DISTRIBUTIONS OF THE PRODUCT AND THE QUOTIENT OF INDEPENDENT KUMMER-BETA VARIABLES

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Abstract. The univariate Kummer-beta family of distributions has been proposed and studied recently by K. W. Ng and Samuel Kotz. This distribution is an univariate extension of the beta distribution. In this article, we derive the distributions of the product and the quotient of independent Kummer-beta variables.

1 Introduction. The beta type I random variable is often used for representing processes with natural lower and upper bounds. For examples, refer to Hahn and Shapiro [4]. Indeed, due to a rich variety of its density shapes, the beta distribution plays a vital role in statistical modeling. The beta distribution is useful for modeling random probabilities and proportions, particularly in the context of Bayesian analysis. Varying within $(0,1)$ the standard beta is usually taken as the prior distribution for the proportion p and forms a conjugate family within the beta prior-Bernoulli sampling scheme. Applications of the densities of the ratio and the product of independent beta variates in the field of stress-strength analysis and availability can be found in Pham-Gia [11]. A natural univariate extension of the beta distribution is the Kummer-beta distribution defined by the density function (Gupta, Cardeño and Nagar [2], Gordy [1], Nagar and Gupta [9] and Ng and Kotz [10]),

$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{x^{\alpha-1}(1-x)^{\beta-1}\exp(-\lambda x)}{{}_1F_1(\alpha;\alpha+\beta;-\lambda)}, \qquad 0<x<1, \tag{1.1}$$

where $\alpha, \beta > 0$, $-\infty < \lambda < \infty$ and ${}_1F_1$ is the confluent hypergeometric function. The Kummer-beta distribution can be seen as bimodal extension of the beta distribution (on a finite interval) and thus can help to describe real world phenomena possessing bimodal characteristics and varying within two finite limits. The Kummer-beta distribution is used in common value auctions where posterior distribution of "value of a single good" is Kummer-beta (Gordy [1]). Several generalizations and properties of the beta distribution are given in Javier and Gupta [5], McDonald and Xu [8] and Johnson, Kotz and Balakrishnana [6].

In this article, we will derive distributions of the product and the ratio of two independent random variables when at least one of them is Kummer-beta.

2 Some Definitions. In this section we will give definitions and results that will be used in the subsequent sections. The generalized hypergeometric functions of one and several variables will be used to derive the density functions of the product and the ratio of the random variables. Throughout this work we will use the Pochammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots$, and $(a)_0 = 1$.

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The generalized hypergeometric function of scalar argument is defined by

$$
{}_pF_q\left(a_1,\dots,a_p;b_1,\dots,b_q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, \tag{2.1}
$$

where a_i, $i=1,\dots,p$; b_j, $j=1,\dots,q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (2.1) are available in the literature, see Luke [7]. From (2.1) it is easy to see that ${}_0F_0(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!}=\exp(x)$, ${}_1F_0(a;x)=\sum_{k=0}^{\infty}(a)_k\frac{x^k}{k!}=(1-x)^{-a}, |x|<1$, and ${}_1F_1(a;b;x)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{x^k}{k!}$. The integral representations of the confluent hypergeometric function and the Gauss hypergeometric function are given as

$$
{}_1F_1(a;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}\exp(zt)\,dt \tag{2.2}
$$

and

$$
{}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b}\,dt \tag{2.3}
$$

respectively, where $\mathrm{Re}(a)>0$ and $\mathrm{Re}(c-a)>0$. By changing t to $1-t$ in (2.2) it is easy to see that

$$
{}_1F_1(a;c;z)=\exp(z)\;{}_1F_1(c-a;c;-z). \tag{2.4}
$$

The Humbert's confluent hypergeometric function Φ_1 is defined by

$$
\Phi_1[a,b;c;z_1,z_2]=\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\frac{(a)_{j_1+j_2}(b)_{j_1}}{(c)_{j_1+j_2}}\frac{z_1^{j_1}z_2^{j_2}}{j_1!\,j_2!},\quad |z_1|<1,\quad |z_2|<\infty. \tag{2.5}
$$

