DOI: 10.2478/v10309-012-0001-z
An. Şt. Univ. Ovidius Constanţa **Vol. 20(1), 2012, 5–14**

Approximate Cycles of the Second Kind in Hilbert space for a Generalized Barbashin-Ezeilo Problem

A. U. Afuwapẹ, M. Y. Balla, Xavier Udo-utun

Abstract

In this work we show that the Volterra integral operator defined on the space of absolutely stable functions induces an asymptotically pseudocontractive operator. We, then, show that Afuwape's [1] generalization of the Barbashin-Ezeilo problem is solvable in a Banach space (but not in Hilbert space $L_2[0,\infty)$). However applying Osilike-Akuchu[10] theorem and recent results (in Hilbert space) of Igbokwe and Udo-utun[8] we formulate conditions for finding approximate cycles of the second kind (in the Hilbert space $W_0^{2,2}[0,\infty)$) to this problem given in the form $x''' + ax'' + g(x') + \varphi(x) = 0$.

1 Introduction

In 2006 using the ideas of nonlocal reduction method, Afuwape[1] generalized and modified the Barbashin-Ezeilo problem by asking: *When will a general third-order equation of the form*

$$x''' + \alpha x'' + g(x') + \varphi(x) \quad = \quad 0 \tag{1.1}$$

have a cycle of the second kind where α is an arbitrary constant $g(y)$ a continuous bounded function and $\varphi(x)$ is a 2π-periodic odd function having zeros 0, x_0 *in* $[0, 2\pi)$ *and at any point* x $[0, 2\pi)$ *satisfies*

$$\varphi^2(x) + [\varphi'(x)]^2 \quad \neq \quad 0. \tag{1.2}$$

Key Words: approximate cycles of the second kind; generalized Barbashin-Ezeilo problem; Volterra Integral operator; pseudocontractive maps in Hilbert spaces.
2010 Mathematics Subject Classification: 34C11, 34D05, 34K12.
Received: February, 2011.
Accepted: November, 2011.

Transforming (1.1) as

$$\begin{cases} \dfrac{dz}{dt} = Az + b\varphi(\sigma) \\ \\ \dfrac{d\sigma}{dt} = c^* z \end{cases} \tag{1.3}$$

where $A = \begin{pmatrix} -\alpha & -\beta \\ 1 & 0 \end{pmatrix}; b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\varphi(\sigma) = [\hat{g}(\dot{\sigma}) + \varphi(\sigma)]$, $g(y) = \beta y + \hat{g}(y)$, with $0 \leq \hat{g}(y) \leq \mu$, the following theorem was proved:

Theorem 1.1. *(Afuwape[1])*
Suppose there exists positive numbers β and μ such that $\beta \leq g(y) \leq \mu + \beta$, with $\alpha^2 > 4\beta$ and $\mu < \lambda_1(\lambda_1^2 - \alpha\lambda_1 + \beta)$ for some constant parameter λ_1.

Suppose also that for some $\lambda_2 > 0$,

$$\alpha - \sqrt{\alpha^2 - 4\beta} < \quad 2\lambda_2 \quad < \alpha + \sqrt{\alpha^2 - 4\beta} \tag{1.4}$$

and $\varphi'(0) > \alpha\beta$. Then (1.1) is Bakaev-stable and has a nontrivial periodic solution in every strip $\Pi = \{[2k\pi, x_0 + 2k\pi); k\mathbb{Z}\}$.

In this paper our objective is to transform the system (1.3) into equivalent Volterra integral equation form and and verify that the resulting operator is an asymptotically pseudocontractive map. This will enable us to apply the Osilike-Akuchu Theorem in [10] and the results of Igbokwe and Udo-utun Theorem in [8] to show that the resulting feedback problem (1.1) has a fixed point in Banach space for all α (and in Hilbert space for $\alpha > 0$).

