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On Power-Associative Nilalgebras of Nilindex and Dimension n

Sobre nilálgebras de potencia asociativa de nilíndice y dimensión n

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Abstract. We investigate the structure of commutative power-associative nilalgebras of dimension and nilindex n.

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Resumen. Investigamos la estructura de nilálgebras conmutativas de potencia asociativa de dimensión y nilíndice n.

Palabras y frases clave. Conmutatividad, potencia asociativa, nilálgebra.

1. Introduction

Commutative power-associative algebras are a natural generalization of associative, alternative and Jordan algebras. An algebra is said to be power-associative if the subalgebra generated by any element is associative. We refer the reader to the paper [1] for more information. In [2] the authors classify Jordan power-associative nilalgebras of nilindex n and dimension $n \geq 4$. In this paper we give the structure constants for power-associative nilalgebras of nilindex n and dimension $n \geq 5$.

Throughout this paper, $\mathfrak{A}$ will be a commutative power-associative nilalgebra of dimension n over a field F of characteristic $\neq 2, 3$ and 5. For every $a \in \mathfrak{A}$ we will denote by $\mathfrak{A}_a$ the subalgebra of $\mathfrak{A}$ generated by a. We define inductively

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the powers of $a \in \mathfrak{A}$ by $a^1 = a$ and $a^k = aa^{k-1}$ for $k > 1$. In a commutative power-associative algebra $\mathfrak{A}$, we have that $a^i a^j = a^{i+j}$ for every a of $\mathfrak{A}$ and all positive integers i, j and hence $\mathfrak{A}_a$ is spanned, as a vector space, by all the powers a^k with k a positive integer. We remember that in a commutative power-associative algebra, the algebra generated by all right multiplications $R_x : \mathfrak{A} \to \mathfrak{A}$, with $x \in \mathfrak{A}_a$, is in fact generated by R_a and R_{a^2}. A commutative algebra is called Engel if every right multiplication of $\mathfrak{A}$ is nilpotent. We will use the process of linearization of identities, which is an important tool in our investigation. Thus, $p(x, y, z, t) = 0$ will be the complete linearization of the fourth power-associative identity $x^4 - (x^2x^2) = 0$. Next, linearizing the identities $x^2x^3 = x(x^2x^2)$ and $x^3x^3 = (x^2)^3$ we get the following new identities

$$x^4y = 2x^3(xy) + x^2(x^2y) + 2x^2\big(x(xy)\big) - 4x\big(x^2(xy)\big), \quad (1)$$
$$x^3(x^2y) + 2x^3\big(x(xy)\big) = 2x^2\big(x^2(xy)\big) + x^4(xy). \quad (2)$$

For every positive integer $r \geq 3$, the identity $p\big(a^{r-2}, a, a, b\big) = 0$ implies the well known multiplication identity

$$R_{a^r} = \frac{1}{3}\big(8R_{a^{r-1}}R_a - 2R_aR_{a^{r-1}} + 4R_{a^2}R_{a^{r-2}} - 2R_a^2R_{a^{r-2}} - R_{a^{r-2}}R_{a^2} - 2R_aR_{a^{r-2}}R_a - 2R_{a^{r-2}}R_a^2\big). \quad (3)$$

We observe that each product in a commutative power-associative algebra $\mathfrak{A}$ with b, one time, and a, s times, can be written as $a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big)$, where $i_1, \dots, i_k$ are positive integers and $i_1 + \cdots + i_k = s$. We get the following relevant facts about the structure of a commutative power-associative algebra $\mathfrak{A}$.

Lemma 1. *Let $a, b \in \mathfrak{A}$ such that $ba \in \mathfrak{A}_a$. Then*

$$\begin{aligned} ba^3 &= -a(ba^2) + 2a^2(ba), \\ ba^4 &= a^2(ba^2), \\ a^3(ba^2) &= a^4(ba). \end{aligned} \quad (4)$$

Furthermore,

(i) *If $ba^2 \in \mathfrak{A}_a$, then $b\mathfrak{A}_a \subset \mathfrak{A}_a$ and $a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big) = a^{s-1}(ba)$ for all positive integers $k, i_1, \dots, i_k$ where $s = i_1 + \cdots + i_k \geq 5$.*

(ii) *If $ba = 0$ and $b\mathfrak{A}_a^3 \subset \mathfrak{A}_a$, then*

$$
\begin{aligned}
ba^3 &= -a(ba^2),\\
a^3(ba^2) &= 0,\\
ba^5 &= -a(ba^4) = 2a^2(ba^3),\\
ba^6 &= -a(ba^5) = a^4(ba^2) = a^2(ba^4), \quad \textit{and}\\
a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big) &= 0,
\end{aligned}
$$

for all positive integers $k, i_1, \dots, i_k$ where $i_1 + \cdots + i_k \geq 7$.

