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SOME EQUIVALENCES BETWEEN HOMOTOPY AND DERIVED CATEGORIES

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Abstract

We prove two triangle equivalences. One is the triangle equivalence between the homotopy category of the bounded below complexes of Ext-injectives objects of a closed by subobjects and co-resolving subcategory $\mathcal{B}$ of an abelian category and the derived category of the bounded below complexes over $\mathcal{B}$. The other triangle equivalence is

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between the homotopy category of the bounded cohomology and bounded below complexes over a strongly closed by cokernels of monomorphisms and auto-orthogonal subcategory of an abelian category $\mathcal{A}$ and the derived category of the bounded cohomology and bounded below complexes over $\mathcal{A}$.

0. Introduction

Derived categories are a “formalization for hyperhomology” (see [16]). They were introduced in the early sixties by Grothendieck and Verdier and they are useful in algebraic geometry and homological algebra. The first applications appeared in duality theory of coherent sets and locally compact sets. These methods of Grothendieck-Verdier have been adapted to the study of partial differential equations by Sato in [15] and Kashiwara in [11]. The derived categories have become a standard language of microbiological analysis (see [12, 14] and [3]). Brylinski and Kashiwara proved Kazhdan-Lusztig conjecture (see [4]) and this enabled the use of derived categories in the theory of representations of Lie groups and finite groups of Chevalley having a crucial role in the abelian subcategory of the derived categories, which have been developed to preserve schemes (see [2]).

Beilinson, Bernstein and Gelfand used derived categories to establish a relationship between coherent schemes in the projective space and representations of some finite dimensional algebras. Their methods have allowed many generalizations (see [5, 9] and [10]).

In the area of representation theory of Artin algebras, the study of the derived category was introduced from an interpretation of the tilted theory given by Happel in [7], which extended the generalization of the classical equivalence of Morita in terms of derived equivalences, which produced several invariants. Thus, if two rings have triangle equivalent derived categories of modules over these rings, then their centers are isomorphic, their Hochschild cohomologies are isomorphic, and in the case of selfinjectives algebras over an algebraically closed field, they have the same type of representation.

In this work, we use similar ideas of Borel et al. (see [3], Chapter 1) and Happel (see [6]) and we show some triangle equivalences between homotopy and derived categories.

This paper is organized as follows: after preliminaries, in Section 2, we will show the triangle equivalence between the homotopy category of the bounded below complexes of the Ext-injectives objects of a strongly closed by cokernels of monomorphisms subcategory $\mathcal{B}$ of an abelian category with enough sufficient injectives and the derived category of the bounded below complexes over $\mathcal{B}$ (see Theorem 2.6). As a corollary, we obtain the well known result (see [3], Chapter 1): the homotopy category of the bounded below complexes of the injectives objects of an abelian category $\mathcal{A}$ with enough sufficient injectives is triangle equivalent to the derived category of bounded below complexes over $\mathcal{A}$. Also, from Theorem 2.6, we get that the homotopy category of the bounded below complexes of Ext-injectives objects of a closed by subobjects and co-resolving subcategory $\mathcal{B}$ of an abelian category is triangle equivalent to the derived category of the bounded below complexes over $\mathcal{B}$ (see Corollary 2.9).

In Section 3, we prove the triangle equivalence between the homotopy category of the bounded cohomology and bounded below complexes over a strongly closed by cokernels of monomorphisms and auto-orthogonal subcategory of an abelian category $\mathcal{A}$ and the derived category of the bounded cohomology and bounded below complexes over $\mathcal{A}$ (see Theorem 3.4). This theorem generalizes Theorem 1.6 proved by Happel in [6].

1. Preliminaries

We begin by recalling some definitions and by fixing some notation. In this section, we just mention basic facts, for a deeper treatment, we refer the readers to [3, 7, 8] and [17].

From now on, $\mathcal{A}$ will be an abelian category and $\mathcal{B}$ a full additive subcategory of $\mathcal{A}$.

