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CONVERGENCE OF SOLUTIONS OF SOME THIRD ORDER SYSTEMS OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS*

BY

ANTHONY UYI AFUWAPE

Abstract. In this paper, we consider the convergence of solutions of solutions of equations of the form $\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X})$, in which $X \in I\!R^n$, $P : I\!R \times I\!R^n \times I\!R^n \times I\!R^n \to I\!R^n$, A, is an $n \times n$ constant matrices and the dots as usual indicate differentiation with respect to t. We shall assume that the functions G and H are of class $C(I\!R^n)$, and satisfy for any X_1, X_2, Y_1, Y_2 in $I\!R_n$

$$\begin{aligned} G(Y_2) &= G(Y_1) + B_g(Y_1, Y_2)(Y_2 - Y_1), \\ H(X_2) &= H(X_1) + C_h(X_1, X_2)(X_2 - X_1), \end{aligned}$$

where $B_g(Y_1, Y_2)$, $C_h(X_1, X_2)$ are $n \times n$ real continuous operators, having positive eigenvalues.

Under different conditions on P, we shall give new and sufficient conditions to establish the convergence of solutions.

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1. Introduction.

Consider a third order system of nonlinear ordinary differential equations of the form

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}) \tag{1.1}$$

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or the equivalent system form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -AZ - G(Y) - H(X) + P(t, X, Y, Z) \end{aligned} \tag{1.2}$$

where $X, Y, Z \in I\!R^n$, A is an $n \times n$ constant matrix; $G(Y), H(X), P(t, X, Y, Z)$ are real vector functions, which are continuous in their respective arguments.

Two solutions $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2)$ of (1.2) are said to *converge to each other* if $\|X_2 - X_1\| \to 0, \|Y_2 - Y_1\| \to 0, \|Z_2 - Z_1\| \to 0$, as $t \to \infty$.

If any pair of solutions of (1.2) satisfy this relation, we shall say that the solutions of (1.2) *converge.*

We note that in the special case of a linear third order system of the form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -AZ - BY - CX \end{aligned} \tag{1.3}$$

convergence results are valid if the Routh-Hurwitz condition

$$\max \lambda_i(C) < \min \lambda_i(A) \min \lambda_i(B) \tag{1.4}$$

is satisfied, where $\lambda_i(D)$ is the eigen-value of matrix D, with A, B, C as $n \times n$ matrices.

In the particular cases when matrices A, B, C are diagonal with positive eigen-values, (1.4) is very obvious. However, when A, B, C are are symmetric or arbitrary, with positive eigen-values, it is still true. On the other hand, when any of them is replaced by a non-linear function, the Routh-Hurwitz condition (1.4) is replaced by a "generalized" Routh-Hurwitz condition. This condition depends on the properties of the non-linear function in the system (see for example [12], [14]).

For example, when CX in (1.3) is replaced by a differentiable function $H(X)$, Tejumola in [10], (for the case when $n = 1$), imposed the conditions

$$h(0) = 0, \quad [h(x_2) - h(x_1)](x_2 - x_1)^{-1} \geq \delta > 0 \text{ for } x_1 \neq x_2,$$
$$h'(x) \leq c \text{ for all } x, \quad 0 < c < ab.$$

This was later in [11] improved upon to the situation when both BY and CX in (1.3) are respectively replaced by non-linear differentiable functions $g(y)$ and $h(x)$ for $n = 1$.

However, following the $n-$dimensional analogue of [10], Afuwape in [1], Afuwape and Omeike in [4] gave an $n-$dimensional improved version of [11].

Another line of thought was initiated by Ezeilo in [8], the case when $n = 1$, and $G(Y) = by$, with $b > 0$, and $h(x)$ not necessarily differentiable was considered, but with $h(0) = 0$, and incrementary ratio $[h(\xi+\eta) - h(\xi)]/\eta$, (with $\eta \neq 0$), lying in a closed subinterval $[\delta, kab]$ $(k < 1)$, of the Routh-Hurwitz interval $(0, ab)$. This was extended to the case when the equations have two non-linear functions in [5]. This is an extension of [7, 9].