It is straightforward to show that

$$
\begin{aligned}
\Phi_1[a,b;c;z_1,z_2] &= \sum_{j_1=0}^{\infty}\frac{(a)_{j_1}(b)_{j_1}}{(c)_{j_1}}\frac{z_1^{j_1}}{j_1!}\,{}_1F_1(a+j_1;c+j_1;z_2)\\
&= \sum_{j_2=0}^{\infty}\frac{(a)_{j_2}}{(c)_{j_2}}\frac{z_2^{j_2}}{j_2!}\,{}_2F_1(a+j_2,b;c+j_2;z_1).
\end{aligned} \tag{2.6}
$$

Using the results

$$
\frac{(a)_{j_1+j_2}}{(c)_{j_1+j_2}}=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 v^{a+j_1+j_2-1}(1-v)^{c-a-1}\,dv,\quad \mathrm{Re}(a,c-a)>0,
$$

for $j_1,j_2=0,1,2,\dots,$ and

$$
\sum_{j_1=0}^{\infty}\frac{(b)_{j_1}(vz_1)^{j_1}}{j_1!}={}_1F_0(b;vz_1)=(1-vz_1)^{-b},\quad |vz_1|<1,
$$

$$
\sum_{j_2=0}^{\infty}\frac{(vz_2)^{j_2}}{j_2!}={}_0F_0(-;vz_2)=\exp(vz_2)
$$

in (2.5), one obtains

$$\Phi_1[a,b;c;z_1,z_2] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a-1}(1-v)^{c-a-1}(1-vz_1)^{-b} \exp(vz_2)\, dv \tag{2.7}$$

where $|z_1| < 1$ and $|z_2| < \infty$. For further results and properties of this function the reader is referred to Srivastava and Karlsson [12].

Finally, we define the gamma, the beta type I and the beta type II distributions. These definitions can be found in any text in mathematical statistics. First we re-define Kummer-beta distribution in a form convenient to use in the derivations of the densities of products and ratios.

Definition 2.1 *A random variable X is said to have a Kummer-beta distribution, denoted by $X \sim KB(\alpha,\beta,\lambda)$, if its p.d.f. is given by*

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}\exp[\lambda(1-x)]}{B(\alpha,\beta)_1F_1(\beta;\alpha+\beta;\lambda)}, \quad 0 < x < 1, \tag{2.8}$$

where $\alpha > 0$, $\beta > 0$, $-\infty < \lambda < \infty$ and

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{2.9}$$

Definition 2.2 *A random variable X is said to have a gamma distribution with parameters $\theta\,(>0), \kappa\,(>0)$, denoted by $X \sim Ga(\theta,\kappa)$, if its p.d.f. is given by*

$$\{\theta^{\kappa}\Gamma(\kappa)\}^{-1} x^{\kappa-1} \exp\left(-\frac{x}{\theta}\right), \quad x > 0. \tag{2.10}$$

Definition 2.3 *A random variable X is said to have a beta type I distribution with parameters (a,b), $a > 0$, $b > 0$, denoted as $X \sim B^I(a,b)$, if its p.d.f. is given by*

$$\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1.$$

Definition 2.4 *A random variable X is said to have a beta type II distribution with parameters (a,b), denoted as $X \sim B^{II}(a,b)$, $a > 0$, $b > 0$, if its p.d.f. is given by*

$$\{B(a,b)\}^{-1}x^{a-1}(1+x)^{-(a+b)}, \quad x > 0.$$

The matrix variate generalizations of the gamma, the beta type I and the beta type II distributions have been defined and studied extensively. For example, see Gupta and Nagar [3].

3 Distribution of The Product. In this section we obtain distributional results for the product of two independent random variables involving Kummer-beta distribution.

