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# BIVARIATE EXTENDED CONFLUENT HYPERGEOMETRIC FUNCTION DISTRIBUTION

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**SYNOPTIC ABSTRACT**

*In this article, we define a bivariate extended confluent hypergeometric function density in terms of extended confluent hypergeometric function. We also derive several of its properties and results in terms of extended beta, extended confluent hypergeometric, and modified Bessel functions.*

**Key Words and Phrases:** beta distribution, bivariate distribution, extended beta function, extended confluent hypergeometric function, quotient, product, Gauss hypergeometric function.

## 1. Introduction

A random variable $X$ is said to have an extended confluent hypergeometric function kind 1 distribution with parameters $(\nu, \alpha, \beta, \sigma)$, denoted by $X \sim \mathrm{ECH}(\nu, \alpha, \beta, \sigma, \text{kind } 1)$, if its probability density function (PDF) is given by (Nagar, Morán-Vásquez, & Gupta, 2012),

$$\frac{B(\alpha, \beta-\alpha)}{\Gamma(\nu) B(\alpha-\nu, \beta-\alpha; \sigma)} x^{\nu-1} \Phi_\sigma(\alpha; \beta; -x), \quad x > 0, \tag{1}$$

where $\nu > 0$, $\beta > \alpha > 0$ if $\sigma > 0$, and $\beta > \nu > 0$, $\alpha > \nu > 0$ if $\sigma = 0$. Further, $B(a, b)$ is the beta function defined by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0,$$

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and $\Phi_\sigma$ is the extended confluent hypergeometric function (Chaudhry, Qadir, Srivastava, & Paris, 2004) defined as

$$\Phi_\sigma(b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \times \exp\left[zt - \frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{2}$$

where $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\sigma \geq 0$. By taking $\sigma = 0$ in (2), one obtains $\Phi_0(b; c; z) = \Phi(b; c; z)$, which means that the classical confluent hypergeometric function (Gradshteyn & Ryzhik, 2007, Sec. 9.21) is a special case of the extended confluent hypergeometric function. Further, the extended beta function $B(b, c; \sigma)$ used in (1) is defined as (Chaudhry, Qadir, Rafique, & Zubair, 1997),

$$B(b, c; \sigma) = \int_0^1 t^{b-1}(1-t)^{c-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{3}$$

where $\sigma > 0$, and if $\sigma = 0$, then we must have $\mathrm{Re}(b) > 0$ and $\mathrm{Re}(c) > 0$.

The extended confluent hypergeometric function kind 1 distribution occurs as the distribution of the ratio of independent gamma and extended beta variables (Nagar et al., 2012). For $\sigma = 0$, the extended confluent hypergeometric function kind 1 distribution slides to a confluent hypergeometric function kind 1 distribution (Gupta & Nagar, 2000). For $\sigma = 0$ and $\alpha = \beta$, the density (1) reduces to a standard gamma density with shape parameter $\nu$ given by

$$\{\Gamma(\nu)\}^{-1} x^{\nu-1} \exp(-x), \quad x > 0,$$

and in this case we will write $X \sim \mathrm{Ga}(\nu)$. The gamma distribution has been used to model amounts of daily rainfall (Aksoy, 2000). In neuroscience, the gamma distribution is often used to describe the distribution of interspike intervals (Robson & Troy, 1987). The gamma distribution is widely used as a conjugate prior in Bayesian statistics. It is the conjugate prior for the precision (i.e., inverse of the variance) of a normal distribution. It is also the conjugate prior for the exponential distribution. It is, therefore, reasonable to say that the extended confluent hypergeometric function kind 1 distribution, which is a generalization of the gamma distribution, can be used as an alternative to gamma quite effectively in analyzing many lifetime data. In the same vein, we can say that the bivariate generalization of the extended confluent hypergeometric function kind 1 distribution proposed in this article can also serve as an alternative to many bivariate gamma distributions. The bivariate gamma distributions have applications in several areas; for example, in the modeling of rainfall using two nearby rain gauges, data obtained from rainmaking experiments, the dependence between annual streamflow and areal precipitation, wind-gust data, and the dependence between rainfall and runoff. These distributions have also been applied in reliability theory, renewal processes, and stochastic routing problems. A comprehensive account of some applications and other aspects of these distributions can be found in Balakrishnan and Lai (2009), Nadarajah (2005), Nadarajah and Kotz (2006), and Nadarajah and Gupta (2006).