2 Preliminaries

Definition 2.1. *The solution $(x(t), \sigma(t))$ of (1.1) is said to be circular if there exists a number $\epsilon > 0$, and a time $\widetilde{t} \geq 0$ such that $\frac{d\sigma(t)}{dt} \geq \epsilon$ for all $t \geq \widetilde{t}$.*

Definition 2.2. *The solution $(x(t), \sigma(t))$ of (1.1) is said to be cycle of the second kind if there exists an integer $j \neq 0$, and a moment $\widetilde{t} > 0$ such that for all $x(\widetilde{t}) = 0$, $\sigma(\widetilde{t}) - \sigma(0) = j\tau$ where $\varphi(\sigma + \tau) = \varphi(\sigma)$.*

Let C_0 denote the space of absolutely stable solutions $\sigma(t)$ of the one-dimensional feedback system

$$\sigma(t) = \eta\Phi(t) + \int_0^t \Phi(t-s)g(\sigma(s))ds$$

then each element $\sigma \;\; C_0$ satisfies $\lim_{t\to\infty} \sigma(t) = 0$.

It has been shown by Corduneanu[3, 4] and Holtzman[7] that $C_0 \subseteq L_2[0,\infty)$ which is a Hilbert space.

We shall show that the operator $(Au)(t) = \int_0^t \Phi(t-s)u(s)ds$ is compact by applying Arzela-Ascoli theorem for suitable functions defined on $[0,\eta)$ and that if A is self- adjoint then the operator $T : C_0 \longrightarrow C_0$ given by

$$(T\sigma)(t) = \eta\Phi(t) + \int_0^t \Phi(t-s)g(\sigma(s))ds = \eta\Phi(t) + (A(g\sigma))(t)$$

has a fixed point if some conditions given below are satisfied. Here, g is an odd function satisfying the sector condition

$$0 \leq \frac{g(\xi_1) - g(\xi_2)}{\xi_1 - \xi_2} \quad \leq \quad k,\ (\xi_1 \neq \xi_2),\ k\ >\ 0 \qquad (2.1)$$

for all $\xi_1,\ \xi_2 \ \ [0,\eta]$.

Let E be an arbitrary real Banach space and let J denote normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f\ \ E^* : \langle x, f\rangle \ =\ \|x\|^2;\ \|x\|^2\ =\ \|f\|^2\}$$

where E^* denotes the dual space of E and $\langle , \rangle$ denotes the generalized duality pairing. If E^* is strictly convex, then J is single-valued. In the sequel, we shall denote single-valued duality mappings by j. In Hilbert spaces, j is the identity.

Let us give some basic definition and remarks that will be used in this work.

Definition 2.3. *In Hilbert spaces H, a self-mapping T of a nonempty subset K of H is* asymptotically pseudocontractive *if it satisfies*

$$\|T^n x - T^n y\| \quad \leq \quad a_n\|x-y\|^2 + \|x - y - (T^n x - T^n y)\|^2 \qquad (2.2)$$

for all $x,\ y\ \ K$ and for some sequence $\{a_n\} \subseteq [0,\infty)$ such that $\lim_{n\to\infty} a_n = 1$.

We remark that the class of asymptotically pseudocontractive mappings contains the important class of *asymptotically nonexpansive* mappings (that is mappings $T : K \longrightarrow K$ such that

$$\|T^n x - T^n y\| \quad \leq \quad a_n\|x-y\|\, \forall\, x,\ y\ K \qquad (2.3)$$

for all $x,\ y\ \ K$ and for some sequence $\{a_n\} \subseteq [0,\infty)$ such that $\lim_{n\to\infty} a_n = 1$).

Definition 2.4. *T is called* asymptotically quasi-nonexpansive *if $F(T) = \{x \quad K : Tx = x\} \neq \emptyset$ and (2.3) is satisfied $\forall\ x \quad K$ and $y \quad F(T)$.*

Definition 2.5. *If there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all $n \geq 1$ and for all $x,\ y \quad K$, then T is called* uniformly Lipschitzian.

