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NON-CENTRAL COMPLEX MATRIX-VARIATE BETA DISTRIBUTION

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Abstract

Let $X_i \sim CW_m(n_i, \Sigma, \Theta_i)$, where $\Theta_i = \text{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, 2$. In this article, the distribution of $U = T^{-1}X_1(T^H)^{-1}$, where $X_1 + X_2 = TT^H$ and T is a complex lower triangular matrix with positive diagonal elements has been derived. The distribution of U is a non-central complex matrix-variate beta distribution. Several properties of this distribution have also been studied.

1. Introduction

Let X be an $m \times m$ random Hermitian positive definite matrix such that all its eigenvalues are in the open interval $(0, 1)$. Then, X is said to have a complex matrix-variate beta distribution with parameters (a, b),

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denoted as $X \sim CB_m^I(a, b)$, if its p.d.f. is given by

$$\frac{\widetilde{\Gamma}_m(a+b)}{\widetilde{\Gamma}_m(a)\widetilde{\Gamma}_m(b)} \det(X)^{a-m} \det(I_m - X)^{b-m}, \tag{1.1}$$

where $a > m-1$, $b > m-1$ and

$$\widetilde{\Gamma}(a) = \pi^{m(m-1)/2} \prod_{i=1}^{m} \Gamma(a-i+1), \ \mathrm{Re}(a) > m-1. \tag{1.2}$$

The complex matrix-variate beta distribution can be derived by using independent Wishart matrices. Let X_1 and X_2 be independent Hermitian positive definite random matrices of order m. Define the transformation $X_1 + X_2 = TT^H$ and $U = T^{-1}X_1(T^H)^{-1}$, where the complex random matrix T is lower triangular with positive diagonal elements. If $X_i \sim CW_m(n_i, \Sigma)$, $i = 1, 2$, then $U \sim CB_m^I(n_1, n_2)$. Further, if $X_2 \sim CW_m(n_2, \Sigma, \Theta)$ and $X_1 \sim W_m(n_1, \Sigma)$, then U follows a non-central complex matrix-variate beta distribution (Khatri [5], Gupta [1] and Gupta and Nagar [2, 3]).

The complex matrix-variate beta distribution arises in various problems in multivariate statistical analysis. Several test statistics in multivariate analysis of variance and covariance are functions of the beta matrix. The complex matrix-variate distributions play an important role in various fields of research. Applications of complex random matrices can be found in multiple time series analysis, nuclear physics and radio communications. A number of results on the distribution of the complex random matrices have also been derived. Distributional results on Gaussian, Wishart, beta, and Dirichlet can be found in Tan [10]. Recently, Nagar and Arias [8] have studied the complex matrix-variate Cauchy distribution.

In this article we will study the distribution of U when both X_1 and X_2 have complex non-central Wishart distribution of the rank one.

2. Some Useful Results

In this section we give definitions and results which will be used in the subsequent sections. Throughout this work we will use the Pochammer symbol $(a)_n$ defined by $(a)_n = a(a+1)\cdots(a+n-1) = (a)_{n-1}(a+n-1)$ for $n = 1, 2, \ldots,$ and $(a)_0 = 1$.

The generalized hypergeometric function of scalar argument is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}, \tag{2.1} $$

where a_i, $i = 1, \ldots, p$; b_j, $j = 1, \ldots, q$ are complex numbers with suitable restrictions and z is a complex variable. Conditions for the convergence of the series in (2.1) are available in the literature, see Luke [6]. From (2.1) it is easy to see that

$$ {}_0F_1(b; x) = \sum_{k=0}^{\infty} \frac{x^k}{(b)_k k!} \text{ and } {}_1F_1(a; b; x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{x^k}{k!}. $$

The Humbert's confluent hypergeometric function in z_1 and z_2 is defined by

$$ \mathbf{\Psi}_2[a; c_1, c_2; z_1, z_2] = \sum_{j_1, j_2 = 0}^{\infty} \frac{(a)_{j_1+j_2} z_1^{j_1} z_2^{j_2}}{(c_1)_{j_1} (c_2)_{j_2} j_1! \, j_2!}, \tag{2.2} $$

where the series expansion is valid for all $z_1 \in \mathbb{R}$ and $z_2 \in \mathbb{R}$. Using the results

$$ (a)_j = \frac{\Gamma(a+j)}{\Gamma(a)} = \frac{1}{\Gamma(a)} \int_0^{\infty} \exp(-t) t^{a+j-1} dt, \ \mathrm{Re}(a) > 0, $$

for $j = 0, 1, 2, \ldots,$ and

$$ \sum_{j_i=0}^{\infty} \frac{(tz_i)^{j_i}}{(c_i)_{j_i} j_i!} = {}_0F_1(c_i; tz_i) $$

