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ULTIMATE BOUNDEDNESS RESULTS FOR SOLUTIONS OF CERTAIN THIRD ORDER NONLINEAR MATRIX DIFFERENTIAL EQUATIONS

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Abstract. We present in this paper ultimate boundedness results for a third order nonlinear matrix differential equations of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}),$$

where A, B are constant symmetric $n \times n$ matrices, $X, H(X)$ and $P(t, X, \dot{X}, \ddot{X})$ are real $n \times n$ matrices continuous in their respective arguments. Our results give a matrix analogue of earlier results of Afuwape [1] and Meng [4], and extend other earlier results for the case in which we do not necessarily require that $H(X)$ be differentiable.

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1. INTRODUCTION

Let $\mathcal{M}$ denote the space of all real $n \times n$ matrices, $I\!R^n$ the real n-dimensional Euclidean space and $I\!R$ the real line $-\infty < t < \infty$. We shall be concerned here with certain properties of solutions of differential equations of the form

$$\dddot{X} + A\ddot{X} + B\dot{X} + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1}$$

where $X : I\!R \longrightarrow \mathcal{M}$ is the unknown, $A, B \in \mathcal{M}$ are constants, $H : \mathcal{M} \longrightarrow \mathcal{M}$ and $P : I\!R \times \mathcal{M} \times \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$, and the dots indicate differentiation with respect to t. We shall assume throughout that $H \in \mathcal{C}(\mathcal{M})$ and $P \in \mathcal{C}(I\!R \times \mathcal{M} \times \mathcal{M} \times \mathcal{M})$.

Definition 1. *The solutions of* (1) *will be said to be ultimately bounded if there exists a constant* $D > 0$ *and if corresponding to any* $\alpha > 0$, *there exists a* $T(\alpha) > 0$ *such that for*

$$\{\|X(t_0)\|^2 + \|\dot{X}(t_0)\|^2 + \|\ddot{X}(t_0)\|^2\} < \alpha \quad \Rightarrow \quad \{\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2\} < D$$

for $t_0 \geq 0$ *and* $t \geq t_0 + T(\alpha)$.

The object of this paper is to prove ultimate boundedness results under some specified conditions on $H(X)$ and $P(t, X, \dot{X}, \ddot{X})$. Specifically, unlike [6], we shall only assume that $H(X) \in \mathcal{C}(\mathcal{M})$ and that for any $X, Y \in \mathcal{M}$, there exists an $n \times n$ real continuous matrix $C(X, Y)$ such that

$$H(X) = H(Y) + C(X, Y)(X - Y). \tag{2}$$

For the special case in which (1) is an $n-$vector equation (so that $X : I\!R \longrightarrow I\!R^n$, $H : I\!R^n \longrightarrow I\!R^n$ and $P : I\!R \times I\!R^n \times I\!R^n \times I\!R^n \longrightarrow I\!R^n$) a number of boundedness, stability and existence of periodic solutions results have been established, see [1, 2, 3, 4, 5] and the references contained therein. The conditions obtained in each of these previous investigations are generalizations of the well-known Routh-Hurwitz conditions

$$a > 0, \quad c > 0, \quad ab - c > 0 \tag{3}$$

for the stability of the trivial solution of the linear differential equation

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0 \tag{4}$$

with constant coefficients, see [7].

The result in this paper is the matrix analogue of the results obtained in [1], [4] and an extension of the matrix result obtained in Tejumola [8] to (1).

The motivation for the present investigation has come from the papers mentioned above. It should be also noted that the condition imposed on $H(X)$ here is different from that imposed in [6].

2. NOTATIONS

Some standard matrix notation will be used. For any $X \in \mathcal{M}$, X^T and x_{ij} $i, j = 1, 2, \ldots, n$ denote the transpose and the elements of X respectively while (c_{ij}) with $c_{ij} = \sum_{\ell=1}^{n} x_{i\ell} y_{\ell j}$ will denote the product matrix XY of the matrices $X, Y \in \mathcal{M}$. $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$ and $X^j = (x_{1j}, x_{2j}, \ldots, x_{nj})$ stand for the ith row and jth column of X respectively and $\underline{X} = (X_1, X_2, \ldots, X_n)$ is the n^2 column vector consisting of the n rows of X.