Proof. Let $a, b \in \mathfrak{A}$ such that $ab \in \mathfrak{A}_a$. From identity $p(a,a,a,b) = 0$ we get immediately $ba^3 = -a(ba^2) + 2a^2(ba)$. Setting $x = a$ and $y = b$ in (1) immediately yields relation $ba^4 = a^2(ba^2)$. Replacing x by a and y by b in (2) we get $a^3(ba^2) = a^4(ba)$.

Now we will prove (i). If $ba^2 \in \mathfrak{A}_a$, then using (3) we can prove inductively on $r \geq 3$ that there exist $\lambda_r, \mu_r \in F$ such that $\lambda_r + \mu_r = 1$ and

$$a^r b = \lambda_r a^{r-2}(ba^2) + \mu_r a^{r-1}(ba). \tag{5}$$

The cases $r = 3, 4$ are proved above. For $r > 4$, we obtain from (3) and the induction hypothesis that $ba^r = (1/3)\big(4a^{r-1}(ba) - a^{r-2}(ba^2) + 2a^2(ba^{r-2}) - 2a(ba^{r-1})\big) = (1/3)\big(4a^{r-1}(ba) - a^{r-2}(ba^2) + 2\big(\lambda_{r-2}a^{r-2}(ba^2) + \mu_{r-2}a^{r-1}(ba)\big) - 2\big(\lambda_{r-1}a^{r-2}(ba^2) + \mu_{r-1}a^{r-1}(ba)\big)\big) = (1/3)\big((-1 + 2\lambda_{r-2} - 2\lambda_{r-1})a^{r-2}(ba^2) + (4 + 2\mu_{r-2} - 2\mu_{r-1})a^{r-1}(ba)\big)$. Thus, if $ba^2 \in \mathfrak{A}_a$, then relation (5) immediately yields relation $b\mathfrak{A}_a \subset \mathfrak{A}_a$. If $i_k = 1$, then $a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big) = a^{s-1}(ba)$ since $ba \in \mathfrak{A}_a$ and $\mathfrak{A}_a$ is an associative algebra. If $ba^2 \in \mathfrak{A}_a$ and $i_k = 2$, then $a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big) = a^{s-2}(ba^2) = a^{s-5}\big(a^3(ba^2)\big) = a^{s-5}\big(a^4(ba)\big) = a^{s-1}(ba)$, since $ba^2 \in \mathfrak{A}_a$ and $a^3(ba^2) = a^4(ba)$. If $ba^2 \in \mathfrak{A}_a$ and $i_k \geq 3$, then we already proved that $ba^{i_k} \in \mathfrak{A}_a$ and hence

$$
\begin{aligned}
a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big) &= a^{s-i_k}(a^{i_k}b) = \\
&a^{s-i_k}\big(\lambda_{i_k}a^{i_k-2}(ba^2) + \mu_{i_k}a^{i_k-1}(ba)\big) = \\
&\lambda_{i_k}a^{s-i_k}\big(a^{i_k-2}(ba^2)\big) + \mu_{i_k}a^{s-i_k}\big(a^{i_k-1}(ba)\big) = \\
&\lambda_{i_k}a^{s-1}(ba) + \mu_{i_k}a^{s-1}(ba) = (\lambda_{i_k} + \mu_{i_k})a^{s-1}(ba) = a^{s-1}(ba).
\end{aligned}
$$

