^1, F_l)$.

Third step (search of additional initial conditions): if it is supposed that $l = 0$ then Eqs. (47) and (48) are transformed to the system (50).

$$\begin{aligned} \alpha_1^{1-r_1} + f_1 &= x_1^4 - \frac{1}{2}x_1^3 - \frac{3}{\Gamma(4-\nu_1)}x_1^{3-\nu_1} + \frac{24}{\Gamma(5-\nu_1)}x_1^{4-\nu_1} \\ \alpha_1^{1-r_1} &= \frac{G_1^1 - G_0^1}{h} \end{aligned} \tag{50}$$

Now applying Theorem 3.2, (51) and (52) are obtained.

$$f_1 - \frac{h^{-r_1}}{\Gamma(2-r_1)}G_1^1 = -\frac{h^{-r_1}}{\Gamma(2-r_1)}G_0^1 + \frac{h^{-r_1}}{\Gamma(1-r_1)}G_0^1 \tag{51}$$

$$G_1^1 + \frac{h^{r_1}}{\Gamma(2+r_1)}(f_0 - f_1) - \frac{h^{r_1}}{\Gamma(1+r_1)}f_0 = 0 \tag{52}$$

Eqs. (50)– (52) compose a system of four unknowns and four equations, that is solved to find f_1 and G_1^1 which are the additional initial conditions necessary to solve (47) to (49). Absolute error is shown in Fig. 1.

Fig. 1 shows the absolute error for different values of ν_1. We consider that the solution is accurate, the maximum error is less than $5.3x10^{-3}$. In this case, the absolute error tends to increase when the fractional order is increased.

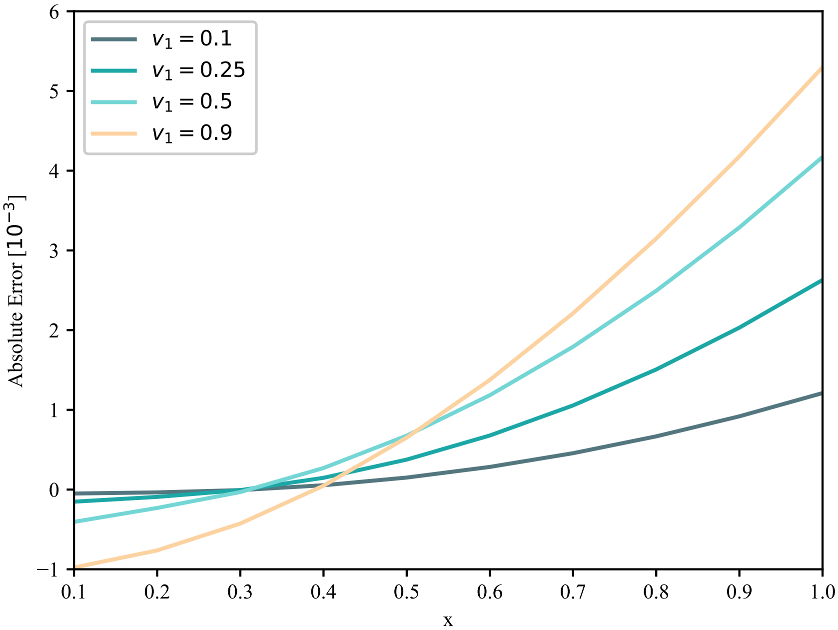

Fig. 1. Absolute Error for different values of ν_1.