Corresponding to the constant matrix $A \in \mathcal{M}$ we define an $n^2 \times n^2$ matrix $\tilde{A}$ consisting of n^2 diagonal $n \times n$ matrix$(a_{ij} I_n)$ (I_n being the unit $n \times n$ matrix) and such that $(a_{ij} I_n)$ belongs to the $ith - n$ row and $jth - n$ column (that is, counting n at a time) of $\tilde{A}$. In the special case $n = 2$, $\tilde{A}$ is the 4×4 matrix

$$\begin{pmatrix} a_{11} I_2 & a_{12} I_2 \\ a_{21} I_2 & a_{22} I_2 \end{pmatrix}.$$

Next we introduce an inner product $\langle ., . \rangle$ and a norm $\| \cdot \|$ on $\mathcal{M}$ as follows. For arbitrary $X, Y \in \mathcal{M}$, $\langle X, Y \rangle = \text{trace } XY^T$. It is easy to check that $\langle X, Y \rangle = \langle Y, X \rangle$ and that $\|X - Y\|^2 = \langle X - Y, X - Y \rangle$ defines a norm of $\mathcal{M}$. Indeed, $\|X\| = |\underline{X}|_{n^2}$ where $| \cdot |_{n^2}$ denotes the usual Euclidean norm in $\mathbb{R}^{n^2}$ and $\underline{X} \in \mathbb{R}^{n^2}$ is as defined above.

Lastly the symbol δ, with or without subscripts, denote finite positive constants whose magnitudes depend only on A, B, H and P. Any δ, with a subscript, retains a fixed identity throughout while the unnumbered ones are not necessarily the same each time they occur.

3. STATEMENT OF RESULTS

It will be assumed throughout the sequel that $H \in \mathcal{C}(\mathcal{M})$ and that $P \in \mathcal{C}(I\!\!R \times \mathcal{M} \times \mathcal{M} \times \mathcal{M})$.

Our main result in this paper is the following, which is a matrix analogue of results in [1], [4].

Theorem 1. *Let $H(0) = 0$ and suppose that*

(i) there exists an $n \times n$ real continuous matrix $C(X, Y)$ for any $X, Y \in \mathcal{M}$ such that (2) is satisfied;

(ii) the matrices $\tilde{A}, \tilde{B}, \tilde{C}(X, Y)$ are associative and commute pairwise. The eigenvalues $\lambda_i(\tilde{A})$ of $\tilde{A}$, $\lambda_i(\tilde{B})$ of $\tilde{B}$ and $\lambda_i(\tilde{C}(X, Y))$ of $\tilde{C}(X, Y)$ $\quad (i = 1, 2, \ldots, n^2)$ satisfy

$$0 < \delta_a \leq \lambda_i(\tilde{A}) \leq \Delta_a \tag{5}$$

$$0 < \delta_b < \lambda_i(\tilde{B}) \leq \Delta_b \tag{6}$$

$$0 < \delta_c < \lambda_i(\tilde{C}(X, Y)) \leq \Delta_c \tag{7}$$

where $\delta_a, \delta_b, \delta_c, \Delta_a, \Delta_b, \Delta_c$ are finite constants. Furthermore,

$$\Delta_c \leq k\delta_a\delta_b, \tag{8}$$

$$\text{where} \qquad k = \min\left\{\frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2}\ ;\ \frac{\alpha(1-\beta)\delta_a}{2(\delta_a+2\alpha)^2}\right\} \tag{9}$$

$\alpha > 0, 0 < \beta < 1$ are some constants,

(iii) P satisfies

$$\|P(t, X, Y, Z)\| \leq \delta_0 + \delta_1(\|X\| + \|Y\| + \|Z\|) \tag{10}$$

for arbitrary $X, Y, Z \in \mathcal{M}$, *where* $\delta_0 \geq 0, \delta_1 \geq 0$ *are constants and* δ_1 *is sufficiently small.*

Then every solution $X(t)$ *of* (1) *satisfies*

$$\|X(t)\| \leq \Delta_1, \quad \|\dot{X}(t)\| \leq \Delta_1, \quad \|\ddot{X}(t)\| \leq \Delta_1 \tag{11}$$

for all t *sufficiently large, where* Δ_1 *is a positive constant the magnitude of which depends only on* $\delta_0, \delta_1, A, B, H$ *and* P.