in (2.2), one obtains

$$\mathbf{\Psi}_2[a; c_1, c_2; z_1, z_2] = \frac{1}{\Gamma(a)}\int_0^\infty \exp(-t)t^{a-1}\, {}_0F_1(c_1; tz_1)\, {}_0F_1(c_2; tz_2)dt \tag{2.3}$$

and

$$\begin{aligned}\mathbf{\Psi}_2[a; c_1, c_2; z_1, z_2] &= \sum_{j_1=0}^\infty \frac{(a)_{j_1} z_1^{j_1}}{(c_1)_{j_1} j_1!} {}_1F_1(a + j_1; c_2; z_2) \\ &= \sum_{j_2=0}^\infty \frac{(a)_{j_2} z_2^{j_2}}{(c_2)_{j_2} j_2!} {}_1F_1(a + j_2; c_1; z_1). \end{aligned}\tag{2.4}$$

For further results and properties of $\mathbf{\Psi}_2$ the reader is referred to Srivastava and Karlsson [9]. Next we will give a result which has been used in Section 4 to derive moment expression.

Lemma 2.1. *For* $\text{Re}(\alpha) > 0$ *and* $\text{Re}(\beta) > 0$,

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\mathbf{\Psi}_2[a; c, \beta; \delta_1 x, \delta_2(1-x)]dx$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\sum_{j=0}^\infty \frac{(a)_j(\alpha)_j\delta_1^j}{(c)_j(\alpha+\beta)_j j!} {}_1F_1(a+j; \alpha+\beta+j; \delta_2). \tag{2.5}$$

Proof. Expanding Ψ_2 using (2.4) and integrating x with the help of beta integral we get the desired result.

Next we state the following notations and results that will be utilized in this and subsequent sections. Let $A = (a_{ij})$ be an $m \times m$ matrix of complex numbers. Then, A' denotes the transpose of A; $\overline{A}$ denotes conjugate of A; A^H denotes conjugate transpose of A; $\text{tr}(A) = a_{11} + \cdots + a_{mm}$; $\text{etr}(A) = \exp(\text{tr}(A))$; $\det(A)$ = determinant of A; $\det(A)_+$ = absolute value of $\det(A)$; $A = A^H > 0$ means that A is Hermitian positive definite and $A^{1/2}$ denotes the unique Hermitian positive definite square root of $A = A^H > 0$. Further, for the partition $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}$, $\det(A_{11}) \neq 0$, the Schur compliment is defined as $A_{22\cdot 1} = A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Lemma 2.2. *Let* $Z(m \times m)$ *and* $W(m \times m)$ *be Hermitian positive definite matrices of functionally independent complex variables and let* $G(m \times m)$ *be a complex nonsingular matrix. The Jacobian of the transformation* $Z = GWG^H$ *is* $J(Z \to W) = \det(G)_+^{2m}$.

Lemma 2.3. *Let X be a Hermitian positive definite matrix. Let T be a complex lower triangular matrix with real positive diagonal elements. If* $X = TT^H$, *then* $J(X \to T) = 2^m \prod_{j=1}^m t_{jj}^{2(m-j)+1}$.

The complex multivariate gamma function, denoted by $\widetilde{\Gamma}_m(a)$, is defined as

$$\widetilde{\Gamma}_m(a) = \int_{A=A^H>0} \operatorname{etr}(-A)\det(A)^{a-m}\, dA, \tag{2.6}$$

where $\operatorname{Re}(a) > m-1$, and the integral is over the space of $m \times m$ Hermitian positive definite matrices. By evaluating the integral in (2.6), the multivariate gamma function can be expressed as product of ordinary gamma functions given by (1.2). The complex multivariate beta function, denoted by $\widetilde{B}_m(a, b)$, is defined by

$$\widetilde{B}_m(a, b) = \int_{0<X=X^H<I_m} \det(X)^{a-m} \det(I_m - X)^{b-m}\, dX, \tag{2.7}$$

where $\operatorname{Re}(a) > m-1$ and $\operatorname{Re}(b) > m-1$. The complex multivariate beta function $\widetilde{B}_m(a, b)$ can be expressed in terms of complex multivariate gamma functions as

$$\widetilde{B}_m(a, b) = \frac{\widetilde{\Gamma}_m(a)\widetilde{\Gamma}_m(b)}{\widetilde{\Gamma}_m(a+b)} = \widetilde{B}_m(b, a). \tag{2.8}$$

Several generalizations of the complex multivariate gamma function and complex multivariate beta function are available in Nagar, Gupta and Sánchez [7]. Finally, we define the Wishart and the non-central Wishart distributions and state some of their properties.