Let H be a Hilbert space, $\rho,\ \bar{\rho} \quad H$ be such that $\bar{\rho} = g(\rho)$ (where g is a continuous function satisfying $0 \leq \frac{g(\xi_1)-g(\xi_2)}{\xi_1-\xi_2} \leq k,\ k > 0$) and $T : H \longrightarrow H$ be given by

$$\begin{aligned} T\rho(t) &= \eta\Phi(t) + \int_0^t \Phi(t-s)g(\rho(s))ds \qquad (2.4) \\ &= \eta\Phi(t) + \int_0^t \Phi(t-s)\bar{\rho}(s)ds. \end{aligned}$$

3 Main Results

Our main result shall be the following theorem

Theorem 3.1. *Let $\alpha > 0$; and $p_1, p_2 = -\frac{\alpha}{2} \mp \sqrt{\frac{\alpha^2}{4} - \beta}$; then a cycle of second kind of the generalized Barbashin-Ezeilo problem (1.1) is the limit in some Hilbert space of the iterative process*

$$\left.\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)T^k y_n \\ y_n &= \beta_n x_{n-1} + (1-\beta_n)T^k x_n \end{aligned}\right\},\ n \geq 1 \qquad (3.1)$$

where $n = k \geq 1$, T is $L-$Lipschitzian operator given by

$$(T\sigma)(t) = -p_1 \int_0^t \int_0^s e^{p_1(t-\xi)}[\hat{g}(\dot{\sigma}(\xi)) + \varphi(\sigma(\xi))]d\xi ds \qquad (3.2)$$

and α_n, β_n satisfy the conditions

$$\begin{array}{llll} (i) & \sum_{n=1}^{\infty}(1-\alpha_n) = \infty & (ii) & \sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty \\ (iii) & \sum_{n=1}^{\infty}(1-\beta_n) < \infty & (iv) & (1-\beta_n)(1-\alpha_n)L^2 < 1. \end{array}$$

Before proving this main result we shall, first, establish the existence and validity of the operator T.

Proposition 3.1. *Let the operator $A(g(\sigma))(t) = \int_0^t \Phi(t-s)g(\sigma(s))ds$ be self-adjoint with eigenfunctions $g(\sigma_j) \quad C_0$ (whenever $\sigma_j \quad C_0$) corresponding to eigenvalues λ_j ($j \geq 1$). The operator T has at least one fixed point if $k^n\|A\| \leq (1 + \nu_n)$ for some sequence $\{\nu_n\}$.*

PROOF
Any bounded sequence of functions $\{\sigma_n\} \subseteq C_0$ is equibounded and equicontinuous on $[0, \eta)$ and by Arzela-Ascoli theorem such sequences are relatively compact. Now, A being a bounded linear operator is continuous so it maps relatively compact sets into relatively compact sets, therefore we infer that A is a compact operator so T maps every closed convex set K into itself. Further,

$$\begin{aligned}
\|T\sigma_i - T\sigma_j\| &= \left\| \int_0^t \Phi(t-s)\left(g(\sigma_i(s)) - g(\sigma_j(s))\right) ds \right\| \\
&= \|A\left(g(\sigma_i) - g(\sigma_j)\right)\| = \|\lambda_i g(\sigma_i) - \lambda_j g(\sigma_j)\| \\
&\leq \max\{|\lambda_i|, |\lambda_j|\}\|k(\sigma_i - \sigma_j)\| \\
&= k\max\{|\lambda_i|, |\lambda_j|\}\|\sigma_i - \sigma_j\|. \qquad (3.3)
\end{aligned}$$

Therefore

$$\|T^n\sigma_i - T^n\sigma_j\| \leq k^n \max\{|\lambda_i|, |\lambda_j|\}\|\sigma_i - \sigma_j\| \qquad (3.4)$$

verifying that T is an asymptotically nonexpansive map if there exists a sequence $\{\nu_n\}$ with $\nu_n \longrightarrow 0$ such that $k^n \max\{|\lambda_i|, |\lambda_j|\} \leq (1 + \nu_n)$ since (3.4) yields $\|T^n\sigma_i - T^n\sigma_j\| \leq a_n\|\sigma_i - \sigma_j\|$ (where $a_n = (1 + \nu_n)$).

Inequality (3.4) follows from condition (2.1) and since A is a compact operator (3.4) becomes

$$\|T^n\sigma_i - T^n\sigma_j\| \leq k^n\|A\|\|\sigma_i - \sigma_j\|. \qquad (3.5)$$

then applying the results of [10], we conclude that the modified implicit process

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_i^k x_n, \; n \geq 1$$

as given by Sun[12] converges to the fixed point of (2.4).