In this work, we shall give the $n-$dimensional generalized form of these results to systems of the form (1.2). The main differences in this and the earlier studies are in the type of Lyapunov functions to be used. While the earlier generalizations ([1, 5, 6]) had Lyapunov functions that had direct terms involving matrix A, the present uses only a property of A.

This also allows us to consider the extension of the convergence results to larger types of systems. For example, when AX in (1.1) is replaced either by $F(\ddot{X})$, $F(\dot{X})\ddot{X}$, or $F(X)\ddot{X}$, or if possible, following [13], by $F(X, \dot{X}, \ddot{X})\ddot{X}$, where only the ultimate boundedness of solutions was studied. This may demand an appropriate suitable conditions imposed on the functions F, so as to have the generalized Routh-Hurwitz conditions on the the sub-interval of $(0, ab)$. Thus in the present study, we are improving on [2], and [5] by using a Lyapunov function that is freer for the type of matrix A, or a non-linear function that may replace AX.

We shall make the following assumptions as initiated in [3], and subsequently used in [13], that

$$G(Y_2) = G(Y_1) + B_g(Y_1, Y_2)(Y_1 - Y_2) \tag{1.5}$$

$$H(X_2) = H(X_1) + C_h(X_1, X_2)(X_2 - X_1) \tag{1.6}$$

where $B_g(Y_1, Y_2), C_h(Y_1, Y_2)$ are $n \times n$ real continuous operators, with positive eigenvalues $\lambda_i(B_g(Y_1, Y_2)),\ \ \lambda_i(C_h(X_1, X_2)), (i = 1, 2, \cdots, n)$ satisfying

$$0 < \delta_g \leq \lambda_i(B_g(Y_1, Y_2)) \leq \Delta_g$$
$$0 < \delta_h \leq \lambda_i(C_h(Y_1, Y_2)) \leq \Delta_h.$$

2. Notations and definitions. The notations adopted in [4] shall be used throughout this paper. That is, the δ's, $\triangle$'s with or without suffixes will be positive constants whose magnitudes depend on matrix A, vector functions G, H and P. The δ 's and $\triangle$'s with numerical or alphabetical suffixes shall retain fixed magnitudes, while those without suffixes are not necessarily the same at each occurrence. Finally, we shall denote the scalar product $\langle X, Y\rangle$ of any vectors X, Y in $\mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^{n} x_i y_i$. In particular, $\langle X, X\rangle = \|X\|^2$.

3. Main results. Our main result in this paper is the following:

Theorem 1. *Suppose that in* (1.2)

(i) $G(0) = H(0) = 0$ *and that* $G, \quad H$ *satisfy* (1.5), (1.6) *respectively;*

(ii) *the constant matrix* A*, operators* $B_g(Y_1, Y_2)$ *and* $C_h(X_1, X_2)$ *are associative and commute pairwise;*

(iii) *the eigen-values of* A*, with* $0 < \delta_a \leq \lambda_i(A) \leq \Delta_a$*, satisfy the generalized Routh-Hurwitz condition*

$$\Delta_h \leq k\delta_a\delta_g, \quad (k < 1); \tag{3.1}$$

and

(iv) $P(t, X, Y, Z)$ *satisfy for any two solutions* (X_1, Y_1, Z_1), (X_2, Y_2, Z_2) *of* (1.2)

$$\begin{aligned} \|P(t, X_2, Y_2, Z_2) &- P(t, X_1, Y_1, Z_1)\| \\ &\leq \phi(t)\{\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2\}^{1/2}, \end{aligned} \tag{3.2}$$

for arbitrary t *and where* $\phi(t)$ *is a continuous function in* t.

$$\int_0^t \phi^{\nu}(\tau)d\tau \leq \triangle_0 t, \tag{3.3}$$

for some ν *in the range* $1 \leq \nu \leq 2$.