Definition 2.1. An $m \times m$ Hermitian positive definite random matrix S is said to have a complex Wishart distribution with parameters

m, $n(\geq m)$ and $\Sigma = \Sigma^H > 0$, denoted by $S \sim CW_m(n, \Sigma)$, if its p.d.f. is given by

$$\{\widetilde{\Gamma}_m(n)\det(\Sigma)^n\}^{-1}\operatorname{etr}(-\Sigma^{-1}S)\det(S)^{n-m}, \quad S = S^H > 0. \tag{2.9}$$

Definition 2.2. An $m \times m$ Hermitian positive definite random matrix S is said to have a *non-central complex Wishart distribution* with parameters m, $n(\geq m)$, $\Sigma = \Sigma^H > 0$ and Θ, denoted by $S \sim CW_m(n, \Sigma, \Theta)$, if its p.d.f. is given by

$$\{\widetilde{\Gamma}_m(n)\det(\Sigma)^n\}^{-1}\operatorname{etr}(-\Theta)\operatorname{etr}(-\Sigma^{-1}S)\det(S)^{n-m}\,{}_0\widetilde{F}_1(n;\,\Theta\Sigma^{-1}S),$$

$$S = S^H > 0,$$

where ${}_0\widetilde{F}_1$ is the Bessel function of the Hermitian matrix argument.

For $\Theta = 0$, the non-central complex Wishart distribution reduces to a complex Wishart distribution. Further, when $\Sigma = I_m$ and $\Theta = \operatorname{diag}(\theta^2, 0, \ldots, 0)$, the p.d.f. of $S = (s_{ij})$ simplifies to

$$\{\widetilde{\Gamma}_m(n)\}^{-1}\exp[-(\theta^2 + \operatorname{tr}S)]\det(S)^{n-m}\,{}_0F_1(n;\,\theta^2 s_{11}), \tag{2.10}$$

where $S = S^H > 0$, $n \geq m$ and ${}_0F_1$ is the Bessel function of the scalar argument.

Definition 2.3. The complex random matrix $T(p \times m)$ is said to have a *complex inverted matrix-variate t-distribution* with parameters $M \in \mathbb{C}^{p\times m}$, $\Sigma(p \times p)$, and $\Omega(m \times m)$, if its p.d.f. is given by

$$\frac{\widetilde{\Gamma}_p(n+m+p-1)}{\pi^{mp}\widetilde{\Gamma}_p(n+p-1)}\det(\Sigma)^{-m}\det(\Omega)^{-p}$$

$$\det(I_p - \Sigma^{-1}(T-M)\Omega^{-1}(T-M)^H)^{n-1}, \; T \in \mathbb{C}^{p\times m}, \tag{2.11}$$

where $I_p - \Sigma^{-1}(T-M)\Omega^{-1}(T-M)^H$, Σ and Ω are Hermitian positive definite matrices.

We shall denote this by $T \sim CIT_{p,m}(n, M, \Sigma, \Omega)$.

Definition 2.4. The doubly non-central beta distribution, denoted by $u \sim B^I(n_1, n_2; \theta_1^2, \theta_2^2)$, is defined by the p.d.f.

$$\frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \exp(-\theta_1^2 - \theta_2^2) u^{n_1 - 1}(1 - u)^{n_2 - 1}$$

$$\times \mathbf{\Psi}_2[n_1 + n_2; n_1, n_2; \theta_1^2 u, \theta_2^2(1 - u)], \ 0 < u < 1.$$

Note that for $\theta_1^2 = \theta_2^2 = 0$, the above distribution slides to a univariate beta distribution with the density function

$$\frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} u^{n_1 - 1}(1 - u)^{n_2 - 1}, \ 0 < u < 1$$

which will be denoted by $u \sim B^I(n_1, n_2)$.

3. Non-central Complex Matrix-variate Beta Distribution

In this section we will derive the probability density function of U when both X_1 and X_2 have non-central complex Wishart distribution of rank one.

Theorem 3.1. *Let* X_1 *and* X_2 *be independent Hermitian positive definite random matrices,* $X_i \sim CW_m(n_i, \Sigma, \Theta_i)$ *where* $\Theta_i = \mathrm{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, 2$. *Define* $X_1 + X_2 = TT^H$ *and* $X_1 = TUT^H$, *where the complex matrix* T *is lower triangular with positive diagonal elements. Then, the p.d.f. of* U *is given by*

$$\{\widetilde{B}_m(n_1, n_2)\}^{-1} \exp[-(\theta_1^2 + \theta_2^2)] \det(U)^{n_1 - m} \det(I_m - U)^{n_2 - m}$$

$$\times \mathbf{\Psi}_2[n_1 + n_2; n_1, n_2; \theta_1^2 u_{11}, \theta_2^2(1 - u_{11})], \ 0 < U = U^H < I_m,$$

where $U = (u_{\alpha\beta})$, *and* $\mathbf{\Psi}_2$ *is the Humbert's confluent hypergeometric function.*