Example 4.2. Consider the RLFDE of (53) with variable coefficients

$$a(x)D^{\nu_2} f(x) + bD^{\nu_1} f(x) + c(x) f(x) = u(x) \tag{53}$$

Where $u(x) = \frac{2}{\Gamma(3-\nu_2)} x^{3-\nu_2} - \frac{1}{\Gamma(2-\nu_2)} x^{2-\nu_2} + \frac{2}{\Gamma(3-\nu_1)} x^{2-\nu_1} - \frac{1}{\Gamma(2-\nu_1)} x^{1-\nu_1} + \sqrt{x}(x^2 - x)$; the coefficients are $a(x) = x$, $b = 1$ and $c(x) = \sqrt{(x)}$; and the fractional orders are $\nu_2 = 0.719$ and $\nu_1 = 0.141$. (53) is subject to initial conditions $I^{\nu_2 - 1} f(0) = 0$ and $I^{\nu_1 - 1} f(0) = 0$. The exact solution is $f(x) = x^2 - x$. The numerical solution f_{l+1} with $h = 0.005$ and the exact solution $f(x)$ are shown in Fig. 2, while the absolute error is shown in Fig. 3.

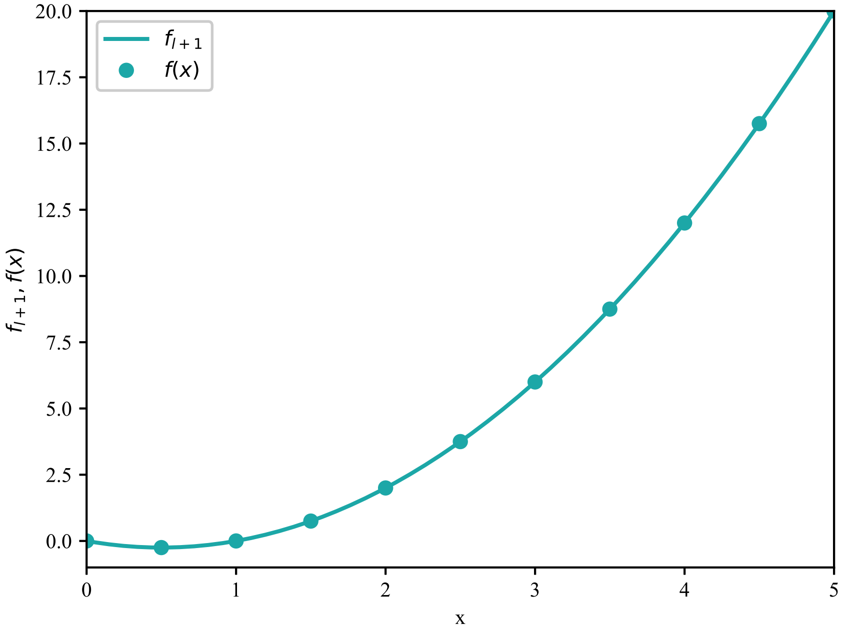

Fig. 2. Numerical (f_{l+1}) and exact solution ($f(x)$) of RLFDE with variable coefficients.