Theorem 3.1 *Let X_1 and X_2 be independent, $X_i \sim KB(\alpha_i,\beta_i,\lambda_i)$, $i = 1,2$. Then, the p.d.f of $Z = X_1X_2$ is given by*

$$\frac{z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{\prod_{i=1}^2\{B(\alpha_i,\beta_i)\,{}_1F_1(\beta_i;\alpha_i+\beta_i;\lambda_i)\}} \sum_{r=0}^{\infty} \frac{\lambda_1^r(1-z)^r}{r!} B(\beta_1+r,\beta_2)$$
$$\times\Phi_1[\beta_2,\alpha_1+\beta_1-\alpha_2+r;\beta_1+\beta_2+r;1-z,\lambda_2(1-z)], \quad 0 < z < 1.$$

Proof: Using the independence, the joint p.d.f. of X_1 and X_2 is given by

$$K_1 x_1^{\alpha_1-1}(1-x_1)^{\beta_1-1}x_2^{\alpha_2-1}(1-x_2)^{\beta_2-1}\exp[\lambda_1(1-x_1)+\lambda_2(1-x_2)] \tag{3.1}$$

where

$$K_1=\prod_{i=1}^{2}\{B(\alpha_i,\beta_i)\,{}_1F_1(\beta_i;\alpha_i+\beta_i;\lambda_i)\}^{-1}.$$

Transforming $Z=X_1X_2$, $X_2=X_2$ with the Jacobian $J(x_1,x_2\rightarrow z,x_2)=1/x_2$ we obtain the joint p.d.f. of Z and X_2 as

$$K_1 z^{\alpha_1-1}x_2^{\alpha_2-\alpha_1-\beta_1}(1-x_2)^{\beta_2-1}(x_2-z)^{\beta_1-1}\exp\left[\lambda_1\left(1-\frac{z}{x_2}\right)+\lambda_2(1-x_2)\right], \tag{3.2}$$

where $0<z<x_2<1$. To find the marginal p.d.f. of Z, we integrate (3.2) with respect to x_2 to get

$$\begin{aligned} K_1 z^{\alpha_1-1}\int_z^1 & x_2^{\alpha_2-\alpha_1-\beta_1}(1-x_2)^{\beta_2-1}(x_2-z)^{\beta_1-1}\\ & \times\exp\left[\lambda_1\left(1-\frac{z}{x_2}\right)+\lambda_2(1-x_2)\right]dx_2. \end{aligned} \tag{3.3}$$

In (3.3) change of variable $w=(1-x_2)/(1-z)$ yields

$$\begin{aligned} K_1 z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}\int_0^1 & w^{\beta_2-1}(1-w)^{\beta_1-1}[1-(1-z)w]^{\alpha_2-\alpha_1-\beta_1}\\ & \times\exp\left\{\lambda_1\frac{(1-w)(1-z)}{1-w(1-z)}+\lambda_2 w(1-z)\right\}dw. \end{aligned} \tag{3.4}$$

Now, expanding $\exp[\lambda_1(1-w)(1-z)/\{1-w(1-z)\}]$ in the integral (3.4) in terms of power series we arrive at

$$\begin{aligned} & K_1 z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}\sum_{r=0}^{\infty}\frac{\lambda_1^r(1-z)^r}{r!}\\ & \times\int_0^1 w^{\beta_2-1}(1-w)^{\beta_1+r-1}[1-(1-z)w]^{-(\alpha_1+\beta_1+r-\alpha_2)}\exp[\lambda_2 w(1-z)]\,dw. \end{aligned}$$