The bivariate generalization of the extended confluent hypergeometric function kind 1 distribution, denoted by $(X_1, X_2) \sim \text{ECH}(\nu_1, \nu_2, \alpha, \beta, \sigma, \text{kind } 1)$, is defined by the density

$$\frac{B(\alpha, \beta - \alpha)}{\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha - \nu_1 - \nu_2, \beta - \alpha; \sigma)} \times x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} \Phi_\sigma(\alpha; \beta; -x_1 - x_2), \tag{4}$$

where $x_1 > 0$, $x_2 > 0$; $\nu_1 > 0$, $\nu_2 > 0$, and $\beta > \alpha > 0$ if $\sigma > 0$; $\nu_1 > 0$, $\nu_2 > 0$, $\beta > \nu_1 + \nu_2$, $\alpha > \nu_1 + \nu_2$ if $\sigma = 0$. By using Kummer's first formula for the extended confluent hypergeometric function (Chaudhry et al., 2004, Equation (6.3)), the density (4) can also be written as

$$\frac{B(\alpha, \beta - \alpha)}{\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha - \nu_1 - \nu_2, \beta - \alpha; \sigma)} x_1^{\nu_1 - 1} x_2^{\nu_2 - 1} \exp[-(x_1 + x_2)] \times \Phi_\sigma(\beta - \alpha; \beta; x_1 + x_2),$$

where $x_1 > 0$ and $x_2 > 0$. For $\sigma = 0$, (4) reduces to a bivariate confluent hypergeometric function kind 1 distribution given by the density

$$\frac{\Gamma(\alpha)\Gamma(\beta-\nu_1-\nu_2)}{\Gamma(\nu_1)\Gamma(\nu_2)\Gamma(\beta)\Gamma(\alpha-\nu_1-\nu_2)} x_1^{\nu_1-1} x_2^{\nu_2-1} \Phi(\alpha;\beta;-x_1-x_2),$$
$$x_1 > 0, \quad x_2 > 0,$$

where $\nu_1 > 0$, $\nu_2 > 0$, $\alpha > \nu_1 + \nu_2$, and $\beta > \nu_1 + \nu_2$. For $\sigma = 0$ and $\alpha = \beta$, the random variables $X_1$ and $X_2$ are independent, $X_1 \sim \text{Ga}(\nu_1)$ and $X_2 \sim \text{Ga}(\nu_2)$.

In this article, we study several properties such as marginal distributions, cumulative distribution function, and measure of the component reliability of the bivariate distribution defined by (4). We also derive distributions of sum, product, and quotients of the components $X_1$ and $X_2$.

## 2. Properties

In this section, we study several properties of the bivariate extended confluent hypergeometric function kind 1 distribution defined in Section 1. First, we define beta type 1, beta type 2, and extended beta type 1 distributions. These definitions can be found in Johnson, Kotz, and Balakrishnan (1995), Chaudhry et al. (1997), and Nagar et al. (2012).

**Definition 2.1** A random variable $X$ is said to have a beta type 1 distribution with parameters $(a, b)$, $a > 0$, $b > 0$, denoted as $X \sim \text{B1}(a, b)$, if its PDF is given by

$$\{B(a,b)\}^{-1} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1.$$

**Definition 2.2** A random variable $X$ is said to have a beta type 2 distribution with parameters $(a, b)$, denoted as $X \sim \text{B2}(a, b)$, $a > 0$, $b > 0$, if its PDF is given by

$$\{B(a,b)\}^{-1} x^{a-1}(1+x)^{-(a+b)}, \quad x > 0.$$

**Definition 2.3** A random variable $X$ is said to have an extended beta type 1 distribution, denoted by $X \sim \text{EB1}(a, b; \lambda)$, if its PDF is given by

$$\{B(a, b; \lambda)\}^{-1} x^{a-1} (1-x)^{b-1} \exp\left[-\frac{\lambda}{x(1-x)}\right], \quad 0 < x < 1,$$

where $B(a, b; \lambda)$ is the extended beta function defined by (3), $\lambda > 0$ and $-\infty < a, b < \infty$.

For $\lambda = 0$ and $a > 0$ $b > 0$, the extended beta type 1 density defined in Definition 2.3 reduces to a beta type 1 density.

Now, we derive marginal distributions.