We will denote by $C(\mathcal{D})$ the complex category over any additive category $\mathcal{D}$, that is, a typical object (called *complex*) is a family $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$, where $X^i \in \mathcal{D}$ and $d_X^i : X^i \to X^{i+1}$ are morphisms in $\mathcal{D}$ (called *differentials* of the complex X) such that $d_X^{i+1}d_X^i = 0$ for all $i \in \mathbb{Z}$. If X and Y are complexes over $\mathcal{D}$, a morphism $f = (f^i)_{i\in\mathbb{Z}} : X \to Y$ of complexes is given by a family $f^i : X^i \to Y^i$ of morphisms in $\mathcal{D}$ such that $d_Y^i f^i = f^{i+1}d_X^i$ for all $i \in \mathbb{Z}$. A complex $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ is a *bounded below complex* if there exists $i_0 \in \mathbb{Z}$ such that $X^{i_0} \neq 0$ and $X^{i_0} = 0$ for all $i < i_0$. A complex $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ is a *bounded complex* if there exists j, $k \in \mathbb{Z}$ with $j \leq k$ such that $X^j \neq 0$, $X^k \neq 0$, $X^i = 0$ for all $i < j$ and $X^i = 0$ for all $k < i$ and the width $w(X)$ of X is defined as $k - j + 1$.

A complex $X = (X^i, d_X^i)_{i\in\mathbb{Z}}$ is a *stalk complex* if there exists $i_0 \in \mathbb{Z}$ such that $X^{i_0} \neq 0$ and $X^{i_0} = 0$ for all $i \neq i_0$. The object X^{i_0} is then called the *stalk* at the i_0-position of the complex X. When we have an object $V \in \mathcal{D}$ we denote by $V[i]$ the stalk complex with V the stalk at the i-position. For $n \in \mathbb{Z}$, we denote by $[n] : C(\mathcal{D}) \to C(\mathcal{D})$ the classical shift functor, that is, if $X \in C(\mathcal{D})$, then the image is the complex denoted by $X[n]$, where $X[n]^i = X^{n+i}$ and the differentials of $X[n]$ are $d_{X[n]}^i = (-1)^n d_X^{n+i}$ for all $i \in \mathbb{Z}$.

We denote by $C^+(\mathcal{D})$ the full subcategory of $C(\mathcal{D})$ formed by bounded below complexes. By $K^+(\mathcal{D})$ and $D^+(\mathcal{D})$ we denote the associated homotopy category and the derived category, respectively. Similarly, $C^b(\mathcal{D})$ is the full subcategory of $C(\mathcal{D})$ formed by the bounded complexes and so we have $K^b(\mathcal{D})$ and $D^b(\mathcal{D})$. Also, $C^{+,b}(\mathcal{D})$ denotes the full subcategory of $C^+(\mathcal{D})$ of the complexes with only finitely many non-zero cohomology groups and so the associated homotopy category $K^{+,b}(\mathcal{D})$ and the derived category $D^{+,b}(\mathcal{D})$.

We will use the following known result, for $i > 0$, $Hom_{D^b(\mathcal{A})}(X, Y) = Ext^i_{\mathcal{A}}(X, Y)$, for X and Y stalk complexes.

The derived category $D(\mathcal{D})$ is the triangulated category obtained from the homotopy category $K(\mathcal{D})$ by localizing with respect to the set of quasi-isomorphisms $(Qis_{\mathcal{D}})$. A *distinguished triangle* in $D(\mathcal{D})$ is a triangle that is isomorphic to the *standard triangle*

$$X \xrightarrow{f} Y \xrightarrow{q} C_f \xrightarrow{p} X[1],$$

where $f : X \to Y$ is a morphism of complexes and C_f denotes the mapping cone of *f*.

We now state the following result about localization of categories. For a multiplicative system $\mathcal{S}_{\mathcal{A}}$ in $\mathcal{A}$, we define $\mathcal{S}_{\mathcal{B}} = Mor(\mathcal{B}) \cap \mathcal{S}_{\mathcal{A}}$, where $Mor(\mathcal{B})$ denotes the set of the morphisms in the category $\mathcal{B}$. If $\mathcal{S}_{\mathcal{B}}$ is a multiplicative system in $\mathcal{B}$, then the localization of $\mathcal{B}$ with respect to $\mathcal{S}_{\mathcal{B}}$ is the category $\mathcal{S}_{\mathcal{B}}^{-1}(\mathcal{B})$ and the canonical functor of localization is $\iota_{\mathcal{S}_{\mathcal{B}}} : \mathcal{B} \to \mathcal{S}_{\mathcal{B}}^{-1}(\mathcal{B})$.