Finally, applying (2.7) and substituting for K_1 we obtain the desired result. ■

Corollary 3.1.1 *Let X_1 and X_2 be independent random variables, $X_1\sim B^I(\alpha_1,\beta_1)$ and $X_2\sim KB(\alpha_2,\beta_2,\lambda)$. Then, the p.d.f. of $Z=X_1X_2$ is*

$$\frac{B(\beta_2,\beta_1)z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)\,{}_1F_1(\beta_2;\alpha_2+\beta_2;\lambda)}\Phi_1[\beta_2,\alpha_1+\beta_1-\alpha_2;\beta_1+\beta_2;1-z,\lambda(1-z)],$$

where $0<z<1$. Further, if $\alpha_2=\alpha_1+\beta_1$, then the p.d.f. of $Z=X_1X_2$ is given by

$$\frac{\Gamma(\alpha_1+\beta_1+\beta_2)z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}}{\Gamma(\alpha_1)\Gamma(\beta_1+\beta_2)\,{}_1F_1(\beta_2;\alpha_1+\beta_1+\beta_2;\lambda)}\,{}_1F_1(\beta_2;\beta_1+\beta_2;\lambda(1-z)),\quad 0<z<1.$$

Corollary 3.1.2 *Let X_1 and X_2 be independent random variables, $X_i \sim B^I(\alpha_i, \beta_i)$, $i = 1, 2$, then the p.d.f. of $Z = X_1X_2$ is*

$$\frac{\Gamma(\alpha_1+\beta_1)\Gamma(\alpha_2+\beta_2)}{\Gamma(\beta_1+\beta_2)\Gamma(\alpha_1)\Gamma(\alpha_2)} z^{\alpha_1-1}(1-z)^{\beta_1+\beta_2-1}{}_2F_1(\beta_2, \alpha_1+\beta_1-\alpha_2; \beta_1+\beta_2; 1-z),$$

where $0 < z < 1$. Further, if $\alpha_2 = \alpha_1 + \beta_1$, then $Z = X_1X_2 \sim B^I(\alpha_1, \beta_1 + \beta_2)$.

Theorem 3.2 *Let the random variables X_1 and X_2 be independent, $X_1 \sim KB(\alpha_1, \beta_1, \lambda)$ and $X_2 \sim B^{II}(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1X_2$ is given by*

$$\frac{B(\beta_1, \alpha_1+\beta_2)z^{\alpha_2-1}(1+z)^{-(\alpha_2+\beta_2)}}{B(\alpha_2, \beta_2)B(\alpha_1, \beta_1)\ {}_1F_1(\beta_1; \alpha_1+\beta_1; \lambda)}\Phi_1\Big[\beta_1, \alpha_2+\beta_2; \alpha_1+\beta_1+\beta_2; \frac{1}{1+z}, \lambda\Big],\ z > 0.$$

Proof: Since X_1 and X_2 are independent, their joint p.d.f. is given by

$$K_2 x_1^{\alpha_1-1}(1-x_1)^{\beta_1-1}\exp[\lambda(1-x_1)]x_2^{\alpha_2-1}(1+x_2)^{-(\alpha_2+\beta_2)}.$$

where

$$K_2 = \{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)\ {}_1F_1(\beta_1; \alpha_1+\beta_1; \lambda)\}^{-1}.$$

Now consider the transformation $Z = X_1X_2$, $W = 1 - X_1$ whose Jacobian is $J(x_1, x_2 \to w, z) = 1/(1-w)$. Thus, we obtain the joint p.d.f. of W and Z as

$$K_2 z^{\alpha_2-1}(1+z)^{-(\alpha_2+\beta_2)}w^{\beta_1-1}(1-w)^{\alpha_1+\beta_2-1}\left[1 - w\Big(\frac{1}{1+z}\Big)\right]^{-(\alpha_2+\beta_2)}\exp(\lambda w) \tag{3.5}$$

where $0 < w < 1$. Now, integrating w using the integral representation of the Humbert's confluent hypergeometric function (2.7) and substituting for K_2 in (3.5), we obtain the desired result. ■

Corollary 3.2.1 *Let X_1 and X_2 be independent random variables, $X_1 \sim B^I(\alpha_1, \beta_1)$ and $X_2 \sim B^{II}(\alpha_2, \beta_2)$. Then, the p.d.f. of $Z = X_1X_2$ is given by*

$$\frac{B(\beta_1, \alpha_1+\beta_2)z^{\alpha_2-1}(1+z)^{-(\alpha_2+\beta_2)}}{B(\alpha_1, \beta_1)B(\alpha_2, \beta_2)}{}_2F_1\Big(\beta_1, \alpha_2+\beta_2; \alpha_1+\beta_1+\beta_2; \frac{1}{1+z}\Big), \quad z > 0.$$

4 Distribution of The Quotient.

In this section we obtain distributional results for the quotient of two independent random variables involving Kummer-beta distribution.