Theorem 3.2. *Let the operator $A(g(\sigma))(t) = \int_0^t \Phi(t-s)g(\sigma(s))ds$ be self-adjoint with eigenfunctions $g(\sigma_j)$ C_0 (whenever σ_j C_0) corresponding to eigenvalues λ_j $(j \geq 1)$. Then the averaging composite implicit iteration process* (3.1)

$$\left.\begin{aligned}
\sigma_{n+1} &= \alpha_n\sigma_n + (1-\alpha_n)T^l\rho_{n+1} \\
\rho_{n+1} &= \beta_n\sigma_n + (1-\beta_n)T_i^k\sigma_{n+1}
\end{aligned}\right\} n \geq 1$$

converges to at least one fixed point of (2.4) *provided $k^n\|A\| \leq (1 + \nu_n)$ for some sequence $\{\nu_n\}$ satisfying $\nu_n \longrightarrow 0$ as $n \longrightarrow \infty$.*

PROOF
The argument is the same as in the proof of Proposition 3.1 so we only need to prove that T is asymptotically pseudocontractive.

$$\begin{aligned}\langle T\sigma_i - T\sigma_j, j(\sigma_i - \sigma_j\rangle &= \langle A(g(\sigma_i) - g(\sigma_j), j(\sigma_i) - \sigma_j\rangle \\ &= \langle \lambda_i g(\sigma_i) - \lambda_j g(\sigma_j), j(\sigma_i - \sigma_j)\rangle \\ &\leq \max\{|\lambda_i|, |\lambda_j|\}\langle k(\sigma_i - \sigma_j), j(\sigma_i - \sigma_j)\rangle \\ &= k\max\{|\lambda_i|, |\lambda_j|\}\|\sigma_i - \sigma_j\|^2.\end{aligned}$$

Therefore, iteratively, we have

$$\langle T^n\sigma_i - T^n\sigma_j, j(\sigma_i - \sigma_j\rangle \leq k^n \max\{|\lambda_i|, |\lambda_j|\}\|\sigma_i - \sigma_j\|^2.$$

This implies that T is asymptotically pseudocontractive, and thus we can apply the results of [8] and [9]. This is the desired result.

4 PROOF OF THEOREM 3.1

We now give the proof of Theorem 3.1 by rewriting (1.1) as

$$\begin{cases}\dot{x_1} = -\alpha x_1 - \beta x_2 - \hat{g}(\dot{\sigma}) - \varphi(\sigma) \\ \dot{x_2} = x_1 \\ \dot{\sigma} = x_2\end{cases}$$

This clearly implies that $x_1 = \ddot{\sigma}$, $x_2 = \dot{\sigma}$ and $-[\hat{g}(\dot{\sigma}) + \varphi(\sigma)] = \dot{x}_1 + \alpha x_1 + \beta x_2$.

Observing that a fundamental matrix solution X is given by

$$X(t) = \begin{pmatrix} e^{p_1 t} & e^{p_2 t} \\ p_1 e^{p_1 t} & p_2 e^{p_2 t}\end{pmatrix},$$

it follows that that the desired cycle of the second kind is given by

$$\sigma(t) = -p_1 \int_0^t \int_0^s e^{p_1(s-\xi)}[\hat{g}(\dot{\sigma}(\xi)) + \varphi(\sigma(\xi))]d\xi ds \tag{4.1}$$

where

$$\begin{aligned} p_1 &= -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} - \beta} \\ p_2 &= -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{4} - \beta}\end{aligned}$$

But the output $\dot{\sigma}(t)$ is given by

$$\dot{\sigma}(t) = -p_1 \int_0^t e^{p_1(t-s)}[\hat{g}(\sigma(s)) + \varphi(\sigma(s))]ds \tag{4.2}$$

which yields

$$\begin{aligned}\dot{\sigma}(t) &= p_1[\ddot{\sigma}(t) + \alpha\dot{\sigma}(t) + \beta\sigma(t)] - p_1[\ddot{\sigma}(0) + \alpha\dot{\sigma}(0) + \beta\sigma(0)]e^{p_1 t} \\ &\quad -p_1^2 \int_0^t e^{p_1(t-s)}[\ddot{\sigma}(s) + \alpha\dot{\sigma}(s) + \beta\sigma(s)]ds.\end{aligned}$$