Let $\mathcal{U} \neq \emptyset$ be a set (or a class) of objects from category $\mathcal{A}$. Then we say that $\mathcal{U}$ *co-generates* a subcategory $\mathcal{B}$ if, for any object $V \in \mathcal{B}$, there exists a monomorphism $V \hookrightarrow U$, with $U \in add(\mathcal{U})$, where $add(\mathcal{U})$ is the smallest of the additive categories containing $\mathcal{U}$. If in addition $\mathcal{U}$ is finite, then it is called a *finite co-generator* of $\mathcal{B}$. In the case, where $\mathcal{U}$ is formed by all injective objects of $\mathcal{A}$ and $\mathcal{U}$ co-generates $\mathcal{A}$, we say that $\mathcal{A}$ *has sufficient injectives*. If $\mathcal{A}$ has sufficient injectives, a subcategory $\mathcal{B}$ is *auto-orthogonal*, if $Ext^i_{\mathcal{B}}(-, -) = 0$ for all $i > 0$. An object $V \in \mathcal{A}$ is called *Ext*-

injective in $\mathcal{B}$, if we have that $Ext^1_{\mathcal{B}}(-, V) = 0$. We denote by $\mathcal{B}^{\perp 1}$ the additive category formed by the Ext-injective objects in $\mathcal{B}$. To a subcategory $\mathcal{B}$, we associate the subcategory

$$\mathcal{B}^{\perp} = \{V \in \mathcal{A} / Ext^i_{\mathcal{B}}(-, V) = 0 \text{ for all } i > 0\}.$$

Definition 1.1. (a) We say that a subcategory $\mathcal{B}$ is *strongly closed by cokernels of monomorphisms* (*s.c.c.m.*) if $\mathcal{B}$ is closed by cokernels of monomorphisms of the form $A \hookrightarrow V$, with $A \hookrightarrow \mathcal{A}$ and $V \in \mathcal{B}$.

(b) Let $\mathcal{B}$ be a full additive subcategory of the abelian category with sufficient injectives $\mathcal{A}$. We say that $\mathcal{B}$ is *co-resolving* if it satisfies the following three conditions:

(i) The category $\mathcal{B}$ contains the injective objects of $\mathcal{A}$.

(ii) The category $\mathcal{B}$ is closed by extensions.

(iii) The category $\mathcal{B}$ is closed by cokernels of monomorphisms.

(c) Let T be an object of an abelian category with sufficient injective and projective objects and Krull-Schmidt $\mathcal{A}$. We say that T is a *tilting object* if it satisfies the following three conditions:

(i) $dp(T) < \infty$, that is, its projective dimension is finite.

(ii) $Ext^i_{\mathcal{A}}(T, T) = 0$ for all $i > 0$.

(iii) For every indecomposable injective object I of $\mathcal{A}$, there exists a long exact sequence of the following form:

$$0 \to I \to T_1 \to T_2 \to \cdots \to T_t \to 0,$$

with $T_i \in add(T)$.

A $\mathcal{B}$-*left approximation* of M with $M \in \mathcal{A}$ is a morphism $f^M : M \to V$, with $V \in \mathcal{B}$ such that, for any morphism $g : M \to V'$, with $V' \in \mathcal{B}$, there exists $g' : V \to V'$ such that $g'f^M = g$. Moreover, we say that $\mathcal{B}$ is a *covariantly finite* subcategory, if for any $M \in \mathcal{A}$, there exists a $\mathcal{B}$-left approximation.

Remark 1.2 (See [1]). For an object $T \in \mathcal{A}$ with the property $Ext^i_{\mathcal{A}}(T, T) = 0$ for all $i > 0$, we associate the subcategory $T^{\perp}$ whose objects B are such that $Ext^i_{\mathcal{A}}(T, B) = 0$ for all $i > 0$. It is known that $T^{\perp}$ is a co-resolving subcategory of $\mathcal{A}$. Furthermore, T is a tilting object if and only if $T^{\perp}$ is covariantly finite and for any object C in $\mathcal{A}$ there exists an exact sequence of finite length, of the following form:

$$0 \to C \to T^0 \to \cdots \to T^n \to 0,$$

with $T^i \in T^{\perp}$.