For (ii), we will assume in what follows that $ba = 0$ and $b\mathfrak{A}_a^3 \subset \mathfrak{A}_a$, that is $ba = 0$ and $ba^k \in \mathfrak{A}_a$ for all positive integers $k \geq 3$. Using (4) we get $ba^3 = -a(ba^2)$ and $a^3(ba^2) = 0$. Now

$$
\begin{aligned}
0 &= p(a,a,a,ba^2)/6 = \\
&a^3(ba^2) + a\big(a^2(ba^2)\big) + 2a\big(a\big(a(ba^2)\big)\big) - 4a^2\big(a(ba^2)\big) = \\
&a(ba^4) - 2a\big(a(ba^3)\big) + 4a^2(ba^3) = a(ba^4) + 2a^2(ba^3)
\end{aligned}
$$

so that $a(ba^4) = -2a^2(ba^3)$. Next, relation (3) for $r = 5$ forces $ba^5 = (1/3)\big(-2a(ba^4) + 4a^2(ba^3) - 2a\big(a(ba^3)\big)\big) = (1/3)\big(-2a(ba^4) + 2a^2(ba^3)\big) = (1/3)\big(-3a(ba^4)\big) = -a(ba^4)$. Setting $x = a$ and $y = ba^2$ in (1) immediately yields relation $a^4(ba^2) = a^2\big(a^2(ba^2)\big)$ and now using second identity of (4) we get $a^4(ba^2) = a^2(ba^4)$. Thus,

$$0 = p(a^2, a^2, a^2, b)/6 = ba^6 + a^2(ba^4) + 2a^2\big(a^2(ba^2)\big) - 4a^4(ba^2) = ba^6 - a^4(ba^2).$$

Now, $a(ba^5) = -a(a(ba^4) = -a^2(ba^4) = -ba^6$.

Taking $x = a$ and $y = ba^2$ in identity (2) we get

$$\begin{aligned} 0 = a^3(ba^4) + 2a^3\big(a\big(a(ba^2)\big)\big) - 2a^2\big(a^2\big(a(ba^2)\big)\big) - a^4\big(a(ba^2)\big) = \\ a^3(ba^4) - a^4\big(a(ba^2)\big) = a\big(a^2(ba^4)\big) + a^4(ba^3) = \\ a(ba^6) + a^4(ba^3) = a(ba^6) + a\big(a\big(a^2(ba^3)\big)\big) = \\ a(ba^6) + a\big(a(ba^5)\big)/2 = a(ba^6) - a(ba^6)/2 = a(ba^6)/2. \end{aligned}$$

Finally, we will prove that $x = a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big)$ vanishes for all $i_1, i_2, \ldots, i_k$ positive integers with $s = \sum_{l=1}^{k} i_l \geq 7$. Using (3), we can prove, by induction on s, that the element $a^{i_1}\big(a^{i_2}\big(\cdots(a^{i_k}b)\cdots\big)\big)$ is spanned by the set of all elements $a^{j_1}\big(a^{j_2}\big(\cdots(a^{j_t}b)\cdots\big)\big)$ with $j_1, \ldots, j_t \in \{1, 2\}$ and $j_1 + \cdots + j_t = s$. Thus, we can assume, without loss of generality, that $i_1, \ldots, i_k \in \{1, 2\}$. If $i_k = 1$, then $x = 0$ since $ba = 0$. If $i_k = 2$ and $i_{k-1} = 1$, then $x = -a^{i_1}\big(a^{i_2}\big(\cdots\big(a^{i_{k-2}}(ba^3)\big)\cdots\big)\big) = -a^{s-7}\big(a\big(a\big(a^2(ba^3)\big)\big)\big) = a^{k-7}\big(a(ba^6)\big)/2 = 0$ since $ba^3 \in \mathfrak{A}_a$. If $i_k = i_{k-1} = 2$, then $x = a^{i_1}\big(a^{i_2}\big(\cdots\big(a^{i_{k-2}}(ba^4)\big)\cdots\big)\big) = a^{s-7}\big(a\big(a^2(ba^4)\big)\big) = a^{s-7}\big(a(ba^6)\big) = 0$. This complete the proof of the lemma. ☑

2. Nilindex n

Throughout this section, $\mathfrak{A}$ will be a commutative power-associative nilalgebra of dimension and nilindex n. Let a be an element in $\mathfrak{A}$ with maximal nilindex. It is well known that $\mathfrak{A}^k = \mathfrak{A}_a^k$, for all $k \geq 2$ (see [2]). Hence

$$\mathfrak{A}\mathfrak{A}_a^j \subset \mathfrak{A}_a^{j+1}, \tag{6}$$

for all $j \geq 1$. Furthermore, $\mathfrak{A}^n = \mathfrak{A}_a^n = 0$ and for each $x \in \mathfrak{A}$, the power x^{n-1} is in the annihilator of $\mathfrak{A}$.