This gives

$$\begin{aligned}\beta\sigma(t) &= -\ddot{\sigma}(t) + \frac{1-\alpha p_1}{p_1}\dot{\sigma}(t) + [\ddot{\sigma}(0) + \alpha\dot{\sigma}(0) + \beta\sigma(0)]e^{p_1 t} \\ &\quad +p_1 \int_0^t e^{p_1(t-s)}[\ddot{\sigma}(s) + \alpha\dot{\sigma}(s) + \beta\sigma(s)]ds \\ \sigma(t) &= \frac{p_1}{\beta}\int_0^t e^{p_1(t-s)}[\ddot{\sigma}(s) + \alpha\dot{\sigma}(s) + \beta\sigma(s)]ds + \frac{1}{\beta}[\ddot{\sigma}(0) + \alpha\dot{\sigma}(0) + \beta\sigma(0)]e^{p_1 t} \\ &\quad -\frac{1}{\beta}\ddot{\sigma}(t) + \frac{1-\alpha p_1}{\beta p_1}\dot{\sigma}(t) \\ &= p_1 \int_0^t e^{p_1(t-s)}\sigma(s)ds + \left[\frac{-\ddot{\sigma}(0) + (\alpha-1)\dot{\sigma}(0)}{p_1(p_1-\alpha)+\beta} + \sigma(0)\right]e^{p_1 t} \\ &\quad -\frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma}(t) + \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma}(t)\end{aligned} \tag{4.3}$$

So analysis of the nonlinear operator (4.1) is equivalent to analysis of the simpler nonlinear operator

$$\begin{aligned}(T\sigma)(t) &= p_1 \int_0^t e^{p_1(t-s)}\sigma(s)ds + \left[\frac{-\ddot{\sigma}(0) + (\alpha-1)\dot{\sigma}(0)}{p_1(p_1-\alpha)+\beta} + \sigma(0)\right]e^{p_1 t} \\ &\quad -\frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma}(t) + \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma}(t)\end{aligned}$$

which anchors on the behavior of the linear operator (where σ is a fixed point of T.)

$$(A\sigma)(t) = p_1 \int_0^t e^{p_1(t-s)}\sigma(s)ds - \frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma}(t) + \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma}(t).$$

We observe that (3.2) implies $\sigma(0) = 0$ while (4.2) gives $\dot{\sigma}(0) = 0$ and so

(4.3) becomes

$$\begin{aligned}(T\sigma)(t) &= p_1\int_0^t e^{p_1(t-s)}\sigma(s)ds - \frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma}(t)\\ &+ \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma}(t) \qquad (4.4)\end{aligned}$$

therefore the operator T is equivalent to A.

Using condition (1.4) we observe that $\alpha - \frac{3\alpha}{2} - \sqrt{\frac{\alpha^2}{2} - \beta} < \lambda_2$. That is $p_1 < \lambda_2$.

Now, for $\alpha > 0$ the operator $T = A$ is defined on the Sobolev space $W_0^{(2,2)}[0,\infty)$ which is a Hilbert space in $L_2[0,\infty)-$space. In this case we can apply Theorem 3.2 to the operator T choosing the iterates from $C_0^\infty[0,\infty)$ where $\sigma_n(t)$ are defined by (3.1).

Clearly, since $\sigma_0 = \sigma(0) = 0$, we have from (3.1) and (4.4) that

$$\begin{aligned}y_1 &= (1-\beta_1)[p_1\int_0^t e^{p_1(t-s)}\sigma_1(s)ds - \frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma_1}(t)\\ &+ \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma_1}(t)]\\ \sigma_1 &= (1-\alpha_1)Ty_1\\ &= (1-\alpha_1)(1-\beta_1)\left[p_1\int_0^t e^{p_1(t-s)}\sigma_1(s)ds - \frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma_1}(t)\right.\\ &\left.+\frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma_1}(t)\right] \qquad (4.5)\\ \sigma_{n+1}(t) &= \alpha_{n+1}\sigma_n(t) + (1-\alpha_{n+1})\left\{\beta_{n+1}\left[p_1\int_0^t e^{p_1(t-s)}\sigma_n(s)ds-\right.\right.\\ &- \frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma_n}(t)\\ &+ \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma_n}(t)] + (1-\beta_{n+1})\left[p_1\int_0^t e^{p_1(t-s)}\sigma_{n+1}(s)ds\right.\\ &- \left.\left.\frac{1}{p_1(p_1-\alpha)+\beta}\ddot{\sigma}_{n+1}(t) + \frac{(1-\alpha)p_1+1}{p_1(p_1-\alpha)+\beta}\dot{\sigma}_{n+1}(t)\right]\right\}. \qquad (4.6)\end{aligned}$$