Proof. Note that the complex random matrix U is invariant under the transformation $X_1 \to \Sigma^{-1/2} X_1 (\Sigma^{-1/2})^H$, $X_2 \to \Sigma^{-1/2} X_2 (\Sigma^{-1/2})^H$, where $\Sigma^{-1/2}$ is a lower triangular matrix such that $\Sigma^{1/2}(\Sigma^{1/2})^H = \Sigma$. Hence, without loss of generality, we can assume that $\Sigma = I_m$, that is, $X_i \sim CW_m(n_i, I_m, \Theta_i)$. Using the independence and (2.10) the joint p.d.f. of X_1 and X_2 is given by

$$\{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_2)\}^{-1}\exp[-(\theta_1^2+\theta_2^2)]\operatorname{etr}[-(X_1+X_2)]$$

$$\times \det(X_1)^{n_1-m}\det(X_2)^{n_2-m}\,{}_0F_1(n_1;\,\theta_1^2 x_{111})\,{}_0F_1(n_2;\,\theta_2^2 x_{211}).$$

Let $X_1 + X_2 = TT^H$, and $X_1 = TUT^H$, where $T = (t_{ij})$ is a complex lower triangular matrix, $t_{ij} > 0$. Then, using Lemmas 2.2 and 2.3, the Jacobian of the transformation is given by $J(X_1, X_2 \to U, T) = 2^m \prod_{i=1}^m t_{ii}^{2(m+1)-i}$. Now making these substitutions and integrating with respect to t_{ij}, $1 \le j \le i \le m$, we get the density of U as

$$2^m \frac{\exp[-(\theta_1^2+\theta_2^2)]}{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_2)}\det(U)^{n_1-m}\det(I_m-U)^{n_2-m} I_1 I_2 \prod_{i=2}^m I_{3i}, \qquad (3.1)$$

where $0 < U = U^H < I_m$. Further, using results on integration and (2.3), it is easy to see that

$$I_1 = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(-\sum_{i>j}^m t_{ij}^2\right)\prod_{i>j}^m dt_{ij} = \pi^{m(m-1)/2},$$

$$I_2 = \int_0^\infty \exp(-t_{11}^2)t_{11}^{n_1+n_2-1}\,{}_0F_1(n_1;\,\theta_1^2 t_{11}^2 u_{111})\,{}_0F_1(n_2;\,\theta_2^2 t_{11}^2(1-u_{111}))dt_{11}$$

$$= 2^{n_1+n_2-1}\Gamma(n_1+n_2)\boldsymbol{\Psi}_2[n_1+n_2;\,n_1,\,n_2;\,\theta_1^2 u_{11},\,\theta_2^2(1-u_{11})],$$

and

$$I_{3i} = \int_0^\infty \exp(-t_{ii}^2)t_{ii}^{n_1+n_2-i}dt_{ii} = 2^{n_1+n_2-i-1}\Gamma(n_1+n_2-i+1).$$

Finally, substituting I_1, I_2 and I_{3i} in (3.1) and simplifying the resulting expression, we obtain the desired result.

The above distribution is designated by $U \sim CB_m^I(n_1, n_2; \theta_1^2, \theta_2^2)$. The distribution of U, in this case, is called *doubly non-central complex matrix-variate beta distribution*. The above density, using (2.4), can also be written as

$$\{\widetilde{B}_m(n_1, n_2)\}^{-1}\exp[-(\theta_1^2 + \theta_2^2)]\det(U)^{n_1-m}\det(I_m - U)^{n_2-m}$$

$$\times \sum_{j=0}^{\infty} \frac{(n_1 + n_2)_j (\theta_1^2 u_{11})^j}{(n_1)_j\, j!} {}_1F_1(n_1 + n_2 + j; n_2; \theta_2^2(1 - u_{11})).$$

Corollary 3.1.1. *Let the random matrices* X_1 *and* X_2 *be independent,* $X_1 \sim CW_m(n_1, \Sigma)$ *and* $X_2 \sim CW_m(n_2, \Sigma, \Theta)$, *where* $\Theta = \mathrm{diag}(\theta^2, 0, \ldots, 0)$. *Define* $X_1 + X_2 = TT^H$ *and* $X_1 = TUT^H$, *where the complex matrix T is lower triangular with positive diagonal elements. Then, the p.d.f. of* $U = (u_{\alpha\beta})$ *is given by*

$$\{\widetilde{B}_m(n_1, n_2)\}^{-1}\exp(-\theta^2)\det(U)^{n_1-m}\det(I_m - U)^{n_2-m}$$

$$\times {}_1F_1(n_1 + n_2; n_2; \theta^2(1 - u_{11})),\ 0 < U = U^H < I_m.$$

The above distribution, designated by $U \sim CB_m^I(n_1, n_2; \theta^2)$, is called the *linear non-central complex matrix-variate beta distribution*. The density function of U given above was first derived by Khatri [5], also see Gupta [1] and James [4].