For a finite list $S = \{a_1, \ldots, a_n\}$ we write $\langle a_1, \ldots, a_n\rangle$ for the subspace consisting of all the linear combinations of elements of S.

Lemma 2. *Let a be an element in $\mathfrak{A}$ with maximal nilindex and k an integer with $1 \leq k \leq n-1$. Then there exists $b_k \in \mathfrak{A} \setminus \mathfrak{A}_a$ such that $b_k a^k = 0$. The annihilator of a^k in $\mathfrak{A}$ is $\big\langle b_k, a^{n-k}, a^{n-k+1}, \ldots, a^{n-1}\big\rangle$.*

Proof. Take $b \in \mathfrak{A} \setminus \mathfrak{A}_a$. Then $\{b, a, a^2, \dots, a^{n-1}\}$ is a basis of $\mathfrak{A}$. By the above lemma, $ba^k \in \mathfrak{A}_a^{k+1}$, so that $ba^k = \lambda_{k+1}a^{k+1} + \cdots + \lambda_{n-1}a^{n-1}$, for $\lambda_{k+1}, \dots, \lambda_{n-1}$ in F. Then $b_k a^k = 0$ for $b_k = b - \lambda_{k+1}a - \cdots - \lambda_{n-1}a^{n-k-1}$.

Finally, let $x = \xi_0 b_k + \xi_1 a + \xi_2 a^2 + \cdots + \xi_{n-1}a^{n-1}$ be an arbitrary element in $\mathfrak{A}$. Then $xa^k = \xi_0 b_k a^k + \xi_1 a^{k+1} + \xi_2 a^{k+2} + \cdots + \xi_{n-k-1}a^{n-1} = \xi_1 a^{k+1} + \xi_2 a^{k+2} + \cdots + \xi_{n-k-1}a^{n-1}$. This proves the lemma. ☑

Corollary 3. *Let $a \in \mathfrak{A}$ be an element in $\mathfrak{A}$ with maximal nilindex. Then there exists $b \in \mathfrak{A} \setminus \mathfrak{A}_a$ such that $ba \in \langle a^2 \rangle$ and $a^{n-2}b = 0$. Furthermore, an element $c \in \mathfrak{A}$ satisfies $ca \in \langle a^2 \rangle$ and $ca^{n-2} = 0$ if and only if $c \in \langle b, a^{n-1} \rangle$.*

We will denote by $\mathcal{P}(\mathfrak{A})$ the set of ordered pairs (a, b) of elements in $\mathfrak{A}$ where a has maximal nilindex, and $b \in \mathfrak{A} \setminus \mathfrak{A}_a$ with $ba \in \langle a^2 \rangle$ and $ba^{n-2} = 0$. By definition and relation (6), we have that

$$\begin{aligned} b^2 &\in \mathfrak{A}^2, \\ ba &= \lambda a^2, \\ ba^k &\subset \langle a^{k+1}, \dots, a^{n-1} \rangle = \mathfrak{A}_a^{k+1} \quad \text{for} \quad k = 2, \dots, n-3, \\ ba^{n-2} &= ba^{n-1} = 0, \end{aligned}$$

for any $(a, b) \in \mathcal{P}(\mathfrak{A})$.

For commutative power-associative nilalgebras of dimension 3 and nilindex 3, we have one family of algebras $A(\alpha) = \langle b, a, a^2 \rangle$, with $b^2 = \alpha a^2$, $ba = 0$, parametrized by $F/(F^*)^2$, that is $A(\alpha)$ is isomorphic to $A(\alpha')$ if and only if there exists $\gamma \in F^*$ such that $\alpha' = \gamma^2\alpha$. We denote $F \setminus \{0\}$ by F^*.

M. Gerstenhaber and H. C. Myung [4] showed that commutative, power-associative nilalgebras of dimension 4 over fields of characteristic $\neq 2$ are nilpotent and determined the isomorphic classes. They found one family of algebras parametrized by $F/(F^*)^2$ and four individual algebras.