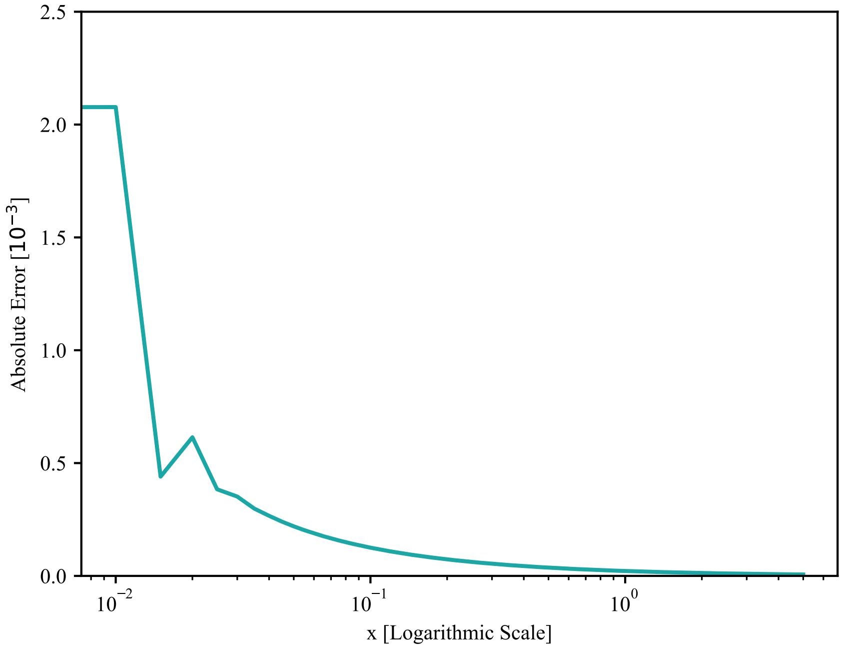

Fig. 3. Absolute error between numerical and exact solutions.

Figs. 2 and 3 show that the proposed method is able to solve RLFDE with multi-order fractional derivatives and variable coefficients, the solution obtained ($f_{(l+1)}$) is accurate to the exact solution, being the maximum absolute error less than $2.2x10^{-3}$.

Example 4.3. Consider the multi-order CFDE (54) that it is known as Bagley-Torvik equation in [5]. This is a classic problem of fractional differential equations, the exact solution of this problem is $f(x) = x + 1$.

$$D^2 f(x) + {}^cD^{\nu_1} f(x) + f(x) = (x + 1) \tag{54}$$

Where $\nu_1 = \frac{3}{2}$ subject to initial conditions $f(0) = f^{(1)}(0) = 1$, the solution is given in the intervals $0 \le x \le 5$ and $0 \le x \le 10$ and for three different h (0.01325, 0.125, 0.5).

Table 1 gives the maximum error obtained using the two methods reported in [19] that are called E_{M_1} with CPU time (CPU_1) and E_{M_2} with (CPU_2). Table 1 also gives the maximum error of the proposed method ($E_{Proposed}$) with CPU time ($CPU_{Proposed}$). Characteristics of the computer used in the simulations are: RAM 8 GB, Intel Core i3 without GPU.

Table 1 shows that the most accurate method is E_{M_2} while E_{M_1} and $E_{Proposed}$ have similar precision. The proposed method demanded less CPU time for all simulations than all the others thus corresponding to the most efficient method in this scenario. For the proposed method, when h is decreased the precision significantly increases without increasing too much its CPU time. In consequence, reduction of the step size is a practical solution for increasing the accuracy of the proposed method.

Table 1
Comparison between the solutions of [19] and the proposed method.

h	Interval	$\lvert E_{M_1}\rvert$	$\lvert E_{M_2}\rvert$	$\lvert E_{Proposed}\rvert$	CPU_1	CPU_2	$CPU_{Proposed}$
0.5	$0 \le x \le 5$	0.3831	0.00741	0.37240	$3.2x10^{-3}$	$8x10^{-3}$	$1.5x10^{-3}$
0.125	$0 \le x \le 5$	0.0265	0.00196	0.02348	$9.4x10^{-3}$	$25x10^{-3}$	$5.8x10^{-3}$
0.01325	$0 \le x \le 5$	0.0028	0.00016	0.00248	$68x10^{-3}$	$185x10^{-3}$	$66x10^{-3}$
0.5	$0 \le x \le 10$	0.3834	0.00709	0.37130	$6.1x10^{-3}$	$20x10^{-3}$	$3.1x10^{-3}$
0.125	$0 \le x \le 10$	0.0254	0.00198	0.02389	$25x10^{-3}$	$68x10^{-3}$	$11x10^{-3}$
0.01325	$0 \le x \le 10$	0.0021	0.00015	0.00104	$134x10^{-3}$	$405x10^{-3}$	$105x10^{-3}$

Example 4.4. Consider the multi-order CFDE with variable coefficients (55) in [23]

$$aD^2 f(x) - b(x)^c D^{\nu_2} f(x) + c(x) D f(x) + e(x)^c D^{\nu_1} f(x) + k(x) f(x) = u(x) \tag{55}$$