Corollary 3.1.2. *Let the* $m \times m$ *independent random matrices* X_1 *and* X_2 *have complex Wishart distribution,* $X_i \sim CW_m(n_i, \Sigma)$, $i = 1, 2$. *Define* $X_1 + X_2 = TT^H$ *and* $X_1 = TUT^H$, *where the complex matrix T is lower triangular with positive diagonal elements. Then,* $U \sim CB_m^I(n_1, n_2)$.

4. Properties

In this section we will study certain properties of the non-central complex matrix-variate beta distribution derived in the last section.

Theorem 4.1. *Let* $U \sim CB_m^I(n_1, n_2; \theta_1^2; \theta_2^2)$. *Partition the matrix* U *as* $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, $U_{11}(q \times q)$. *Then,* U_{11} *and* $U_{22\cdot 1}$ *are independent,* $U_{11} \sim CB_q(n_1, n_2; \theta_1^2, \theta_2^2)$, $U_{22\cdot 1} \sim CB_{m-q}(n_1 - q, n_2)$ *and* $T \mid (U_{11}, U_{22\cdot 1}) \sim CIT_{m-q,q}(n_2 - m + 1, 0, I_{m-q} - U_{22\cdot 1}, I_q - U_{11})$.

Proof. From the partition of U, we have

$$\det(U) = \det(U_{11})\det(U_{22\cdot 1}) \tag{4.1}$$

and

$$\det(I_m - U) = \det(I_q - U_{11})\det(I_{m-q} - U_{22\cdot 1} - U_{21}U_{11}^{-1}(I_q - U_{11})^{-1}U_{12}). \tag{4.2}$$

Making the transformation $U_{11} = U_{11}$, $T = U_{21}U_{11}^{-1/2}$ and $U_{22\cdot 1} = U_{22} - U_{21}U_{11}^{-1}U_{12} = U_{22} - TT^H$ with Jacobian $J(U_{11}, U_{22}, U_{21} \to U_{11}, U_{22\cdot 1}, T) = \det(U_{11})^{m-q}$ and substituting (4.1) and (4.2) in the density of U, we get the joint density of $U_{11}, U_{22\cdot 1}$, and T as

$$\frac{\tilde{\Gamma}_q(n_1 + n_2)\exp[-(\theta_1^2 + \theta_2^2)]}{\tilde{\Gamma}_q(n_1)\tilde{\Gamma}_q(n_2)} \det(U_{11})^{n_1 - q} \det(I_q - U_{11})^{n_2 - q}$$

$$\times \mathbf{\Psi}_2[n_1 + n_2; n_1, n_2; \theta_1^2 u_{11}, \theta_2^2(1 - u_{11})]$$

$$\times \frac{\tilde{\Gamma}_{m-q}(n_1 + n_2 - q)}{\tilde{\Gamma}_{m-q}(n_1 - q)\tilde{\Gamma}_{m-q}(n_2)} \det(U_{22\cdot 1})^{n_1 - m} \det(I_{m-q} - U_{22\cdot 1})^{n_2 - (m-q)}$$

$$\times \frac{\tilde{\Gamma}_{m-q}(n_2)}{\pi^{q(m-q)}\tilde{\Gamma}_{m-q}(n_2 - q)} \det(I_q - U_{11})^{q-m} \det(I_{m-q} - U_{22\cdot 1})^{-q}$$

$$\times \det(I_{m-q} - (I_{m-q} - U_{22\cdot 1})^{-1} T (I_q - U_{11})^{-1} T^H)^{n_2 - m}. \tag{4.3}$$

Now, from (4.3) it is clear that U_{11} and $U_{22\cdot 1}$ are independent, $U_{11} \sim CB_q(n_1, n_2; \theta_1^2, \theta_2^2)$, $U_{22\cdot 1} \sim CB_{m-q}(n_1 - q, n_2)$ and $T \mid (U_{11}, U_{22\cdot 1}) \sim CIT_{m-q,q}(n_2 - m + 1, 0, I_{m-q} - U_{22\cdot 1}, I_q - U_{11})$.

Corollary 4.1.1. *Let* $U \sim CB_m^I(n_1, n_2)$. *Partition the matrix* U *as* $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, $U_{11}(q \times q)$. *Then,* U_{11} *and* $U_{22\cdot 1}$ *are independent,* $U_{11} \sim CB_q(n_1, n_2)$, $U_{22\cdot 1} \sim CB_{m-q}(n_1 - q, n_2)$ *and* $T \mid (U_{11}, U_{22\cdot 1}) \sim CIT_{m-q,q}(n_2 - m + 1, 0, I_{m-q} - U_{22\cdot 1}, I_q - U_{11})$.