Theorem 4. *If $\mathfrak{A}$ is a commutative power-associative nilalgebra over F with dimension and nilindex 4, then $\mathfrak{A}$ has a pair $(a, b) \in \mathcal{P}(\mathfrak{A})$ where the nontrivial and nonzero product belong to one and only one of the list below:*

$$\begin{array}{rll} A_1(\alpha) : b^2 = \alpha a^2 & & (\alpha \in F) \\ A_2 : b^2 = a^3 & & \\ A_3 : & & ba = a^2 \\ A_4 : b^2 = a^3 & & ba = a^2 \\ A_5 : b^2 = a^2 & & ba = a^2 \end{array}$$

where $A_1(\alpha)$ is isomorphic to $A_1(\alpha')$ if and only if there exists $\gamma \in F^*$ such that $\alpha' = \gamma^2\alpha$.

A description of commutative power-associative nilalgebras of dimension 5 was given by I. Correa and A. Suazo in [2] in the Jordan case, and by L. Elgueta and A. Suazo in [3] for algebras that are not Jordan.

Lemma 5. *If $\mathfrak{A}$ is a commutative power-associative nilalgebra over the field F with dimension and nilindex 5 and $(a,b) \in \mathcal{P}(\mathcal{A})$, then $b^2 \in \mathfrak{A}_a^3$ and $ba^2 - 2a(ba) \in \mathfrak{A}_a^4$.*

In Theorem 6, we will show a classification of such algebras without proof.

Theorem 6. *If $\mathfrak{A}$ is a commutative power-associative nilalgebra of dimension and nilindex 5, then $\mathfrak{A}$ has a basis $\{b, a, a^2, a^3, a^4\}$ with $(a,b) \in \mathcal{P}(\mathcal{A})$, and the other nonzero products belong to one and only one of the types listed below.*

$$\begin{array}{llll}
A_1(\alpha) : b^2 = a^3 + \alpha a^4, & ba = a^2, & ba^2 = 2a^3, & (\alpha \in F), \\
A_2(\alpha) : b^2 = \alpha a^4, & ba = a^2, & ba^2 = 2a^3, & (\alpha \in F), \\
A_3(\alpha) : b^2 = \alpha a^4, & & ba^2 = a^4, & (\alpha \in F), \\
A_4(\alpha) : b^2 = \alpha a^4, & & & (\alpha \in F), \\
A_5 : b^2 = a^3, & & ba^2 = a^4, & \\
A_6 : b^2 = a^3. & & &
\end{array}$$

Furthermore, we have the following conditions for two algebras in such a class to be isomorphic. For $i \in \{2,4\}$ we have that $A_i(\alpha) \cong A_i(\alpha')$ if and only if there exists $\gamma \in F^*$ such that $\alpha' = \gamma^2 \alpha$. Next, $A_3(\alpha) \cong A_3(\alpha')$ if and only if $\alpha = \alpha'$. Finally, we have that $A_1(\alpha) \cong A_1(\alpha')$ if and only if there exists $\gamma \in F^*$ such that

$$\alpha' = \frac{16\alpha - \gamma^4 + 1}{16\gamma^4}.$$

We observe that the algebras $A_4(\alpha)$ are associative. The algebras $A_3(\alpha)$, A_5 and A_6 are Jordan and are not associative. On the other hand, the algebras $A_1(\alpha)$ and $A_2(\alpha)$ are not Jordan.

Lemma 7. *Let $\mathfrak{A}$ be a commutative power-associative nilalgebra over the field F with dimension and nilindex 6. Take $(a,b) \in \mathcal{P}(\mathfrak{A})$. Then there exist scalars $\alpha, \beta, \lambda, \lambda_1, \lambda_2 \in F$ such that*

$$\begin{aligned}
b^2 &= \alpha a^4 + \beta a^5, \\
ba &= \lambda a^2, \\
ba^2 &= \lambda_1 a^4 + \lambda_2 a^5, \\
ba^3 &= 2\lambda a^4 - \lambda_1 a^5.
\end{aligned} \tag{7}$$

Reciprocally, if $\alpha, \beta, \lambda, \lambda_1$ and λ_2 are scalars and $\mathfrak{B}$ is a commutative algebra with basis $\{b, a, a^2, a^3, a^4, a^5\}$ and products $ba^4 = ba^5 = a^k = 0$, $a^i a^j = a^{i+j}$ for *all $k \geq 6$ and for all positive integers i, j and (7), then $\mathfrak{B}$ is a power-associative nilalgebra of dimension and nilindex 6. Furthermore, $\mathfrak{A}$ is Jordan if and only if $\lambda = 0 = \lambda_1$.*

Proof. By (6), we know that $b^2 \in \mathfrak{A}^2 = \mathfrak{A}_a^2$ and $ba^k \in \mathfrak{A}^{k+1} = \mathfrak{A}_a^{k+1}$ for $k = 2, 3$.