2. Triangle Equivalence for Co-resolving Subcategory

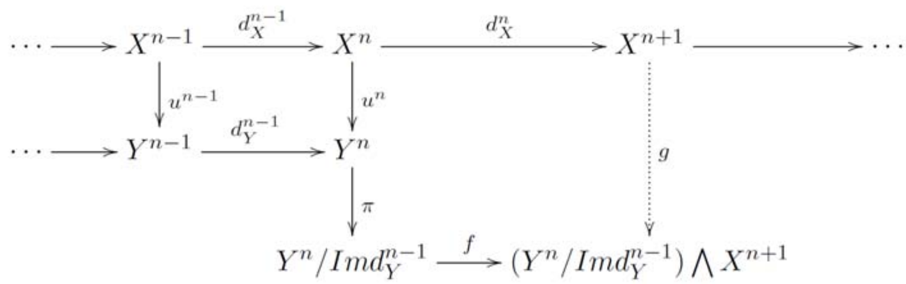
In this section, we prove the first triangle equivalence. We will denote by $\mathcal{B}'$ the intersection of $\mathcal{B}$ with the category of Ext-injective objects in $\mathcal{B}$, that is, $\mathcal{B}' = \mathcal{B}^{\perp_1} \cap \mathcal{B}$.

The following theorem allows us to obtain a resolution for a bounded below complex, in the sense of having a quasi-isomorphism for another complex.

Theorem 2.1. *Let $\mathcal{B}$ be an s.c.c.m. If $\mathcal{B}'$ co-generates $\mathcal{B}$, then for any $X \in C^+(\mathcal{B})$, there exists $Y \in C^+(\mathcal{B}')$ and a quasi-isomorphism of X to Y.*

Proof. If $X \in C^+(\mathcal{B})$, without loss of generality, we can suppose that $X^n = 0$ whenever $n < 0$. We shall construct a complex Y with entries $Y^n \in \mathcal{B}'$ and a quasi-isomorphism $u : X \to Y$. To do that, let us suppose that the complex has been constructed at grade $n \geq 0$ in such a way that u^q is a monomorphism for $q \leq n$ and a quasi-isomorphism for $q \leq n - 1$.

Consider the following diagram:



where $(Y^n/Imd_Y^{n-1}) \wedge X^{n+1}$ is an amalgamated sum of πu^n and d_X^n.

To finalize the proof, it is sufficient to note that $(Y^n/Imd_Y^{n-1}) \wedge X^{n+1} \in \mathcal{B}$ because $\mathcal{B}$ is s.c.c.m. and to follow the same proofs of part (1) of Lemma 4.6 in Chapter I in [8] (or Theorem 7.5, Section 7, Chapter I in [3]).

□

It is easy to see that if a category $\mathcal{C}$ is $\mathcal{B}^{\perp_1}$ or $\mathcal{B}^{\perp}$, then as full subcategories $K^*(\mathcal{C})$ with $(* = \phi, +, b)$, they are triangulated subcategories of $K^*(\mathcal{A})$ and that the multiplicative system $Qis_{\mathcal{C}} = Qis_{\mathcal{A}} \cap Mor(K^*(\mathcal{C}))$ is a multiplicative system in $K^*(\mathcal{C})$ compatible with the triangulated structure (see [3], Chapter 1).

Lemma 2.2. *Let $\mathcal{B}$ be an s.c.c.m. If $X \in C(\mathcal{B})$ for an acyclic complex in $C(\mathcal{A})$ and $Y \in C^+(\mathcal{B}^{\perp_1})$, then $Hom_{K(\mathcal{B})}(X, Y) = 0$.*

Proof. Letting $u \in Hom_{C(\mathcal{B})}(X, Y)$, we shall prove that $u \sim 0$. Without loss of generality, we can suppose that the entries of complex Y satisfy $y^p = 0$ for $p < 0$. Suppose also that a homotopy k is defined at k^p for $p \leq q$, with $q \geq 0$.

Define $k_1^{q+1} : Ind_X^q \to Y^q$ by $k_1^{q+1}(x) = u^q(x') - d_Y^{q-1}k^q(x')$ for $x \in Imd_X^q$ with $x' \in X^q$ such that $d_X^q(x') = x$.

k_1^{q+1} is well defined, in fact, if $x'' \in X^q$ is such that $d_X^q(x'') = x$, then there exists $x''' \in X^{q-1}$ such that $d_X^{q-1}(x''') = x' - x''$. Therefore, $u^q(x' - x'') - d_Y^{q-1}k^q(x' - x'') = d_Y^{q-1}u^{q-1}(x''') - d_Y^{q-1}(u^{q-1}(x''') - d_Y^{q-2}k^{q-1}(x''')) = 0$, thus $u^q(x') - d_Y^{q-1}k^q(x') = u^q(x'') - d_Y^{q-1}k^q(x'')$.