Then all solutions of (1.2) *converge.*

A particular case of Theorem 1 is when inequality (3.2) is replaced by

$$(3.4)\ \|P(t,X_2,Y_2,Z_2) - P(t,X_1,Y_1,Z_1)\| \leq \Delta_1\{\|X_2-X_1\|^2+\|Y_2-Y_1\|^2+\|Z_2-Z_1\|^2\}^{1/2},$$

where $\triangle_1$ is a finite constant. The following is immediate

Corollary 1. *Let the conditions of Theorem 1 hold with inequality* (3.2) *replaced by* (3.4)*. Then, there exists a finite constant* $\delta_1 > 0$ *such that if the constant* $\Delta_1 < \delta_1$*, the solutions of* (1.2) *converge.*

4. Preliminary results and the function V. In this section, we shall state the algebraic results required in the proof of our result. The proofs are not given since they are found in [2, 4, 5].

Lemma 1. *Let D be a real symmetric $n \times n$ matrix. Then for any X in $\mathbb{R}^n$*

$$\delta_d\|X\|^2 \leq \langle DX, X\rangle \leq \triangle_d\|X\|^2$$

where δ_d, $\triangle_d$ are respectively the least and greatest eigenvalues of D.

Lemma 2. *Let Q, D be any two real $n \times n$ commuting symmetric matrices. Then*

(*i*) *the eigenvalues $\lambda_i(QD)$, $(i = 1, 2, \ldots, n)$ of the product matrix QD are all real and satisfy*

$$\min_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D) \leq \lambda_i(QD) \leq \max_{1\leq j,k\leq n} \lambda_j(Q)\lambda_k(D);$$

(*ii*) *the eigenvalues $\lambda_i(Q+D)$, $(i = 1, 2, \ldots, n)$ of the sum of matrices Q and D are all real and satisfy*

$$\left\{\min_{1\leq j\leq n} \lambda_j(Q) + \min_{1\leq k\leq n} \lambda_k(D)\right\} \leq \lambda_i(Q+D) \leq \left\{\max_{1\leq j\leq n} \lambda_j(Q) + \max_{1\leq k\leq n} \lambda_k(D)\right\},$$

where $\lambda_j(Q)$ and $\lambda_k(D)$ are respectively the eigenvalues of Q and D.

Now, we define the main tool in the proof of our results. This is the scalar function $V = V(X, Y, Z)$ defined for any $X, Y, Z \in I\!R^n$ by

$$(4.1) 2V = \beta(1-\beta)\delta_g^2\|X\|^2 + \beta\delta_g\|Y\|^2 + \alpha\delta_g\delta_a^{-1}\|Y\|^2 + \alpha\delta_a^{-1}\|Z\|^2 + \|Z + \delta_a Y + (1-\beta)\delta_g X\|^2$$

with $0 < \beta < 1$, and $\alpha > 0$. This is a modification of a Lyapunov function used in [2, 4].

This function has the following properties:

Lemma 3. *Assume that all the assumptions on A, $G(Y)$, and $H(X)$ hold. Then there exists positive constants δ_2 and δ_3 such that*

$$(4.2) \quad \delta_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) \leq 2V(X, Y, Z) \leq \delta_3(\|X\|^2 + \|Y\|^2 + \|Z\|^2),$$

for arbitrary X, Y, Z in $I\!R^n$.

These inequalities follow from (4.1) if we choose

$$\delta_2 = \min\{\beta(1-\beta)\delta_g^2; (\beta + \alpha\triangle_a^{-1})\delta_g; \alpha\triangle_a^{-1}\}$$

and

$$\delta_3 = \max\{\delta_g(1-\beta)(1+\delta_g+\triangle_a); \delta_g(\beta+\alpha\delta_a^{-1}) + \triangle_a(1+\delta_g+\triangle_a); 1 + \alpha\delta_a^{-1} + \delta_g(1-\beta) + \triangle_a\}.$$

5. Proof of Theorem 1. It was proved in [3] that the solutions of (1.2) are ultimately bounded.

Let $(X_1(t), Y_1(t), Z_1(t))$, $(X_2(t), Y_2(t), Z_2(t))$ be any two ultimately bounded solutions of (1.2).