Where $u(x) = -a - \frac{b(x)}{\Gamma(3-\nu_2)} x^{2-\nu_2} - c(x)x - \frac{e(x)}{\Gamma(3-\nu_1)} x^{2-\nu_1} + k(x)(2 - \frac{1}{2}x^2)$; the coefficients are $a = 1$, $b(x) = x^{\frac{1}{2}}$, $c(x) = x^{\frac{1}{3}}$, $e(x) = x^{\frac{1}{4}}$ and $k(x) = x^{\frac{1}{5}}$; and the orders or fractional derivatives are $\nu_1 = 0.333$ and $\nu_2 = 1.234$. (55) is subject to initial condition $f(0) = 2$ and $f^{(1)}(0) = 0$, under this initial conditions the solution of the problem is $f(x) = 2 - \frac{1}{2}x^2$.

Fig. 4 shows that the solution obtained using the proposed numerical method is close to the exact solution of the CFDE (55), also the accuracy of the method increases by the increasing of the step size h. Fig. 4 proves that with small values of h,

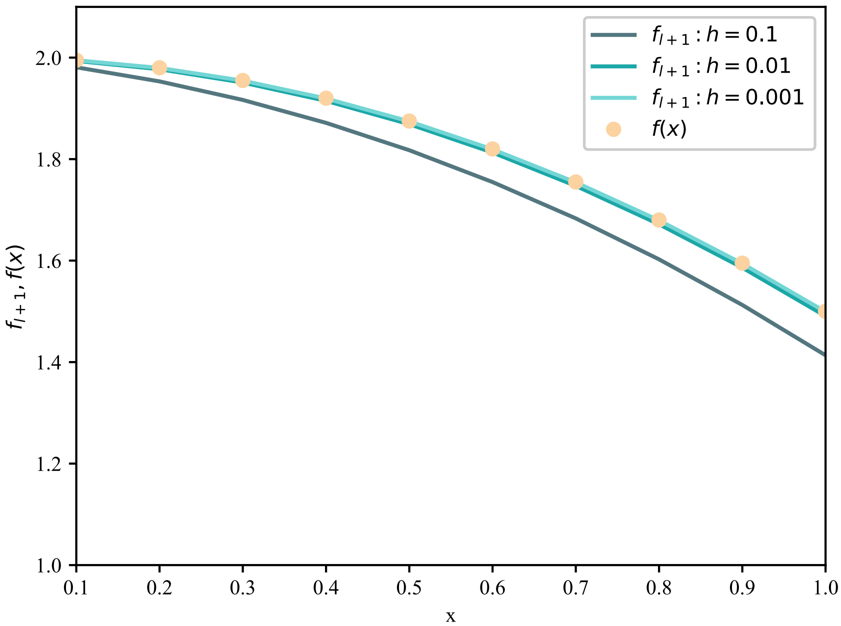

Fig. 4. Numerical solution obtained (f_{l+1}) for different step sizes and the exact solution ($f(x)$).