Since $Y^q \in \mathcal{B}^{\perp_1}$, we can to extend k_1^{q+1} to $k^{q+1} : X^{q+1} \to Y^q$ which satisfies the required condition. □

Lemma 2.3. *If* $\mathcal{B}$ *is s.c.c.m. and* $u \in Hom_{C^+(B^{\perp_1})}(X, Y)$ *is a quasi-isomorphism, then u is a homotopic equivalence.*

Proof. Let us consider a standard triangle

$$X \xrightarrow{u} Y \xrightarrow{q} C_u \xrightarrow{p} X[1].$$

Since u is a quasi-isomorphism, it is easy to see that C_u is an acyclic complex (see [3], Chapter 1), therefore, by Lemma 2.2, p is homotopic to zero. Therefore, there exists a homotopy k such that $dk + kd = p$, thus, we have a morphism $v = kq \in Hom_{C^+(B^{\perp_1})}(Y, X)$.

For each $n \in \mathbb{Z}$, we define $s^n = k^{n-1}q_X^n : Y^{n+1} \to Y^n$, where q_X is the inclusion morphism of X (with entries Y^n) in C_u. We have that $1_{Y^{n+1}} - v^{n+1}u^{n+1} = d_X^n s^{n+1} + s^{n+2}d_X^{n+1}$, thus, $vu \sim 1_X$. Clearly, v is a quasi-isomorphism and applying the above reasoning for v, we obtain a morphism $w \in Hom_{C^+(B^{\perp_1})}(X, Y)$ such that $wv \sim 1_Y$. Therefore, $uv \sim 1_Y$. □

Proposition 2.4. *The canonical functor of localization*

$$\iota_{\mathcal{B}'}^+ : K^+(\mathcal{B}') \to D^+(\mathcal{B}')$$

is an isomorphism of categories.

Proof. By Lemma 2.3, the identity functor $1_{K^+(\mathcal{B}')}$ transforms quasi-isomorphisms in isomorphisms, so, by the universal property of functor $\iota^+_{\mathcal{B}'}$, there exists a unique functor $G : D^+(\mathcal{B}') \to K^+(\mathcal{B}')$ such that $G\iota^+_{\mathcal{B}'} = 1_{K^+(\mathcal{B}')}$.

The universal property of functor $\iota^+_{\mathcal{B}'}$ shows that $\iota^+_{\mathcal{B}'}G = 1_{D^+(\mathcal{B}')}$. □

We have the following diagram:

$$\begin{array}{ccc} K^+(\mathcal{B}') & \overset{J}{\hookrightarrow} & K^+(\mathcal{B}) \\ \downarrow \iota^+_{\mathcal{B}'} & & \downarrow \iota^+_{\mathcal{B}} \\ D^+(\mathcal{B}') & & D^+(\mathcal{B}), \end{array}$$

where J is the inclusion functor and $\iota^+_{\mathcal{B}'}$ is the canonical functor of localization. The universal property of $\iota^+_{\mathcal{B}'}$ allows to define $F : D^+(\mathcal{B}') \to D^+(\mathcal{B})$ such that $F\iota^+_{\mathcal{B}'} = \iota^+_{\mathcal{B}}J$.

Proposition 2.5. *The functor* $F : D^+(\mathcal{B}') \to D^+(\mathcal{B})$ *is a triangle equivalence.*

Proof. Theorem 2.1 implies that F is a triangle equivalence (see $(1.7.2)^{op}$ in [13]). □

Finally, we have the following triangle equivalence.

Theorem 2.6. *Let* $\mathcal{B}$ *be an s.c.c.m. of an abelian category with sufficient injectives* $\mathcal{A}$ *and such that* $\mathcal{B}'$ *co-generates* $\mathcal{B}$. *Then the functor* $G = F\iota^+_{\mathcal{B}'} : K^+(\mathcal{B}') \to D^+(\mathcal{B})$ *is a triangle equivalence.*

In particular, if $\mathcal{B} = \mathcal{A}$, *then we have that* $K^+(\mathcal{I})$ *and* $D^+(\mathcal{I})$ *are triangle equivalents, where* $\mathcal{I}$ *is the full additive subcategory of injective objects of* $\mathcal{A}$.