Corollary 4.1.2. *Let* u_{11} *be the first element on the principal diagonal of U. Then,* $u_{11} \sim B^I(n_1, n_2; \theta_1^2, \theta_2^2)$.

Theorem 4.2. *Let* $U \sim CB_m^I(n_1, n_2; \theta_1^2, \theta_2^2)$ *and* $U = TT^H$, *where* $T = (t_{ij})$ *is a complex lower triangular matrix with* $t_{ii} > 0$, $i = 1, \ldots, m$. *Partition T as*

$$T = \begin{pmatrix} T_{11} & \mathbf{0} \\ \boldsymbol{t}^H & t_{mm} \end{pmatrix}. \tag{4.4}$$

Then t_{mm}, $\boldsymbol{y} = (1 - t_{mm}^2)^{-1/2}(I_{m-1} - T_{11}T_{11}^H)^{-1/2}\boldsymbol{t}$ *and* T_{11} *are independently distributed,* $t_{mm}^2 \sim B^I(n_1 - m + 1, n_2)$, $\boldsymbol{y} \sim CIt_{m-1}(n_2 - m + 1, 0, I_{m-1}, 1)$ *and the distribution of* T_{11} *is same as that of T with m replaced by* $m - 1$.

Proof. Making the transformation $U = TT^H$, with the Jacobian of the transformation $J(X \to T) = 2^m \prod_{i=1}^m t_{ii}^{2(m-i)+1}$ in the density of U, we get the p.d.f. of T as

$$2^m \{\widetilde{B}_m(n_1, n_2)\}^{-1} \exp(-\theta_1^2 - \theta_2^2) \prod_{i=1}^m (t_{ii}^2)^{n_1 - i + 1/2}$$

$$\times \det(I_m - TT^H)^{n_2 - m} \boldsymbol{\Psi}_2[n_1 + n_2; n_1, n_2; \theta_1^2 t_{11}^2, \theta_2^2(1 - t_{11}^2)], \tag{4.5}$$

where $-\infty < t_{ij} < \infty$, $i > j$, $i, j = 1, \ldots, m$, and $t_{ii} > 0$, $i = 1, \ldots, m$. From (4.4), we have

$$\begin{aligned}\det(I_m - TT^H) &= \det\begin{pmatrix} I_{m-1} - T_{11}T_{11}^H & -T_{11}\boldsymbol{t} \\ -\boldsymbol{t}^H T_{11}^H & 1 - t_{mm}^2 \end{pmatrix} \\ &= (1 - t_{mm}^2)\det(I_{m-1} - T_{11}T_{11}^H) \\ &\quad \times (1 - (1 - t_{mm}^2)^{-1}\boldsymbol{t}^H(I_{m-1} - T_{11}T_{11}^H)^{-1}\boldsymbol{t}). \end{aligned} \tag{4.6}$$

Substituting (4.6) in (4.5) we get the joint density of T_{11}, $\boldsymbol{t}$ and t_{mm} as

$$f(T_{11}, \boldsymbol{t}, t_{mm}) = f_1(T_{11}) f_2(t_{mm}) f_3(\boldsymbol{t} \mid T_{11}, t_{mm}),$$

where

$$f_1(T_{11}) = 2^{m-1}\{\widetilde{B}_{m-1}(n_1, n_2)\}^{-1} \exp(-\theta_1^2 - \theta_2^2) \prod_{i=1}^{m-1} (t_{ii}^2)^{n_1 - i + 1/2}$$

$$\times \det(I_{m-1} - T_{11}T_{11}^H)^{n_2 - m + 1}$$

$$\boldsymbol{\Psi}_2[n_1 + n_2; n_1, n_2; \theta_1^2 t_{11}^2, \theta_2^2(1 - t_{11}^2)], \tag{4.7}$$

$$f_2(t_{mm}) = 2\frac{\Gamma(n_1 + n_2 - m + 1)}{\Gamma(n_1 - m + 1)\Gamma(n_2)} (t_{mm}^2)^{n_1 - m + 1/2} (1 - t_{mm}^2)^{n_2 - 1} \tag{4.8}$$

and

$$f_3(\boldsymbol{t} \mid T_{11}, t_{mm}) = \frac{\pi^{-(m-1)}\Gamma(n_2)}{\Gamma[n_2 - (m-1)]} (1 - t_{mm}^2)^{-(m-1)} \det(I_{m-1} - T_{11}T_{11}^H)^{-1}$$

$$\times (1 - (1 - t_{mm}^2)^{-1} \boldsymbol{t}^H (I_{m-1} - T_{11}T_{11}^H)^{-1} \boldsymbol{t})^{n_2 - m}. \tag{4.9}$$

Now transforming $x_m = t_{mm}^2$ and $\boldsymbol{y} = (1 - t_{mm}^2)^{-1/2}(I_{m-1} - T_{11}T_{11}^H)^{-1/2}\boldsymbol{t}$, with the Jacobian $J(\boldsymbol{t}, t_{mm} \to \boldsymbol{y}, x_m) = (2x_m^{1/2})^{-1}(1 - x_m)^{m-1} \det(I_{m-1} - T_{11}T_{11}^H)$, we get the desired result.