Because $(a, b) \in \mathcal{P}(\mathcal{A})$, we have that $ba^4 = 0$ and there exists $\lambda \in F$ such that $ba = \lambda a^2$. By (4) we have that $a^2(ba^2) = ba^4 = 0$ so that $ba^2 \in \mathfrak{A}_a^4$. Let $\lambda_1, \lambda_2 \in F$ such that $ba^2 = \lambda_1 a^4 + \lambda_2 a^5$. Now Lemma 1 forces $ba^3 = -a(ba^2) + 2a^2(ba) = -a(ba^2) + 2\lambda a^4$ and hence $ba^3 = 2\lambda a^4 - \lambda_1 a^5$. Next, $0 = p(a, a, b, b)/4 = a(ab^2) + b(ba^2) + 2a\big(b(ba)\big) + 2b\big(a(ba)\big) - 4(ba)^2 - 2a^2b^2 = -a^2b^2$ so that $b^2 \in \mathfrak{A}_a^4$. This completes the proof of the first part of the lemma.

Reciprocally, let $x = \xi b + y$ be an element in $\mathfrak{B}$, where $y = \sum_{i=1}^{5} \xi_i a^i$. Then

$$\begin{aligned} &x^2 \equiv y^2 + 2\xi\xi_1\lambda a^2 \mod \langle a^4, a^5\rangle, \\ &x^3 \equiv y^3 + 2\xi_1^2\xi\lambda a^3 + \xi\xi_1\big(\xi_1\lambda_1 + 2\xi\lambda\lambda_1 + 6\xi_2\lambda\big)a^4 \mod \langle a^5\rangle, \\ &x^4 = (x^2)^2 = y^4 + 4\xi\xi_1^2\lambda(\xi_1 + \xi\lambda)a^4 + (8\xi\xi_1^2\xi_2\lambda)a^5, \\ &x^5 = \xi_1^3(\xi_1 + 2\xi\lambda)^2 a^5, \end{aligned} \tag{8}$$

and hence $\mathfrak{B}$ is a power-associative nilalgebra of nilindex 6.

Finally, we observe that $(a^2b)a - a^2(ba) = \lambda_1 a^5 - \lambda a^4$ and hence $\lambda = 0 = \lambda_1$ if $\mathfrak{A}$ is Jordan. Reciprocally, if $\lambda = 0 = \lambda_1$, then $(b - \lambda_2 a^3)A^2 = 0$ and Theorem 2.1 of [3] implies that $\mathfrak{A}$ is Jordan. This completes the proof of the lemma. ☑

Lemma 8. *Let $\mathfrak{A}$ be a commutative power-associative nilalgebra over the field F with dimension and nilindex 7. Take $(a, b) \in \mathcal{P}(\mathfrak{A})$. Then $ba = 0$ and there exist scalars $\alpha, \beta, \lambda, \lambda_1, \lambda_2 \in F$ such that*

$$\begin{array}{rclcccc} b^2 &=& \lambda^2 a^4 &+& \alpha a^5 &+& \beta a^6, \\ ba^2 &=& \lambda a^4 &+& \lambda_1 a^5 &+& \lambda_2 a^6, \\ ba^3 &=& &-& \lambda a^5 &-& \lambda_1 a^6, \\ ba^4 &=& & & & & \lambda a^6. \end{array} \tag{9}$$

Reciprocally, if $\alpha, \beta, \lambda, \lambda_1$ and λ_2 are scalars and $\mathfrak{B}$ is a commutative algebra with basis $\{b, a, a^2, a^3, a^4, a^5, a^6\}$ and products $ab = 0 = ba^5 = ba^6 = a^k$, $a^i a^j = a^{i+j}$ for all $k \geq 7$ and for all positive integers i, j and (9), then $\mathfrak{B}$ is a power-associative nilalgebra of dimension and nilindex 7.

Furthermore, $\mathfrak{A}$ is Jordan if and only if $\lambda = 0 = \lambda_1$.

Proof. Because $(a, b) \in \mathcal{P}(\mathcal{A})$ and (6), we know that $b^2 \in \mathfrak{A}^2 = \mathfrak{A}_a^2$, $ba = \lambda_0 a^2$, $ba^k \in \mathfrak{A}^{k+1} = \mathfrak{A}_a^{k+1}$ for $k = 2, 3, 4$, $ba^5 = 0$ and $ba^6 = 0$.