Proof. It is easy to see that G preserves distinguished triangles. By Proposition 2.4 and Proposition 2.5, G is a triangle equivalence. □

Corollary 2.7. *If* $\mathcal{B}$ *is a subcategory s.c.c.m. of an abelian category with sufficient injectives* $\mathcal{A}$ *and such that* $\mathcal{B}^{\perp} \bigcap \mathcal{B}$ *co-generates* $\mathcal{B}$, *then* $K^{+}(\mathcal{B}^{\perp} \bigcap \mathcal{B})$ *and* $D^{+}(\mathcal{B})$ *are triangle equivalent*.

Corollary 2.8. *If* $\mathcal{B}$ *is co-resolving and closed by subobjects, then* $K^{+}(\mathcal{B}')$ *is triangle equivalent to* $D^{+}(\mathcal{B})$.

In particular, $K^{+}(\mathcal{B}^{\perp})$ *and* $D^{+}(\mathcal{B})$ *are triangle equivalent*.

As an application of the above, we have the following.

Corollary 2.9. *Let* T *be a tilting object of an abelian category with sufficient injectives and Krull-Schmidt* $\mathcal{A}$. *If* $T^{\perp}$ *is closed by subobjects, then* $K^{+}(T^{\perp})$ *and* $D^{+}(add(T))$ *are triangle equivalent*.

Proof. The result easily follows from Remark 1.2 and Corollary 2.8. □

All the results described above can be dualized by taking generators, Ext-projective objects, $\mathcal{B}$ resolving, $\mathcal{B}$ strongly closed by kernels of epimorphisms (s.c.k.e.), $\mathcal{A}$ with sufficient projectives, homotopy and derived categories of bounded above complexes.

3. Equivalence for Auto-orthogonal Subcategory

This section is devoted to the second triangle equivalence.

The following result is due to Happel (see [6], Lemma 1.1) and it is essential for our results.

Lemma 3.1. *If* $\mathcal{B}$ *is auto-orthogonal, then*

$$Hom_{K^b(\mathcal{B})}(X, Y) \cong Hom_{D^b(\mathcal{A})}(X, Y).$$

Lemma 3.2. *If* $\mathcal{B}$ *is an auto-orthogonal and s.c.c.m., then* $K^{b}(\mathcal{B})$ *and* $K^{+,b}(\mathcal{B})$ *are triangle equivalent*.

Proof. Let X be in $K^{+,b}(\mathcal{B})$. Then we can choose n such that $H^i(X) = 0$ for $i > n$ and so we define $Y \in K^b(\mathcal{B})$ and $f : Y \to X$ as follow: $Y^i = X^i$ for $i < n+1$, $Y^{n+1} = Imd_X^n$ (which is in $\mathcal{B}$ because $\mathcal{B}$ is s.c.c.m.), $Y^i = 0$ for $i > n+1$ and f is the natural inclusion. It is easy to see that f is a quasi-isomorphism, but $\mathcal{B}$ is an auto-orthogonal and s.c.c.m., therefore, we can do exactly of the same proofs of Lemma 2.2 and Lemma 2.3 and we conclude that f is an isomorphism in $K^{+,b}(\mathcal{B})$. □

Proposition 3.3. *Let $\mathcal{B}$ be an auto-orthogonal and s.c.c.m. Then*

$$Hom_{K^{+,b}(\mathcal{B})}(X, Y) \cong Hom_{D^{+,b}(\mathcal{A})}(X, Y).$$

Proof. It is very well known that $D^b(\mathcal{A})$ and $D^{+,b}(\mathcal{A})$ are triangle equivalent, so the result easily follows from Lemma 3.1 and Lemma 3.2. □

The previous result implies the following triangular equivalence.

Theorem 3.4. *If $\mathcal{B}$ co-generates $\mathcal{A}$, is auto-orthogonal, and s.c.c.m., then $K^{+,b}(\mathcal{B})$ and $D^{+,b}(\mathcal{A})$ are triangle equivalent.*

Proof. The result easily follows from Proposition 3.3 and Theorem 2.1.□

As an application of this theorem, we have the following.

Corollary 3.5. *Let T be a tilting object of an abelian category with sufficient injectives and Krull-Schmidt $\mathcal{A}$, ${}^{\perp}T = \{V \in \mathcal{A}/Exi^i_{add(T)}(V, -) = 0$ for all $i > 0\}$, and $\mathcal{C} = {}^{\perp}T \bigcap T^{\perp}$. If $\mathcal{C}$ is closed by subobjects, then $K^{+,b}(\mathcal{C})$ is triangle equivalent to $D^{+,b}(\mathcal{A})$.*

Proof. The result easily follows from Remark 1.2 and Theorem 3.4. □

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