Theorem 4.3. *Let* $U \sim CB_m^I(n_1, n_2; \theta_1^2, \theta_2^2)$ *and* $U = TT^H$, *where* $T = (t_{ij})$ *is a complex lower triangular matrix with* $t_{ii} > 0$, $i = 1, \ldots, m$. *Then,* $t_{11}^2, \ldots, t_{mm}^2$ *are independently distributed,* $t_{ii}^2 \sim B^I(n_1 - i + 1, n_2)$, $i = 2, \ldots, m$ *and* $t_{11}^2 \sim B^I(n_1, n_2; \theta_1^2, \theta_2^2)$.

Proof. From Theorem 4.2, it is known that $t_{mm}^2 \sim B^I(n_1 - m + 1, n_2)$ and is independent of T_{11}, which has the same distribution as T with m replaced by $m - 1$. Further partitioning T_{11} yields $t_{m-1, m-1}^2 \sim B^I(n_1 - m + 2, n_2)$. Repeated application of this procedure completes the proof of the theorem.

Corollary 4.3.1. *Let* $U \sim CB_m^I(n_1, n_2)$ *and* $U = TT^H$, *where* $T = (t_{ij})$ *is a complex lower triangular matrix with* $t_{ii} > 0$, $i = 1, \ldots, m$. *Then* $t_{11}^2, \ldots, t_{mm}^2$ *are independently distributed,* $t_{ii}^2 \sim B^I(n_1 - i + 1, n_2)$, $i = 1, \ldots, m$.

Now, using Theorems 3.1 and 4.3, we obtain the following result.

Theorem 4.4. *Let* X_1 *and* X_2 *be independent Hermitian positive definite random matrices,* $X_i \sim CW_m(n_i, \Sigma, \Theta_i)$, *where* $\Theta_i = \operatorname{diag}(\theta_i^2, 0, \ldots, 0)$, $i = 1, 2$. *Then the distribution of* $\Lambda = \det(X_1)/\det(X_1 + X_2)$ *is the distribution of* $\prod_{i=1}^m v_i$, *where* $v_1, \ldots, v_m$ *are independent real beta variates with density functions given by*

$$\frac{\Gamma(n_1 + n_2 - i + 1)}{\Gamma(n_2)\Gamma(n_1 - i + 1)} v_i^{n_1 - i}(1 - v_i)^{n_2 - 1}, \; 0 < v_i < 1, \tag{4.10}$$

for $i = 2, 3, \ldots, m$ *and*

$$\frac{\Gamma(n_1 + n_2)}{\Gamma(n_2)\Gamma(n_1)} \exp[-(\theta_1^2 + \theta_2^2)] v_1^{n_1 - 1}(1 - v_1)^{n_2 - 1}$$

$$\times \Psi_2[n_1 + n_2; n_1, n_2; \theta_1^2 v_1, \theta_2^2(1 - v_1)], \; 0 < v_1 < 1,$$

where Ψ_2 *is the Humbert's confluent hypergeometric function.*

Theorem 4.5. *If* $U \sim CB_m^I(n_1, n_2; \theta_1^2, \theta_2^2)$, *then*

$$E[\det(U)^h] = \frac{\widetilde{\Gamma}_m(n_1 + n_2)\widetilde{\Gamma}_m(n_1 + h)}{\widetilde{\Gamma}_m(n_1)\widetilde{\Gamma}_m(n_1 + n_2 + h)} \exp[-(\theta_1^2 + \theta_2^2)]$$

$$\times \sum_{j=0}^{\infty} \frac{(n_1 + n_2)_j (n_1 + h)_j \theta_1^{2j}}{(n_1)_j (n_1 + n_2 + h)_j \, j!}$$

$$\times {}_1F_1(n_1 + n_2 + j; n_1 + n_2 + h + j; \theta_2^2),$$

where ${}_1F_1$ *is the confluent hypergeometric function.*

Proof. From the p.d.f. of U given in Theorem 3.1, we have

$$E[\det(U)^h]$$

$$= \frac{\tilde{\Gamma}_m(n_1+n_2)\tilde{\Gamma}_m(n_1+h)}{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_1+n_2+h)}\exp[-(\theta_1^2+\theta_2^2)]$$

$$\times \int_{0<U=U^H<I_m}\left[\frac{\tilde{\Gamma}_m(n_1+n_2+h)}{\tilde{\Gamma}_m(n_1+h)\tilde{\Gamma}_m(n_2)}\det(U)^{n_1+h-m}\det(I_m-U)^{n_2-m}\right]$$

$$\times \mathbf{\Psi}_2[n_1+n_2;\, n_1,\, n_2;\, \theta_1^2 u_{11},\, \theta_2^2(1-u_{11})]dU.$$