Combining the above relations and (i) of Lemma 1 we have that $\lambda_0 a^6 = a^4(ba) = ba^5 = 0$ so that $\lambda_0 = 0$ and hence $ba = 0$. Also, by (i) of Lemma 1 we have $a^3(ba^2) = ba^5 = 0$ and hence $ba^2 \in \mathfrak{A}_a^4$. Thus, we have $ba^2 = \lambda a^4 + \lambda_1 a^5 + \lambda_2 a^6$, for $\lambda, \lambda_1, \lambda_2 \in F$. Using (4) we have that $ba^3 = -a(ba^2) + a^2(ba) = -a(ba^2) = -\lambda a^5 - \lambda_1 a^6$ and $ba^4 = a^2(ba^2) = \lambda a^6$. Now

$$\begin{aligned} 0 = p(a,a,b,b)/4 = \\ & a(ab^2) + b(ba^2) + 2a\big(b(ba)\big) + 2b\big(a(ba)\big) - 2a^2b^2 - 4(ab)^2 = \\ & \quad a(ab^2) + b(ba^2) - 2a^2b^2 = -a(ab^2) + b(ba^2) = -a(ab^2) + \lambda^2 a^6, \end{aligned}$$

since $a(ab^2) = a^2b^2$ and $b(ba^2) = b(\lambda a^4 + \lambda_1 a^5 + \lambda_2 a^6) = \lambda^2 a^6$. Thus, we have proved that $a(ab^2) = \lambda^2 a^6$. This completes the proof of the first part of the lemma.

Reciprocally, let $x = \xi b + y$ be an element in $\mathfrak{B}$, where $y = \sum_{i=1}^{6} \xi_i a^i$. Then

$$\begin{aligned} x^2 &\equiv y^2 + \xi\lambda(\xi\lambda + 2\xi_2)a^4 \mod \langle a^5, a^6 \rangle, \\ x^3 &\equiv y^3 + \xi\xi_1^2\lambda a^4 + \xi\xi_1(\xi\lambda^2 + \xi_1\lambda_1)a^5 \mod \langle a^6 \rangle, \\ x^4 &= (x^2)^2 = y^4 + 2\xi\xi_1^2\lambda(\xi\lambda + 2\xi_2)a^6, \\ x^5 &= y^5 + \xi\xi_1^4\lambda a^6, \\ x^6 &= y^6 = \xi_1^6 a^6, \end{aligned}$$

so that $\mathfrak{B}$ is a power-associative nilalgebra of nilindex 7.

Finally, if $\mathfrak{A}$ is Jordan, then $0 = (a^2b)a - a^2(ba) = (a^2b)a = \lambda a^5 + \lambda_1 a^6$, so that $\lambda = 0 = \lambda_1$. Reciprocally, if $\lambda = 0 = \lambda_1$, then $(b - \lambda_2 a^4)\mathfrak{A}^2 = 0$ and Theorem 2.1 of [3] implies that $\mathfrak{A}$ is Jordan. This proves the lemma. ☑

Theorem 9. *Let $\mathfrak{A}$ be a commutative power-associative nilalgebra over the field F with dimension and nilindex n and $n \geq 8$. Take $(a,b) \in \mathcal{P}(\mathfrak{A})$. Then*

$$\begin{aligned} ba &= 0, \\ a^2b^2 &= 0, \\ a^3(ba^2) &= 0, \\ ba^3 &= -a(ba^2), \\ ba^4 &= a^2(ba^2), \\ ba^k &= 0, \end{aligned} \tag{10}$$

for all $k \geq 5$.

Reciprocally, if $\alpha, \beta, \lambda, \lambda_1$ and λ_2 are scalars in F and $\mathfrak{B}$ is a commutative algebra with basis $\{b, a, a^2, a^3, \ldots, a^{n-1}\}$ and products $ba = 0$, $a^n = 0$, $a^i a^j = a^{i+j}$, for all positive integers i, j, and

$$
\begin{array}{rcrcrcrl}
b^2 & = & & & \alpha\, a^{n-2} & + & \beta\, a^{n-1}, & \\
ba^2 & = & \lambda\, a^{n-3} & + & \lambda_1 a^{n-2} & + & \lambda_2 a^{n-1}, & \\
ba^3 & = & & - & \lambda a^{n-2} & - & \lambda_1 a^{n-1}, & \\
ba^4 & = & & & & & \lambda a^{n-1}, & \\
ba^k & = & 0, & & & & & \forall k \geq 5,
\end{array}
\tag{11}
$$

then $\mathfrak{B}$ *is a power-associative nilalgebra of dimension and nilindex* n.