**Theorem 2.1** *Let* $(X_1, X_2) \sim \mathrm{ECH}(\nu_1, \nu_2, \alpha, \beta, \sigma, \text{kind } 1)$. *Then,* $X_1 \sim \mathrm{ECH}(\nu_1, \alpha - \nu_2, \beta - \nu_2, \sigma, \text{ kind } 1)$ *and* $X_2 \sim \mathrm{ECH}(\nu_2, \alpha - \nu_1, \beta - \nu_1, \sigma, \text{kind } 1)$.

**Proof.** Replacing $\Phi_\sigma(\alpha; \beta; -x_1 - x_2)$ by its equivalent integral representation in (4) and integrating $x_2$ in the resulting expression by applying (2), the marginal density of $X_1$ is derived. □

The cumulative distribution function (CDF) of $(X_1, X_2)$ is derived as

$$\begin{aligned} F_{X_1,X_2}(x_1, x_2) &= \frac{B(\alpha, \beta-\alpha)}{\Gamma(\nu_1)\Gamma(\nu_2) B(\alpha-\nu_1-\nu_2, \beta-\alpha; \sigma)} \\ &\quad \times \int_0^{x_1}\int_0^{x_2} u_1^{\nu_1-1} u_2^{\nu_2-1} \Phi_\sigma(\alpha; \beta; -u_1-u_2)\, \mathrm{d}u_1\, \mathrm{d}u_2. \\ &= \frac{x_1^{\nu_1} x_2^{\nu_2}}{\Gamma(\nu_1+1)\Gamma(\nu_2+1) B(\alpha-\nu_1-\nu_2, \beta-\alpha; \sigma)} \\ &\quad \times \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1} \\ &\quad \times \exp\left[-\frac{\sigma}{t(1-t)}\right] \Phi(\nu_1; \nu_1+1; -x_1 t) \\ &\quad \times \Phi(\nu_2; \nu_2+1; -x_2 t)\, \mathrm{d}t, \end{aligned}$$

where the last line has been obtained by replacing $\Phi_\sigma(\alpha; \beta; -u_1 - u_2)$ by its integral representation and integrating out $u_1$ and $u_2$. Now, expanding $\Phi(\nu_1; \nu_1+1; -x_1 t)\Phi(\nu_2; \nu_2+1; -x_2 t)$ in the series form using Gradshteyn and Ryzhik (2007, Equation (9.210.1)) and integrating $t$, we arrive at

$$F_{X_1,X_2}(x_1, x_2) = \frac{x_1^{\nu_1} x_2^{\nu_2}}{\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha-\nu_1-\nu_2, \beta-\alpha;\sigma)} \times \sum_{j_1, j_2=0}^{\infty} \frac{x_1^{j_1} x_2^{j_2} (-1)^{j_1+j_2}}{(\nu_1+j_1)(\nu_2+j_2)\, j_1! j_2!} \times B(\alpha+j_1+j_2, \beta-\alpha;\sigma).$$

Further, using (4), the joint $(r, s)$-th moment is obtained as

$$\mathrm{E}(X_1^r X_2^s) = \frac{\Gamma(\nu_1+r)\Gamma(\nu_2+s)}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{B(\alpha-\nu_1-\nu_2-r-s, \beta-\alpha;\sigma)}{B(\alpha-\nu_1-\nu_2, \beta-\alpha;\sigma)},$$

where $\nu_1+r>0$, $\nu_2+s>0$ if $\sigma>0$ and $\nu_1+r>0$, $\nu_2+s>0$, $\beta>\nu_1+\nu_2+r+s>0$, and $\alpha>\nu_1+\nu_2+r+s>0$ if $\sigma=0$.

If $(X_1, X_2)$ has a bivariate extended confluent hypergeometric function kind 1 distribution, then from Theorem 3.1, we have $X_1/X_2 \sim \mathrm{B2}(\nu_1, \nu_2)$ and $X_1/(X_1+X_2) \sim \mathrm{B1}(\nu_1, \nu_2)$. Further, by observing that the measure of the component reliability $R = Pr(X_1 < X_2)$ can be written as $R = Pr(X_1/(X_1+X_2) < 1/2)$ and using the beta type 1 density with parameters $\nu_1$ and $\nu_2$, one obtains $R = B_{1/2}(\nu_1, \nu_2)/B(\nu_1, \nu_2)$, where $B_z(a, b)$ is the incomplete beta function defined in terms of the Gauss hypergeometric function $F$ as $B_z(a, b) = z^a F(a, 1-b; a+1; z)/a$.