The condition (10) can be relaxed to

$$\|P(t, X, Y, Z)\| \leq \theta_1(t) + \theta_2(t)(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}} \tag{12}$$

where $\theta_1(t)$ and $\theta_2(t)$ are continuous functions of t satisfying

$$0 \leq \theta_1(t) < \alpha_0 \quad \text{for all } t \text{ in } I\!R \tag{13}$$

and

$$0 \leq \theta_2(t) < \alpha_1 \quad \text{for all } t \text{ in } I\!R. \tag{14}$$

It will, however, be convenient to deal first with Theorem 1 in its present form and later (see Section 6) to indicate what modification are necessary to convert the methods to the case which the matrix P satisfies (12).

We can obtain some other results on Eq. (1). A particular case which extends Corollary 1 in [1] to the case considered is the following:

Corollary 1. *Suppose that* $P = 0$ *and that the conditions (i) and (ii) of Theorem 1 above hold. Suppose further that* $H(0) = 0$, *then every solution of* (1) *satisfies*

$$\|X(t)\|^2 + \|\dot{X}(t)\|^2 + \|\ddot{X}(t)\|^2 \to 0 \tag{15}$$

as $t \to \infty$.

4. SOME PRELIMINARY RESULTS

In this section, we shall state some standard algebraic results required in the proofs.

Lemma 1. [1] *Let D be a real symmetric $\ell \times \ell$ matrix, then for any $X \in I\!R^\ell$ we have*

$$\delta_d \|X\|^2 \le \langle DX, X\rangle \le \Delta_d \|X\|^2,$$

where δ_d, Δ_d are the least and greatest eigenvalues of D, respectively.

Lemma 2. [2] *Let Q, D be any two real $\ell \times \ell$ commuting symmetric matrices. Then*

(i) the eigenvalues $\lambda_i(QD)$ $(i = 1, 2, \ldots, \ell)$ of the product matrix QD are all real and satisfy

$$\max_{i \le j,k \le \ell} \lambda_j(Q)\lambda_k(D) \ge \lambda_i(QD) \ge \min_{1 \le j,k \le \ell} \lambda_j(Q)\lambda_k(D);$$

(ii) the eigenvalues $\lambda_i(Q + D)$ $(i = 1, 2, \ldots, \ell)$ of the sum of matrices Q and D are real and satisfy

$$\left\{\max_{i \le j \le \ell} \lambda_j(Q) + \max_{1 \le k \le \ell} \lambda_k(D)\right\} \ge \lambda_i(Q + D) \ge \left\{\min_{1 \le j \le \ell} \lambda_j(Q) + \min_{1 \le k \le \ell} \lambda_k(D)\right\}.$$

5. PROOF OF RESULTS

Our main tool in the proof of the results is the scalar Lyapunov function

$$V : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to I\!R$$

adapted from [4] and defined for any function $X, Y, Z \in \mathcal{M}$ by

$$\begin{aligned} 2V \;=\; & \{\langle \beta(1-\beta)BX, BX\rangle + \langle 2\alpha A^{-1}BY, Y\rangle + \langle \beta BY, Y\rangle \\ & +\langle \alpha A^{-1}Z, Z\rangle + \langle \alpha(Z + AY), Y + A^{-1}Z\rangle \\ & \langle Z + AY + (1-\beta)BX, Z + AY + (1-\beta)BX\rangle\} \end{aligned} \tag{16}$$

where $\alpha > 0,\ \ 0 < \beta < 1$ are some constants.

Lemma 3. *Assume that all the conditions on matrices A, B and $H(X)$ in Theorem 1 are satisfied. Then, there exist positive constants δ_2 and δ_3 such that*

$$\delta_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq 2V \leq \delta_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \tag{17}$$

Proof of Lemma 3. See [6, pages 191-192].

Proof of Theorem 1

Let us for convenience replace Eq. (1) by the equivalent system of differential equation

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -AZ - BY - H(X) + P(t, X, Y, Z). \end{aligned} \tag{18}$$

To prove our results it therefore suffices to prove that

$$\|X\|^2 + \|Y\|^2 + \|Z\|^2 \leq \Delta_1$$

for any solution (X, Y, Z) of (18).

The proof of the ultimate boundedness result depends on our being able to prove that V satisfies

(i) $V(X, Y, Z) \to \infty$ as $\|X\|^2 + \|Y\|^2 + \|Z\|^2 \to \infty$ and

(ii) $\frac{dV}{dt} \leq -1$

along paths of any solution (X, Y, Z) of (18) for which $\|X\|^2 + \|Y\|^2 + \|Z\|^2$ is large enough.