Define

$$(5.1) \qquad W(t) = V(X_1(t) - X_2(t), Y_1(t) - Y_2(t), Z_1(t) - Z_2(t)),$$

Then, from (4.2), following the method used in [5] we have

$$(5.2) \qquad \delta_2(S(t)) \leq 2W(t) \leq \delta_3(S(t))$$

where $S(t) = \{\|X_2(t) - X_1(t)\|^2 + \|Y_2(t) - Y_1(t)\|^2 + \|Z_2(t) - Z_1(t)\|^2\}$.

In view of inequalities (4.2), it suffices to prove that

$$(5.3) \qquad W(t_2) \leq W(t_1)\exp\{-\delta_2(t_2 - t_1) + \delta_3\int_{t_1}^{t_2}\phi^{\nu}(\tau)d\tau\}$$

for $t_2 \geq t_1$.

On differentiating (5.1), with respect to t, and using (1.2), we obtain after some simplifications that $\dot{W}(t) = -W_1(t) + W_2(t)$, where

$$\begin{aligned} W_1(t) &= (1-\beta)\delta_g\langle X_2 - X_1, H(X_2) - H(X_1)\rangle \\ &\quad +\delta_a\langle Y_2 - Y_1, [G(Y_2) - G(Y_1)] - \delta_g(1-\beta)(Y_2 - Y_1)\rangle \\ &\quad +\alpha\langle Z_2 - Z_1, Z_2 - Z_1\rangle + (1+\alpha\delta_a^{-1})\langle Z_2 - Z_1, H(X_2) - H(X_1)\rangle \\ &\quad +\delta_a\langle Y_2 - Y_1, H(X_2) - H(X_1)\rangle \\ &\quad +(1-\beta)\delta_g\langle X_2 - X_1, [G(Y_2) - G(Y_1)] - \delta_g(Y_2 - Y_1)\rangle \\ &\quad +(1+\alpha\delta_a^{-1})\alpha\langle Z_2 - Z_1, [G(Y_2) - G(Y_1)] - \delta_g(Y_2 - Y_1)\rangle \end{aligned}$$

and $W_2(t) = \langle\ (1-\beta)\delta_g(X_2 - X_1) + \delta_a(Y_2 - Y_1) + (1+\alpha\delta_a^{-1})(Z_2 - Z_1),\ \theta\rangle$ with $\theta = P(t, X_2, Y_2, Z_2) - P(t, X_1, Y_1, Z_1)$.

A re-arrangement of $W_1(t)$, following the methods used in [5, 13], we can easily conclude that

$$W_1(t) \geq 2\delta_4(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2)$$

and

$$W_2(t) \leq \delta_5(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2)^{\frac{1}{2}}\|\theta\|$$

using the assumptions (1.5) (1.6) and properties of $H(X), G(Y)$, and $P(t, X, Y, Z)$ as given in the statement of the theorem. We thus have

$$\begin{aligned} \text{(5.4)}\quad \dot{W}(t) &\leq -2\delta_4(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2) \\ &\quad +\delta_5(\|X_2 - X_1\|^2 + \|Y_2 - Y_1\|^2 + \|Z_2 - Z_1\|^2)^{\frac{1}{2}}\|\theta\|. \end{aligned}$$

The remaining part of the proof of Theorem 1 follows from [2]. Let ν be any constant such that $1 \leq \nu \leq 2$ and set $\mu = 1 - \frac{1}{2}\nu$, so that $0 \leq \mu \leq \frac{1}{2}$. Consider inequality (5.4) in the form

$$\dot{W} + \delta_4 S \leq \delta_5 S^{\frac{1}{2}}\|\theta\| - \delta_4 S, \tag{5.5}$$

$$\dot{W} + \delta_4 S \leq \delta_5 S^{\mu} W^*, \tag{5.6}$$

where $W^* = S^{(\frac{1}{2}-\mu)}(\|\theta\| - \delta_4\delta_5^{-1}S^{\frac{1}{2}})$.