Theorem 4.1 *Let the random variables U and V be independent. Further, $U \sim KB(\alpha, \beta, \lambda)$ and $V \sim Ga(\theta, \kappa)$. Then, the p.d.f. of $Z_1 = V/U$ is given by*

$$\frac{\Gamma(\alpha+\kappa)\Gamma(\alpha+\beta)z_1^{\kappa-1}\exp(-z_1/\theta)}{\theta^\kappa\Gamma(\kappa)\Gamma(\alpha)\Gamma(\alpha+\beta+\kappa){}_1F_1(\beta; \alpha+\beta; \lambda)}{}_1F_1\left(\beta; \alpha+\beta+\kappa; \lambda+\frac{z_1}{\theta}\right), \quad z_1 > 0.$$

Proof: The joint p.d.f. of U and V is given by

$$K_3 v^{\kappa-1}u^{\alpha-1}(1-u)^{\beta-1}\exp\Big[\lambda(1-u) - \frac{v}{\theta}\Big] \tag{4.6}$$

where

$$K_3 = \{\theta^\kappa\Gamma(\kappa)B(\alpha, \beta)\ {}_1F_1(\beta; \alpha+\beta; \lambda)\}^{-1}. \tag{4.7}$$

Now, transforming $Z_1 = V/U$, $V = V$ with the Jacobian $J(u, v \to z, v) = v/z_1^2$ we obtain the joint p.d.f. of of Z and V as

$$K_3 z_1^{-\alpha-1} v^{\alpha+\kappa-1} \left(1 - \frac{v}{z_1}\right)^{\beta-1} \exp\left[\left(\lambda + \frac{z_1}{\theta}\right)\left(1 - \frac{v}{z_1}\right) - \frac{z_1}{\theta}\right], \tag{4.8}$$

where $0 < v < z_1 < \infty$. Now, integrating v, we get the marginal density of Z as

$$\begin{aligned} &K_3 z_1^{-\alpha-1} \exp\left(-\frac{z_1}{\theta}\right) \int_0^{z_1} v^{\alpha+\kappa-1} \left(1 - \frac{v}{z_1}\right)^{\beta-1} \exp\left[\left(\lambda + \frac{z_1}{\theta}\right)\left(1 - \frac{v}{z_1}\right)\right] dv \\ &= K_3 z_1^{\kappa-1} \exp\left(-\frac{z_1}{\theta}\right) \int_0^1 (1-w)^{\alpha+\kappa-1} w^{\beta-1} \exp\left[\left(\lambda + \frac{z_1}{\theta}\right) w\right] dw \end{aligned} \tag{4.9}$$

where the last line has been obtained by substituting $w = 1 - v/z_1$. Finally, using integral representation (2.2) in (4.9), substituting for K_3 and simplifying the resulting expression, we obtain the desired result. ∎

Corollary 4.1.1 *Let U and V be independent random variables distributed as Kummer-beta and gamma with parameters (α, β, λ) and (θ, κ), respectively. Then, the p.d.f. of $Z_2 = U/V$ is given by*

$$\frac{\Gamma(\alpha+\kappa)\Gamma(\alpha+\beta) z_2^{-\kappa-1} \exp(-1/\theta z_2)}{\theta^\kappa \Gamma(\kappa)\Gamma(\alpha)\Gamma(\alpha+\beta+\kappa) {}_1F_1(\beta; \alpha+\beta; \lambda)} {}_1F_1\left(\beta; \alpha+\beta+\kappa; \lambda + \frac{1}{\theta z_2}\right), \; z_2 > 0.$$

Next, in the following theorem, we consider the case where both the random variables are distributed as Kummer-beta.