So when $\alpha > 0$, the equation (3.2) and (1.4) ensures that $\sigma = T\sigma$ is asymptotically stable for $p_1 < 0$. This gives the required iterates to be asymptotically stable.

We shall complete the proof by showing that each iterate σ_n is a cycle of the second kind. This is obvious since from equation (4.5) if σ_1 chosen as a cycle of the the second kind then σ_n in (4.4) are cycles of the second kind for all n $\mathbb{N}$. This concludes the proof of theorem 3.1.

Remark 4.1. *For $\alpha < 0$ we have $p_1 > 0$ then, the solutions are bounded cycles and the iterates σ_n are required to be bounded only but not asymptotically stable. This means that the problem is not solvable in Hilbert space since the iterates are not square integrable on $[0, \infty)$. Here we apply Proposition 3.1 and results from [10] to obtain some fixed point.*

Acknowledgment

This research was supported by University of Antioquia Research Grant CODI No. IN596CE.

References

[1] A. U. AFUWAPE, *Remarks on Barbashin-Ezeilo Problem on Third-order Nonlinear Differential Equations*, J. Math. Anal. Appl. 317(2006) 613 - 619.

[2] E. A. BARBASHIN,*Introduction to the Theory of Stability*, Wolters-Noordhoff, Groningen, 1970, MR. 4 8737.

[3] C. CORDUNEANU, *Stability problems for some classes of feedback systems*, Equations differentielles et funcionelles non lineaires, (Actes Conference Internat, "Equa - Diff 73", Brussels/ Louvain-la-Neuves, 1073) 397-405.

[4] C. CORDUNEANU, *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973

[5] J. O. C. EZEILO, *Research Problem 12, Periodic Solutions of Differential Equations*, Bull. Amer. Math. Soc. 3(1966) 470.

[6] D. H. GRIFFEL , *Applied Functional Analysis*, John Wiley and Sons, New York.(1988)

[7] JACK M. HOLTZMAN, *Nonlinear System Theory - A Functional Analysis Approach*, Bell Telephone Laboratories, Inc., Whippany, New Jersey, (1970).

[8] D.I.IGBOKWE AND XAVIER UDO-UTUN , *Composite Implicit Iteration Process in Hilbert Spaces for Common Fixed Points of a Finite Family of Asymptotically Pseudocontractive Maps*, J. of Pure and App. Math.: Advances and Applications. Vol.3 (2010) 105 - 121.

[9] M.O.Osilike, *Implicit Iteration Process for Common Fixed Point of a Finite Family of Strictly Pseudo-contractive Maps*, J. Math. Anal. Appl. 294 (2004) 73 - 81.

[10] M.O.Osilike, B.G.Akuchu, *Common Fixed Points of a Finite Family of Asymptotically Pseudocontractive Maps*, J. Fixed Points and Appl. 2(2004) 81 - 99.

[11] T.O.Saposnikova, *Sobolev Spaces*, Springer-Verlag, Berlin (1985).

[12] Z.H.Sun, *Strong Convergence of an Implicit Iteration Process for a Finite Family of Asymptotically Quasi-nonexpansive Mappings*, J. Math. Anal. Appl. 286(2003) 351 - 358.

A. U. Afuwapẹ,
Instituto de Matemáticas,
Universidad de Antioquia,
Calle 67, No. 53-108, Medellín AA 1226,Colombia.
Email: aafuwape@yahoo.co.uk

M. Y. Balla
Department of Mathematics and Statistics,
University of Maiduguri,
Nigeria.
Email: myballa@yahoo.com

Xavier Udo-utun
Department of Mathematics and Statistics,
University of Maiduguri,
Nigeria.
Email: xvior@yahoo.com