Property (i) is obviously taken care by Lemma 3. Thus, we are only left to prove property (ii) for V. Let (X, Y, Z) be any solution of (18). Then, the total derivative of V with respect to t along this solution path is

$$\dot{V} = -U_1 - U_2 - U_3 + U_4 \tag{19}$$

where

$$\begin{aligned} U_1 &= \langle \frac{1-\beta}{2} BX, H(X)\rangle + \langle \beta ABY, Y\rangle + \langle \frac{\alpha}{2} Z, Z\rangle \\ U_2 &= \langle \frac{1-\beta}{2} BX, H(X)\rangle + \langle \alpha Z, Z\rangle + \langle (A+\alpha I)Y, H(X)\rangle \\ U_3 &= \langle \frac{1-\beta}{2} BX, H(X)\rangle + \langle \frac{\alpha}{2} Z, Z\rangle + \langle (I+2\alpha A^{-1})Z, H(X)\rangle \\ U_4 &= \langle (1-\beta)BX + (A+\alpha I)Y + (A+\alpha I)Y + (I+2\alpha A^{-1})Z, P(t,X,Y,Z)\rangle. \end{aligned}$$

Because of the representation of $H(X)$ as

$$H(X) = H(0) + C(X,0)X \tag{20}$$

from (2) and if $H(0) = 0$ with condition (7) satisfied, we obtain

$$\begin{aligned} \langle \frac{1-\beta}{2} BX, H(X)\rangle &= \langle \frac{1-\beta}{2} BX, C(X,0)X\rangle \\ &= \frac{1-\beta}{2} \sum_{i=1}^{n} |BC(X,0)X^i|_n^2 \\ &\geq \frac{1-\beta}{2} \delta_b \delta_c \|X\|^2, \end{aligned} \tag{21a}$$

$$\begin{aligned} \langle \beta ABY, Y\rangle &= \beta \sum_{i=1}^{n} |ABY^i|_n^2 \\ &\geq \beta \delta_a \delta_b \|Y\|^2, \end{aligned} \tag{21b}$$

and

$$\langle \frac{\alpha}{2} Z, Z\rangle = \frac{\alpha}{2} \sum_{i=1}^{n} |Z^i|_n^2 \geq \frac{\alpha}{2} \|Z\|^2. \tag{21c}$$

The estimates above are valid since $\sum_{i=1}^{n} |X^i|_n^2 = \sum_{i=1}^{n} |X_i|_n^2 = |\underline{X}|_{n^2}^2$ for any $X \in \mathcal{M}$.

Combining these estimates (21a)-(21c), we clearly have

$$\begin{aligned} U_1 &\geq \frac{1}{2}(1-\beta)\delta_b\delta_c\|X\|^2 + \beta\delta_a\delta_c\|Y\|^2 + \frac{\alpha}{2}\|Z\|^2 \\ &\geq \delta_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2), \end{aligned} \tag{22}$$

where $\delta_4 = \min \frac{1}{2}\{(1-\beta)\delta_b\delta_c,\ 2\beta\delta_a\delta_c,\ \alpha\}$.

Next, we give estimates for $\langle (A+\alpha I)Y, H(X)\rangle$ and $\langle (I+2\alpha A^{-1})Z, H(X)\rangle$.

For some $k_1 > 0$, $k_2 > 0$, conveniently chosen later, we have

$$\begin{aligned} \langle (A+\alpha I)Y, H(X)\rangle &= \|k_1(A+\alpha I)^{\frac{1}{2}}Y + 2^{-1}k_1^{-1}(A+\alpha I)^{\frac{1}{2}}H(X)\|^2 \\ &\quad -\langle k_1^2(A+\alpha I)Y, Y\rangle - 4^{-1}k_1^{-2}\langle (A+\alpha I)H(X), H(X)\rangle \end{aligned}$$

and

$$\begin{aligned}\langle (I+2\alpha A^{-1})Z, H(X)\rangle &= \|k_2(I+2\alpha A^{-1})^{\frac{1}{2}}Z + 2^{-1}k_2^{-1}(I+2\alpha A^{-1})^{\frac{1}{2}}H(X)\|^2 \\ &\quad -\langle k_2^2(I+2\alpha A^{-1})Z, Z\rangle \\ &\quad -\langle 4^{-1}k_2^{-2}(I+2\alpha A^{-1})H(X), H(X)\rangle,\end{aligned}$$