Theorem 4.2 *Let the random variables X_1 and X_2 be independent, $X_i \sim KB(\alpha_i, \beta_i, \lambda_i)$, $i = 1, 2$. Then, the p.d.f. of $Z = X_1/X_2$ is given by*

$$\frac{B(\alpha_1+\alpha_2, \beta_2) \exp(\lambda_1+\lambda_2) z^{\alpha_1-1}}{\prod_{i=1}^2 \{B(\alpha_i, \beta_i)\, {}_1F_1(\beta_i; \alpha_i+\beta_i; \lambda_i)\}} \Phi_1[\alpha_1+\alpha_2, 1-\beta_1; \alpha_1+\alpha_2+\beta_2; z, -(\lambda_1 z + \lambda_2)]$$

for $0 < z \le 1$, and

$$\frac{B(\alpha_1+\alpha_2, \beta_1) \exp(\lambda_1+\lambda_2) z^{-\alpha_2-1}}{\prod_{i=1}^2 \{B(\alpha_i, \beta_i)\, {}_1F_1(\beta_i; \alpha_i+\beta_i; \lambda_i)\}} \Phi_1\left[\alpha_1+\alpha_2, 1-\beta_2; \alpha_1+\alpha_2+\beta_1; \frac{1}{z}, -\left(\lambda_1 + \frac{\lambda_2}{z}\right)\right]$$

for $z > 1$.

Proof: The joint p.d.f. of X_1 and X_2 is given by (3.1). Consider the transformation $Z = X_1/X_2$, $X_2 = X_2$ whose Jacobian is $|J| = x_2$. Thus, using (3.1), we obtain the joint p.d.f. of Z and X_2 as

$$K_1 z^{\alpha_1-1} x_2^{\alpha_1+\alpha_2-1} (1 - z x_2)^{\beta_1-1} (1 - x_2)^{\beta_2-1} \exp[\lambda_1(1 - x_2 z) + \lambda_2(1 - x_2)] \tag{4.10}$$

where $0 < x_2 < 1$ for $0 < z \le 1$, and $0 < x_2 < 1/z$ for $z > 1$. For $0 < z \le 1$, the marginal p.d.f. of Z is obtained by integrating (4.10) over $0 < x_2 < 1$. Thus, the p.d.f. of Z, for $0 < z \le 1$, is obtained as

$$\begin{aligned} &\frac{K_1 z^{\alpha_1-1}}{\exp[-(\lambda_1+\lambda_2)]} \int_0^1 v^{\alpha_1+\alpha_2-1} (1 - zv)^{\beta_1-1} (1-v)^{\beta_2-1} \exp[-(\lambda_1 z + \lambda_2) v]\, dv. \\ &= \frac{K_1 B(\alpha_1+\alpha_2, \beta_2) z^{\alpha_1-1}}{\exp[-(\lambda_1+\lambda_2)]} \Phi_1[\alpha_1+\alpha_2, 1-\beta_1; \alpha_1+\alpha_2+\beta_2; z, -(\lambda_1 z + \lambda_2)] \end{aligned} \tag{4.11}$$

where the last line has been obtained by applying (2.7). Finally, substituting K_1 in the density (4.11) we get the desired result. For $z > 1$, the density of z is given by

$$K_1 z^{\alpha_1-1}\int_0^{1/z} v^{\alpha_1+\alpha_2-1}(1-zv)^{\beta_1-1}(1-v)^{\beta_2-1}\exp[\lambda_1(1-vz)+\lambda_2(1-v)]\,dv. \tag{4.12}$$