The first factor in the above integral (terms in brackets) is the complex matrix-variate beta density with parameters n_1+h and n_2. The second factor involves only u_{11}. Thus, integrating terms in the bracket over all the elements of U, except the first element, we obtain

$$E[\det(U)^h] = \frac{\tilde{\Gamma}_m(n_1+n_2)\tilde{\Gamma}_m(n_1+h)}{\tilde{\Gamma}_m(n_1)\tilde{\Gamma}_m(n_1+n_2+h)}\exp[-(\theta_1^2+\theta_2^2)]$$

$$\times \frac{\Gamma(n_1+n_2+h)}{\Gamma(n_1+h)\Gamma(n_2)}\int_{0<u_{11}<1} u_{11}^{n_1+h-1}(1-u_{11})^{n_2-1}$$

$$\times \mathbf{\Psi}_2[n_1+n_2;\, n_1,\, n_2;\, \theta_1^2 u_{11},\, \theta_2^2(1-u_{11})]du_{11}.$$

Now, evaluation of the above integral using (2.5) yields the desired result.

By noting that $I_m - U \sim CB_m^I(n_2,\, n_1;\, \theta_2^2,\, \theta_1^2)$, and using the above theorem, we obtain

$$E[\det(I_m-U)^h] = \frac{\tilde{\Gamma}_m(n_1+n_2)\tilde{\Gamma}_m(n_2+h)}{\tilde{\Gamma}_m(n_2)\tilde{\Gamma}_m(n_1+n_2+h)}\exp[-(\theta_1^2+\theta_2^2)]$$

$$\times \sum_{s=0}^{\infty}\frac{(n_1+n_2)_s(n_2+h)_s(\theta_2^2)^s}{(n_1+n_2+h)_s(n_2)_s\, s!}$$

$$\times {}_1F_1(n_1+n_2+s;\, n_1+n_2+h+s;\, \theta_1^2).$$

In the next theorem we give asymptotic expansion for the non-central matrix-variate beta distribution derived in Section 3.

Theorem 4.6. *Let* $U \sim CB_m^I(n_1, n_2; \theta_1^2; \theta_2^2)$ *and* $W = n_2 U$. *Then* W *is asymptotically distributed as a non-central complex Wishart; more specifically*

$$\lim_{n_2 \to \infty} f(W) = \frac{\exp(-\theta_1^2 - \operatorname{tr} W) \det(W)^{n_1 - m} {}_0F_1(n_1; \theta_1^2 w_{11})}{\widetilde{\Gamma}_m(n_1)},$$

where $f(W)$ *denotes the density of the matrix* W.

Proof. Transforming $W = n_2 U$, with Jacobian $J(U \to W) = n_2^{-m^2}$ in the density of U given in Theorem 3.1, the density $f(W)$ of W is obtained as

$$\frac{\widetilde{\Gamma}_m(n_1 + n_2)}{\widetilde{\Gamma}_m(n_2)} \exp[-(\theta_1^2 + \theta_2^2)] n_2^{-mn_1} \left\{ \frac{\det(W)^{n_1 - m}}{\widetilde{\Gamma}_m(n_1)} \right\} \det\left(I_m - \frac{1}{n_2} W \right)^{n_2 - m}$$

$$\times \Psi_2\left[n_1 + n_2; n_1, n_2; \frac{\theta_1^2 w_{11}}{n_2}, \theta_2^2\left(1 - \frac{w_{11}}{n_2}\right)\right],$$

where $W_i = (w_{\alpha\beta})$. Now, using

$$\lim_{n_2 \to \infty} \frac{\widetilde{\Gamma}_m(n_1 + n_2)}{\widetilde{\Gamma}_m(n_2)} n_2^{-mn_1} = 1,$$

$$\lim_{n_2 \to \infty} \det\left(I_m - \frac{W}{n_2} \right)^{n_2 - m} = \operatorname{etr}(-W),$$

$$\lim_{n_2 \to \infty} \Psi_2\left[n_1 + n_2; n_1, n_2; \frac{\theta_1^2 w_{11}}{n_2}, \theta_2^2\left(1 - \frac{w_{11}}{n_2}\right)\right] = \exp(\theta_2^2) {}_0F_1(n_1; \theta_1^2 w_{11}),$$

we obtain the desired result.

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