thus,

$$\begin{aligned}U_2 &= \|k_1(A+\alpha I)^{\frac{1}{2}}Y + 2^{-1}k_1^{-1}(A+\alpha I)^{\frac{1}{2}}H(X)\|^2 \\ &\quad +\langle 4^{-1}(1-\beta)BX - 4^{-1}k_1^{-2}\langle (A+\alpha I)H(X), H(X)\rangle \\ &\quad +\langle [\alpha B - k_1^2(A+\alpha I)]Y, Y\rangle,\end{aligned}$$

and

$$\begin{aligned}U_3 &= \|k_2(I+2\alpha A^{-1})^{\frac{1}{2}}Z + 2^{-1}k_2^{-1}(I+2\alpha A^{-1})^{\frac{1}{2}}H(X)\|^2 \\ &\quad +\langle 4^{-1}(1-\beta)BX - 4^{-1}k_2^{-2}(I+2\alpha A^{-1})H(X), H(X)\rangle \\ &\quad +\langle [\frac{\alpha}{2}I - k_2^2(I+2\alpha A^{-1})]Z, Z\rangle.\end{aligned}$$

By Lemmas 1 and 2, and using (20), we obtain

$$\begin{aligned}U_2 &\geq \{\underline{X}^T[4^{-1}(1-\beta)\tilde{B} - 4^{-1}k_1^{-2}(\alpha\tilde{I}+\tilde{A})\tilde{C}(X,0)]\tilde{C}(X,0)\underline{X} \\ &\quad +\underline{Y}^T[\alpha\tilde{B} - k_1^2(\alpha\tilde{I}+\tilde{A})]\underline{Y}\}\end{aligned}$$

and

$$\begin{aligned}U_3 &\geq \{\underline{X}^T[4^{-1}(1-\beta)\tilde{B} - 4^{-1}k_2^{-2}(\tilde{I}+2\alpha\tilde{A}^{-1})\tilde{C}(X,0)]\tilde{C}(X,0)\underline{X} \\ &\quad +\underline{Z}^T[\frac{\alpha}{2}\tilde{B} - k_2^2(\tilde{I}+2\alpha\tilde{A}^{-1})]\underline{Z}\}\end{aligned}$$

Furthermore, by using Lemmas 1 and 2, and (5)-(7), we obtain

$$U_3 \geq \{\frac{1}{4}\delta_c\left[(1-\beta)\delta_b - k_2^{-2}(1+2\alpha\delta_a^{-1})\Delta_c\right]\|X\|^2 + \left[\frac{\alpha}{2} - k_2^2(1+2\alpha\delta_a^{-1})\right]\|Z\|^2\}$$

Thus, we obtain, for all X, Y in $\mathcal{M}$,

$$U_2 \geq 0 \tag{23a}$$

if $k_1^2 \leq \dfrac{\alpha\delta_b}{\alpha+\Delta_a}$ with

$$\Delta_c \leq \frac{k_1^2(1-\beta)\delta_b}{(\alpha+\Delta_a)} \leq \frac{\alpha(1-\beta)\delta_b^2}{(\alpha+\Delta_a)^2}, \tag{24a}$$

and for all X, Z in $\mathcal{M}$,

$$U_3 \geq 0 \tag{23b}$$

if $k_2^2 \leq \dfrac{\alpha\delta_a}{2(2\alpha+\delta_a)}$ with

$$\Delta_c \leq \frac{k_2^2(1-\beta)\delta_a\delta_b}{(2\alpha+\delta_a)} \leq \frac{\alpha(1-\beta)\delta_a^2\delta_b^2}{2(2\alpha+\delta_a)^2}. \tag{24b}$$

Combining all the inequalities in (23) and (24), we have for all X, Y, Z in $\mathcal{M}, U_2 \geq 0$ and $U_3 \geq 0$, if

$$\Delta_c \leq k\delta_a\delta_b$$

with

$$k = \min\left\{\frac{\alpha(1-\beta)\delta_b}{\delta_a(\alpha+\Delta_a)^2}; \frac{\alpha(1-\beta)\delta_a}{2(2\alpha+\delta_a)^2}\right\} < 1.$$