Furthermore, $\mathfrak{B}$ *is Jordan if and only if* $\lambda = 0 = \lambda_1$.

Proof. Because $(a, b) \in \mathcal{P}(\mathfrak{A})$, we know that $\mathfrak{A}^2 \subset \mathfrak{A}_a^2$, $ba = \lambda_0 a^2$, $ba^k \subset \mathfrak{A}_a^{k+1}$ for $k = 2, \ldots, n-3$, $ba^{n-2} = 0$ and $ba^{n-1} = 0$. By (i) of Lemma 1 we have that $\lambda_0 a^{n-1} = a^{n-3}(ba) = ba^{n-2} = 0$ so that $\lambda_0 = 0$ and hence $ba = 0$. Also, by (i) of Lemma 1 we have

$$
\begin{aligned}
a^3(ba^2) &= a^4(ba) = 0, \\
ba^k &= a^{k-1}(ba) = 0 \quad \text{for} \quad k \geq 5.
\end{aligned}
$$

Using identities of (4) we have that $ba^3 = -a(ba^2) + a^2(ba) = -a(ba^2)$ and $a^2(ba^2) = ba^4$. Now

$$
\begin{aligned}
0 = p(a, a, b, b)/4 = \\
a(ab^2) + b\big(b(a^2)\big) + 2a\big(b(ba)\big) + 2b\big(a(ba)\big) - 2a^2b^2 - 4(ab)^2 = \\
a(ab^2) - 2a^2b^2 = -a(ab^2)
\end{aligned}
$$

so that $a(ab^2) = 0$. This completes the proof of the first part of the lemma.

Reciprocally, let $x = \xi b + y$ be an elements in $\mathfrak{B}$, where $y = \sum_{i=1}^{n-1} \xi_i a^i$. Then

$$
\begin{aligned}
x^2 &\equiv y^2 + 2\,\xi\,\xi_2\,\lambda\, a^{n-3} \mod \big\langle a^{n-2}, a^{n-1}\big\rangle, \\
x^3 &\equiv y^3 + \xi\,\xi_1^2\big(\lambda a^{n-3} + \lambda_1 a^{n-2}\big) \mod \big\langle a^{n-1}\big\rangle, \\
x^4 &= (x^2)^2 = y^4 + 4\,\xi\,\xi_1^2\,\xi_2\,\lambda\, a^{n-1}, \\
x^5 &= y^5 + \xi\,\xi_1^4\,\lambda\, a^{n-1}, \\
x^k &= y^k \quad \text{for all} \quad k > 5, \\
x^{n-1} &= y^{n-1} = \xi_1^{n-1} a^{n-1},
\end{aligned}
$$

so that $\mathfrak{B}$ is a power-associative nilalgebra of nilindex $n-1$.

Finally, if $\mathfrak{B}$ is Jordan, then $0 = (a^2b)a - a^2(ba) = (a^2b)a = \lambda a^{n-1} + \lambda_1 a^{n-1}$, so that $\lambda = 0 = \lambda_1$. Reciprocally, if $\lambda = 0 = \lambda_1$, then $(b - \lambda_2 a^{n-3})\mathfrak{B}^2 = \{0\}$ and Theorem 2.1 of [3] implies that $\mathfrak{B}$ is Jordan. This completes the proof of the theorem. ☑

We therefore have the following result.

***Remark* 10.** Let $\mathfrak{A}$ be a commutative nilalgebra of dimension and nilindex n. Take $(a, b) \in \mathcal{P}(\mathfrak{A})$ and $\lambda \in F$ such that $ab = \lambda a^2$. If $x = \xi b + \sum_{i=1}^{n-1} \xi_i a^i$ is an element of $\mathfrak{A}$, then:

(i) for $n = 5, 6$ we have that x has nilindex n if and only if $\xi_1(\xi_1 + 2\xi\lambda) \neq 0$;

(ii) for $n \geq 7$, we have that x has nilindex n if and only if $\xi_1 \neq 0$.

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