Substituting $w = vz$ in the above expression we obtain

$$\frac{K_1 z^{-\alpha_2-1}}{\exp\left[-(\lambda_1+\lambda_2)\right]}\int_0^1 w^{\alpha_1+\alpha_2-1}(1-w)^{\beta_1-1}\left(1-\frac{w}{z}\right)^{\beta_2-1}\exp\left[-\left(\lambda_1+\frac{\lambda_2}{z}\right)w\right]\,dw.$$

Finally, using integral representation of Humbert's confluent hypergeometric function (2.7) and substituting for K_1, we obtain the p.d.f. of Z for $z > 1$. ■

Corollary 4.2.1 *Let the random variables X_1 and X_2 be independent, $X_i \sim B^I(\alpha_i,\beta_i)$, $i = 1, 2$. Then, the p.d.f. of $Z = X_1/X_2$ is given by*

$$\frac{B(\alpha_1+\alpha_2,\beta_2)z^{\alpha_1-1}}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\,{}_2F_1(\alpha_1+\alpha_2, 1-\beta_1;\alpha_1+\alpha_2+\beta_2; z),\quad 0 < z \le 1$$

and

$$\frac{B(\alpha_1+\alpha_2,\beta_1)z^{-\alpha_2-1}}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\,{}_2F_1\left(\alpha_1+\alpha_2, 1-\beta_2;\alpha_1+\alpha_2+\beta_1; \frac{1}{z}\right),\quad z > 1.$$

Theorem 4.3 *Let the random variables X_1 and X_2 be independent, $X_i \sim KB(\alpha_i,\beta_i,\lambda_i)$ $i = 1, 2$. Then, the p.d.f. of $T = X_1/(X_1+X_2)$ is given by*

$$\begin{aligned} h(t) &= \frac{B(\alpha_1+\alpha_2,\beta_2)\exp(\lambda_1+\lambda_2)t^{\alpha_1-1}(1-t)^{-\alpha_1-1}}{\prod_{i=1}^2\{B(\alpha_i,\beta_i)\,{}_1F_1(\beta_i;\alpha_i+\beta_i;\lambda_i)\}} \\ &\quad\times\Phi_1\left[\alpha_1+\alpha_2, 1-\beta_1;\alpha_1+\alpha_2+\beta_2; \frac{t}{1-t}, -\left(\frac{\lambda_1 t}{1-t}+\lambda_2\right)\right]\end{aligned}$$

for $0 < t \le 1/2$, and

$$\begin{aligned} h(t) &= \frac{B(\alpha_1+\alpha_2,\beta_1)\exp(\lambda_1+\lambda_2)t^{-\alpha_2-1}(1-t)^{\alpha_2-1}}{\prod_{i=1}^2\{B(\alpha_i,\beta_i)\,{}_1F_1(\beta_i;\alpha_i+\beta_i;\lambda_i)\}} \\ &\quad\times\Phi_1\left[\alpha_1+\alpha_2, 1-\beta_2;\alpha_1+\alpha_2+\beta_1; \frac{1-t}{t}, -\left(\frac{\lambda_2(1-t)}{t}+\lambda_1\right)\right]\end{aligned}$$

for $1/2 < t < 1$.

Proof: The joint p.d.f. of X_1 and X_2 is given by (3.1). Now consider the transformation $T = X_1/(X_1+X_2)$, $Y = X_1+X_2$ whose Jacobian is $|J| = y$. Thus, using (3.1), we find the joint p.d.f. of T and Y as

$$\begin{aligned} g(t,y) &= K_1\exp(\lambda_1+\lambda_2)t^{\alpha_1-1}(1-t)^{\alpha_2-1}y^{\alpha_1+\alpha_2-1}(1-yt)^{\beta_1-1} \\ &\quad\times[1-y(1-t)]^{\beta_2-1}\exp[-(\lambda_1 t+\lambda_2(1-t))y].\end{aligned} \tag{4.13}$$