In Theorem 2.2, we derive the bivariate extended confluent hypergeometric function kind 1 distribution using independent extended beta type 1 and gamma variables.

**Theorem 2.2** *Let* $X_1, X_2$, *and* $X_3$ *be independent,* $X_i \sim \mathrm{Ga}(\kappa_i)$, $i = 1, 2$, *and* $X_3 \sim \mathrm{EB1}(a, b;\sigma)$. *Then,* $(X_1/X_3, X_2/X_3) \sim \mathrm{ECH}(\kappa_1, \kappa_2, a+\kappa_1+\kappa_2, a+b+\kappa_1+\kappa_2, \sigma, \text{kind } 1)$.

**Proof.** Using independence, the joint density of $X_1$, $X_2$, and $X_3$ is given as

$$\frac{x_1^{\kappa_1-1} x_2^{\kappa_2-1} x_3^{a-1} (1-x_3)^{b-1} \exp\left[-(x_1+x_2) - \sigma/x_3(1-x_3)\right]}{\Gamma(\kappa_1)\Gamma(\kappa_2)B(a, b;\sigma)}, \quad (5)$$

where $x_1 > 0$, $x_2 > 0$, and $0 < x_3 < 1$. Now, transforming $Z_1 = X_1/X_3$, $Z_2 = X_2/X_3$ with the Jacobian $J(x_1, x_2 \to z_1, z_2) = x_3^2$ in (5), the joint density of $Z_1$, $Z_2$, and $X_3$ is obtained as

$$\frac{z_1^{\kappa_1-1} z_2^{\kappa_2-1} x_3^{\kappa_1+\kappa_2+a-1}(1-x_3)^{b-1}\exp[-(z_1+z_2)x_3 - \sigma/x_3(1-x_3)]}{\Gamma(\kappa_1)\Gamma(\kappa_2)B(a,b;\sigma)},$$

where $z_1 > 0$, $z_2 > 0$, and $0 < x_3 < 1$. Now, the result follows by integrating $x_3$ by applying (2). □

## 3. Distributions of Sum, Product, and Quotients

In statistical distribution theory it is well known that if $X_1$ and $X_2$ are independent, $X_1 \sim \mathrm{Ga}(\nu_1)$, and $X_2 \sim \mathrm{Ga}(\nu_2)$, then

- $X_1/X_2 \sim \mathrm{B2}(\nu_1, \nu_2)$,
- $X_1/(X_1 + X_2) \sim \mathrm{B1}(\nu_1, \nu_2)$,
- $X_1 + X_2 \sim \mathrm{Ga}(\nu_1 + \nu_2)$, and
- $2\sqrt{X_1 X_2} \sim \mathrm{Ga}(2\nu_1)$ if $\nu_2 = \nu_1 + 1/2$.

In this section, we derive similar results when the joint density of $X_1$ and $X_2$ is given by (4).

**Theorem 3.1** *Let* $(X_1, X_2) \sim \mathrm{ECH}(\nu_1, \nu_2, \alpha, \beta, \sigma, \text{kind } 1)$. *Then,* $X_1/(X_1+X_2)$, *and* $X_1+X_2$ *are independent,* $X_1/(X_1+X_2) \sim \mathrm{B1}(\nu_1, \nu_2)$, *and* $X_1 + X_2 \sim \mathrm{ECH}(\nu_1 + \nu_2, \alpha, \beta, \sigma, \text{kind } 1)$. *Further,* $X_1/X_2 \sim \mathrm{B2}(\nu_1, \nu_2)$ *and is independent of* $X_1 + X_2$.

**Proof.** Substituting $Z = X_1/(X_1 + X_2)$ and $S = X_1 + X_2$ with the Jacobian $J(x_1, x_2 \to z, s) = s$ in (4), we obtain the joint PDF of $Z$ and $S$ as

$$\frac{B(\alpha, \beta-\alpha)}{\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha-\nu_1-\nu_2, \beta-\alpha;\sigma)} \times z^{\nu_1-1}(1-z)^{\nu_2-1}s^{\nu_1+\nu_2-1}\Phi_\sigma(\alpha;\beta;-s),$$

where $0 < z < 1$ and $s > 0$. Now, from the above factorization it is clear that $Z$ and $S$ are independent, $Z \sim \mathrm{B1}(\nu_1, \nu_2)$, and $S \sim \mathrm{ECH}(\nu_1 + \nu_2, \alpha, \beta, \sigma, \text{kind } 1)$. □