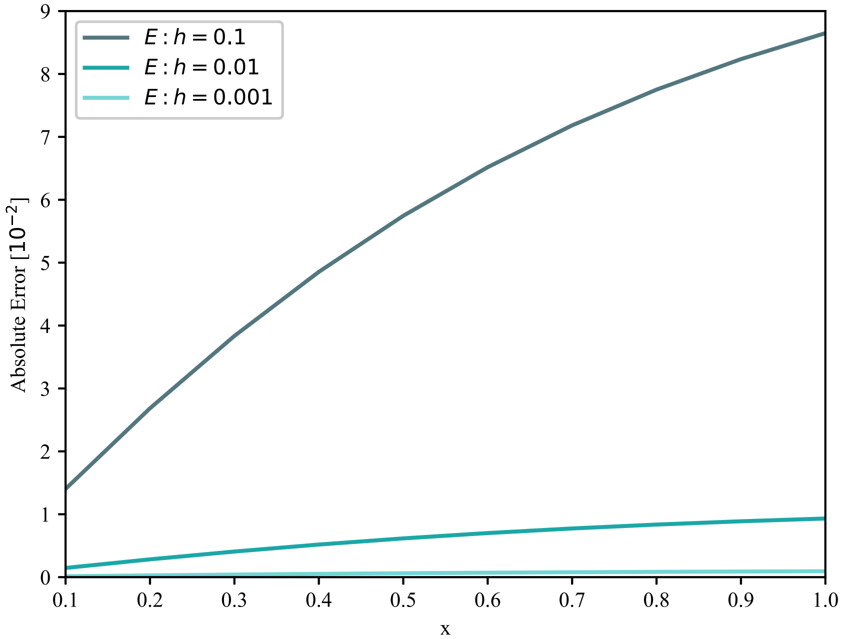

Fig. 5. Absolute error between numerical and exact solutions for different values of h.

the solution f_{l+1} is approaching to the exact solution $f(x)$. In addition to these, Fig. 5 shows that the absolute error sharply decreases when h decreases.

5. Conclusions

The main contribution of this paper is a novel numerical method for solving FDE with Riemann-Liouville's or Caputo's Fractional Derivatives. The strength of the method consists in the solution of FDE with multi-order fractional derivatives and variable coefficients. The proposed method has not restriction in the type of Differential equation to be solved, being possible to be applied to a wide range of FDE.

Two theorems and two corollaries were proposed as important contributions of this paper. Theorem 3.1 and Corollary 3.1.1 transform Riemann-Liouville's and Caputo's fractional derivatives respectively to recurrence equations, these allow solving a FDE like a system of algebraic equations. While, Theorem 3.2 and Corollary 3.2.1 allow finding the additional initial conditions necessary for applying the method.

In CFDE examples, the obtained solution was compared with solutions obtained using other methods, the precision was very similar. Although the proposed method was not the most accurate, it was the most efficient method with fewer CPU times. Also, the pseudo-code is presented in Appendix A to show the easy computational implementation of the proposed method.

Considering that there are not reported examples for numerical solution of RLFDE yet, solutions were only compared with exact solutions. It was demonstrated that the method is accurate for solving linear, multi-order and variable coefficients RLFDE, so that the proposed method is presented as the first numerical method that solves RLFDE which includes multi-order fractional derivatives and variable coefficients.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

David E. Betancur-Herrera: Conceptualization, Methodology, Software, Writing - original draft, Validation, Formal analysis, Data curation. **Nicolás Muñoz-Galeano:** Formal analysis, Writing - original draft, Writing - review & editing, Visualization, Supervision.

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Appendix A. Pseudo-code for solving RLFDE or CFDE with the proposed method

1. start
2. initialize $x_a = 0,\ x_b = b$ and $l = 1$
3. introduce simulation parameters h, v_j with $j = 0, 1, \ldots, m$
4. introduce initial conditions $(G^1(0), \ldots, G^j(0), \ldots, G^m(0); \alpha^{1-r_j}(0), \ldots, \alpha^{k_j-r_j}(0), \ldots, \alpha^{n_j-r_j}(0), \ldots, \alpha^{n_m-r_m}(0))$ for RLFDE or $(f(0), \beta^1(0), \ldots, \beta^{k_j}(0), \ldots, \beta^{n_m-1}(0))$ for CFDE
5. calculate additional initial conditions $(G^1(h), \ldots, G^j(h), \ldots, G^m(h))$ using (13), (14) and Theorem 3.2 for RLFDE or $(\beta_0^{n_m}$ and $\beta_1^{n_m})$ using (15), (16) and Corollary 3.2.1 for CFDE
6. while $x \leq (x_b - h)$ 7. calculate f_{l+1} using (13), (14) and Theorem 3.1 for RLFDE or using (15), (16) and Corollary 3.1.1 for CFDE
8. calculate $x = (l + 1)h$
9. end

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