Now, to evaluate the p.d.f. of T, we integrate (4.13) with respect to y. For $0 < t \le 1/2$, the density $h(t)$ of T is derived as

$$\begin{aligned} h(t) &= K_1\exp(\lambda_1+\lambda_2)t^{\alpha_1-1}(1-t)^{\alpha_2-1}\int_0^{1/(1-t)} y^{\alpha_1+\alpha_2-1}(1-yt)^{\beta_1-1} \\ &\quad\times[1-y(1-t)]^{\beta_2-1}\exp[-(\lambda_1 t+\lambda_2(1-t))y]\,dy.\end{aligned} \tag{4.14}$$

Substituting $w = y(1-t)$ in (4.14), we obtain

$$\begin{aligned} h(t) &= K_1 \exp(\lambda_1+\lambda_2) t^{\alpha_1-1}(1-t)^{-\alpha_1-1}\int_0^1 w^{\alpha_1+\alpha_2-1}(1-w)^{\beta_2-1} \\ &\quad \times \left[1-w\left(\frac{t}{1-t}\right)\right]^{\beta_1-1}\exp\left[-\left(\frac{\lambda_1 t}{1-t}+\lambda_2\right)w\right]dw \\ &= K_1 \exp(\lambda_1+\lambda_2) t^{\alpha_1-1}(1-t)^{-\alpha_1-1} B(\alpha_1+\alpha_2,\beta_2) \\ &\quad \times \Phi_1\left[\alpha_1+\alpha_2, 1-\beta_1;\alpha_1+\alpha_2+\beta_2;\frac{t}{1-t}, -\left(\frac{\lambda_1 t}{1-t}+\lambda_2\right)\right] \end{aligned}$$

where the last line has been obtained by using (2.7). For $1/2 < t < 1$, we have

$$\begin{aligned} h(t) &= K_1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\exp(\lambda_1+\lambda_2)\int_0^{1/t} y^{\alpha_1+\alpha_2-1}(1-yt)^{\beta_1-1} \\ &\quad \times [1-y(1-t)]^{\beta_2-1}\exp[-(\lambda_1 t+\lambda_2(1-t))y]\,dy \\ &= K_1 \exp(\lambda_1+\lambda_2) t^{-\alpha_2-1}(1-t)^{\alpha_2-1}\int_0^1 w^{\alpha_1+\alpha_2-1}(1-w)^{\beta_1-1} \\ &\quad \times \left[1-\frac{(1-t)w}{t}\right]^{\beta_2-1}\exp\left[-\left(\lambda_1+\lambda_2\frac{1-t}{t}\right)w\right]dw \end{aligned}$$

where we have used the substitution $w = yt$. Finally, using (2.7) and resorting to K_2, we get the density of T for $1/2 < t < 1$. ■

Corollary 4.3.1 *Let the random variables X_1 and X_2 be independent, $X_i \sim B^I(\alpha_i,\beta_i)$ $i = 1, 2$. Then, the p.d.f. of $T = X_1/(X_1+X_2)$ is given by*

$$\frac{B(\alpha_1+\alpha_2,\beta_2)t^{\alpha_1-1}(1-t)^{-\alpha_1-1}}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\, {}_2F_1\left(\alpha_1+\alpha_2, 1-\beta_1;\alpha_1+\alpha_2+\beta_2;\frac{t}{1-t}\right)$$

for $0 < t \le 1/2$, and

$$\frac{B(\alpha_1+\alpha_2,\beta_1)t^{-\alpha_2-1}(1-t)^{\alpha_2-1}}{B(\alpha_1,\beta_1)B(\alpha_2,\beta_2)}\, {}_2F_1\left(\alpha_1+\alpha_2, 1-\beta_2;\alpha_1+\alpha_2+\beta_1;\frac{1-t}{t}\right)$$

for $1/2 < t < 1$.

Finally, it may be remarked here that, by using (2.6), alternative expressions for the densities that are given in terms of Φ_1 can be obtained in series involving Gauss hypergeometric function or confluent hypergeometric function.

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