**Theorem 3.2** *Let* $(X_1, X_2) \sim \mathrm{ECH}(\nu_1, \nu_2, \alpha, \beta, \sigma, \text{kind } 1)$. *Then, the PDF of* $Y = 2\sqrt{X_1 X_2}$ *is given by*

$$\frac{2\,(y/2)^{\nu_1+\nu_2-1}}{\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha-\nu_1-\nu_2, \beta-\alpha;\sigma)} \times \int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left[-\frac{\sigma}{t(1-t)}\right]K_{\nu_1-\nu_2}(yt)\,\mathrm{d}t, \quad (6)$$

*where* $y > 0$ *and* $K_\nu$ *is the modified Bessel function of the second kind. Further, if* $\nu_1 = \nu$ *and* $\nu_2 = \nu+n+1/2$, *where* n *is a nonnegative integer, then the pdf of* Y *simplifies to*

$$\frac{\sqrt{\pi}\,\Gamma(\beta-\alpha)\,(y/2)^{2\nu+n-1}}{\Gamma(\nu)\Gamma(\nu+n+1/2)B(\alpha-2\nu-n-1/2, \beta-\alpha;\sigma)} \times \sum_{m=0}^{n}(2y)^{-m}\frac{\Gamma(n+m+1)\Gamma(\alpha-m-1/2)}{\Gamma(n-m+1)\Gamma(\beta-m-1/2)\,m!} \times \Phi_\sigma\left(\alpha-m-\frac{1}{2};\beta-m-\frac{1}{2};-y\right), \quad y > 0.$$

*Furthermore, if* $n = 0$, *then* $2\sqrt{X_1 X_2} \sim \mathrm{ECH}(2\nu, \alpha - 1/2, \beta - 1/2, \sigma, \text{kind } 1)$.

**Proof.** Transforming $Y = 2\sqrt{X_1 X_2}$ and $X_1 = X_1$ with the Jacobian $J(x_1, x_2 \to x_1, y) = y/2x_1$ in (4) and integrating $x_1$, we obtain the marginal PDF of $Y$ as

$$\frac{B(\alpha, \beta-\alpha)\,y^{2\nu_2-1}}{2^{2\nu_2-1}\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha-\nu_1-\nu_2, \beta-\alpha;\sigma)} \times \int_0^\infty x_1^{\nu_1-\nu_2-1}\Phi_\sigma\left(\alpha;\beta;-x_1-\frac{y^2}{4x_1}\right)\mathrm{d}x_1.$$

Replacing $\Phi_\sigma\left(\alpha;\beta;-x_1-y^2/4x_1\right)$ by its equivalent integral representation and changing the order of integration, the previous density is rewritten as

$$\frac{(y/2)^{2\nu_2-1}}{\Gamma(\nu_1)\Gamma(\nu_2)B(\alpha-\nu_1-\nu_2,\beta-\alpha;\sigma)}$$
$$\times\int_0^1 t^{\alpha-1}(1-t)^{\beta-\alpha-1}\exp\left[-\frac{\sigma}{t(1-t)}\right]$$
$$\times\int_0^\infty x_1^{\nu_1-\nu_2-1}\exp\left[-\left(x_1+\frac{y^2}{4x_1}\right)t\right]\,\mathrm{d}x_1\,\mathrm{d}t.$$

Now, using the integral (Gradshteyn and Ryzhik (2007, Equation (3.471.9))

$$\int_0^\infty \exp\left(-at-\frac{b}{t}\right)t^{\nu-1}\,\mathrm{d}t = 2\left(\frac{b}{a}\right)^{\nu/2}K_\nu(2\sqrt{ab}),$$
$$\mathrm{Re}(a)>0,\quad \mathrm{Re}(b)>0,$$

where $K_\nu \equiv K_{-\nu}$ is the modified Bessel function of the second kind, we obtain (6). To prove the second part we put $\nu_1=\nu$ and $\nu_2=\nu+n+1/2$ in (6), write $K_{n+1/2}(ty)$ in a finite series using the result

$$K_{n+1/2}(z)=\sqrt{\frac{\pi}{2z}}\exp(-z)\sum_{m=0}^{n}(2z)^{-m}\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)\,m!}$$

given in Gradshteyn and Ryzhik (2007, Equation 8.468) and integrate $t$ using (2). □

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