Finally, we are left with U_4. Since $P(t, X, Y, Z)$ satisfies inequality (10), by Schwarz's inequality, we obtain

$$\begin{aligned} |U_4| &\leq \{(1-\beta)\Delta_b\|X\| + (\alpha+\Delta_a)\|Y\| + (1+2\alpha\delta_a^{-1})\|Z\|\}\|P(t,X,Y,Z)\| \\ &\leq \delta_5(\|X\|+\|Y\|+\|Z\|)[\delta_0+\delta_1(\|X\|+\|Y\|+\|Z\|)] \\ &\leq 3\delta_1\delta_5(\|X\|^2+\|Y\|^2+\|Z\|^2) + 3^{\frac{1}{2}}\delta_0\delta_5(\|X\|^2+\|Y\|^2+\|Z\|^2)^{\frac{1}{2}}, \end{aligned} \tag{25}$$

where $\delta_5 = \max\{(1-\beta)\Delta_b;\ \alpha+\Delta_a;\ 1+2\alpha\delta_a^{-1}\}$.
Combining inequalities (22), (23) and (25) in (19), we obtain

$$\dot{V} \leq -2\delta_6(\|X\|^2+\|Y\|^2+\|Z\|^2) + \delta_7(\|X\|^2+\|Y\|^2+\|Z\|^2)^{\frac{1}{2}}, \tag{26}$$

where $\delta_6 = \frac{1}{2}(\delta_4 - 3\delta_1\delta_5)$, $\delta_1 < 3^{-1}\delta_5^{-1}\delta_4$, $\delta_7 = 3^{\frac{1}{2}}\delta_0\delta_5$.
If we choose $(\|X\|^2+\|Y\|^2+\|Z\|^2)^{\frac{1}{2}} \geq \delta_8 = \delta_7\delta_6^{-1}$, inequality (26) implies that

$$\dot{V} \leq -\delta_6(\|X\|^2+\|Y\|^2+\|Z\|^2). \tag{27}$$

Then, there exists δ_9 such that

$$\dot{V} \leq -1 \quad \text{if} \quad \|X\|^2+\|Y\|^2+\|Z\|^2 \geq \delta_9^2.$$

The remainder of the proof of Theorem 1 may now be obtained by the use of the estimates (17) and (27) and an adaptation of the Yoshizawa [9] type reasoning employed in [4]. □

6. THE OTHER FORM OF P

We can now turn to the case mentioned in Section 3, in which the matrix P satisfies inequality (12) instead of (10). The proof of our result in this case follows the lines indicated in Section 5 above, except for some minor modifications. The main modification occurs in our estimate for $|U_4|$ defined in (19). If matrix $P(t, X, Y, Z)$ satisfies inequality (12), then

$$\begin{aligned} |U_4| &\leq \delta_{10}(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}\|P(t, X, Y, Z)\| \\ &\leq \delta_{10}\left\{\theta_2(t)(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \theta_1(t)(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}\right\}, \end{aligned}$$

where

$$\delta_{10} = 3^{\frac{1}{2}} \max\{(1-\beta)\Delta_b; \alpha + \Delta_a; 1 + 2\alpha\delta_a^{-1}\}.$$

Now, by (13), $\delta_{10}\theta_1(t) < \delta_{10}\alpha_0$ and by (14), $\delta_{10}\theta_2(t) < \delta_{10}\alpha_1$ for all t in $I\!R$. Thus, we have

$$\dot{V} \leq -2\delta_{11}(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_{12}(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}},$$

where $\delta_{11} = \frac{1}{2}(\delta_4 - \delta_{10}\alpha_1)$, $\alpha_1 < \delta_4\delta_{10}^{-1}$ and $\delta_{12} = \delta_{10}\alpha_0$. Following the procedure indicated in Section 5, we then conclude that $\dot{V} \leq -1$ for $(\|X\|^2+\|Y\|^2+\|Z\|^2)^{\frac{1}{2}} \geq \delta_{13}$.

7. PROOF OF COROLLARY 1

If $P = 0$, then in the proof of Theorem 1, $U_4 = 0$ and if hypotheses *(i)* and *(ii)* of Theorem 1 hold then we have

$$\dot{V} \leq -\delta V(t),$$

for some constant $\delta > 0$. By integrating and with the aid of inequalities (17), we can easily conclude that (15) is valid as $t \to \infty$. This completes the proof of Corollary 1. □

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