Considering the two cases (i) $\|\theta\| \leq \delta_4\delta_5^{-1}S^{\frac{1}{2}}$ and (ii) $\|\theta\| > \delta_4\delta_5^{-1}S^{\frac{1}{2}}$ separately, we find that in either case, there exists some constant $\delta_6 > 0$ such that $W^* \leq \delta_6\|\theta\|^{2(1-\mu)}$. Thus, using (4.2), inequality (5.6) becomes

$$\frac{dW}{dt} + \delta_4 S \leq \delta_7 S^{\mu}\phi^{2(1-\mu)}S^{(1-\mu)}, \tag{5.7}$$

where $\delta_7 \geq 2\delta_5\delta_6$. This immediately gives

$$\frac{dW}{dt} + (\delta_8 - \delta_9\phi^{\nu}(t))W \leq 0, \tag{5.8}$$

after using (5.2) on W, with δ_8, δ_9 as some positive constants.

On integrating (5.8) from t_1 to t_2, $(t_2 \geq t_1)$, we obtain

$$W(t_2) \leq W(t_1)\exp\left\{-\delta_8(t_2 - t_1) + \delta_9\int_{t_1}^{t_2}\phi^{\nu}(\tau)d\tau\right\}. \tag{5.9}$$

Thus, we obtain (5.3), with

$$\delta_2 = \delta_8 \quad \text{and} \quad \delta_3 = \delta_9. \tag{5.10}$$

From inequality (5.3), if

$$\int_{t_1}^{t_2}\phi^{\nu}(\tau)d\tau \leq \delta_2\delta_3^{-1}(t_2 - t_1),$$

then, the exponential index remains negative for all $(t_2 - t_1) \geq 0$. As $t = (t_2 - t_1) \to \infty$, we have $W(t) \to 0$. That is, $\|X_2(t) - X_1(t)\| \to 0$, $\|\dot{X}_2(t) - \dot{X}_1(t)\| \to 0$ and $\|\ddot{X}_2(t) - \ddot{X}_1(t)\| \to 0$ as $t \to \infty$.

This completes the proof of Theorem 1.

6. Remarks

Remark 1. (*i*) If in (1.1), $G(\dot{X}) = B\dot{X}$, then we shall have the equation which was discussed in [1];

(ii) If in (1.1), $H(X) = CX$, then we shall have an independent result on equation

$$\dddot{X} + A\ddot{X} + G(\dot{X}) + CX = P(t, X, \dot{X}, \ddot{X}); \tag{6.1}$$

This again will require that the generalized Routh-Hurwitz condition be $\max\lambda_i(C) < \min\lambda_i(A)\min\lambda_i(B_g(Y_1, Y_2))$, with$(i = 1, 2, \cdots, n)$.

Remark 2. Let $P(t, X, \dot{X}, \ddot{X}) = 0$ and hypothesis (i) of Theorem 1 hold with inequality (3.1). Then, the trivial solution $X(t) = 0$ of (1.1) is exponentially stable in the large.

Remark 3. We can equally use the present Lyapunov function defined in (4.1) easily for the convergence of solutions systems of the form

$$\dddot{X} + F(\ddot{X}) + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}); \tag{6.2}$$

if condition (iii) of Theorem 1 is replaced by the assumptions that

$$F(Z_2) = F(Z_1) + A_f(Z_1, Z_2)(Z_2 - Z_1) \tag{6.3}$$

for some continuous operator $A_f(Z_1, Z_2)$, such that

$$0 < \delta_a \leq \lambda_i(A_f(Z_1, Z_2)) \leq \Delta_a \tag{6.4}$$

with (3.1) valid as before for $F(Z) \in C(\mathbb{R}^n)$.

Remark 4. This same remark holds true if we consider systems of the form

$$\dddot{X} + F(\dot{X})\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}); \tag{6.5}$$

or

$$\dddot{X} + F(X, \dot{X})\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}); \tag{6.6}$$

or

$$\dddot{X} + F(X, \dot{X}, \ddot{X})\ddot{X} + G(\dot{X}) + H(X) = P(t, X, \dot{X}, \ddot{X}); \tag{6.7}$$

after making the appropriate assumptions as (6.3) and (6.4) on the non-linearities following the ideas of [13], which is the adaptation of that of [3].

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Departmento de Matemáticas,
Universidad de Antioquia,
Calle 67, No. 53-108, AA1226 Medellín,
COLOMBIA
aafuwape@yahoo.co.uk