PROPERTIES OF THE HYPERGEOMETRIC FUNCTION TYPE I DISTRIBUTION

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Abstract

The hypergeometric function type I distribution with the pdf proportional to $x^{\nu-1}(1-x)^{\gamma-1} {}_2F_1(\alpha, \beta; \gamma; 1-x)$ occurs as the distribution of the product of two independent beta variables. In this article, we study several properties and stochastic representations of this distribution.

1. Introduction

The random variable X is said to have beta distribution, denoted by $X \sim B(a, b)$, if its probability density function (pdf) is given by

$$\{B(a, b)\}^{-1} x^{a-1}(1-x)^{b-1}, \quad 0 < x < 1, \tag{1}$$

where $a > 0$, $b > 0$, and $B(a, b)$ is beta function given by

$$B(a, b) = \Gamma(a)\Gamma(b)\{\Gamma(a+b)\}^{-1}.$$

Beta distribution is widely used in statistical modeling of bounded random variables. Applications of densities of the product and ratio of independent beta variates in the field of reliability can be found in Pham-Gia [8] and Pham-Gia and Turkkan [9]. Several univariate generalizations of this distribution are given in Gordy [1], McDonald and Xu [5], Nagar and Zarrazola [6] and Ng and Kotz [7]. For an extensive review on beta distributions the reader is referred to Johnson et al. [3]. Recently, Gupta and Nagar [2, p. 298] introduced a univariate generalization of (1) involving the Gauss hypergeometric function. Their generalization of the beta distribution has the pdf

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} x^{\nu-1}(1 - x)^{\gamma-1} \, {}_2F_1(\alpha, \beta; \gamma; 1 - x), \quad 0 < x < 1, \quad (2)$$

where $\nu > 0$, $\gamma > 0$, $\gamma + \nu > \alpha + \beta$, and $_2F_1$ is the Gauss hypergeometric function defined by (Luke [4]),

$$_2F_1(a, b; c; z) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{z^r}{r!},$$

where z is a complex variable, a, b and c can take arbitrary real or complex values (provided that $c \neq 0, -1, -2, ...$) and $(a)_n = a(a + 1) \cdots (a + n - 1) = (a)_{n-1}(a + n - 1)$ for $n = 1, 2, ...,$ and $(a)_0 = 1$. If either a or b is zero or a negative integer, the series terminates after a finite number of terms, and its sum is then a polynomial in z. Except for this case, the radius of convergence of the hypergeometric series is 1. We will call the above distribution *hypergeometric function type* I *distribution* and denote it by $H^I(\nu, \alpha, \beta, \gamma)$. This distribution occurs as the distribution of the product of two independent beta variables. For $\alpha = \gamma$, the density (2) reduces to a beta distribution given by

$$\{B(\nu - \beta, \gamma)\}^{-1} x^{\nu-\beta-1}(1 - x)^{\gamma-1}, \quad 0 < x < 1,$$

and for $\beta = \gamma$, the hypergeometric function type I density slides to

$$\{B(\nu - \alpha, \gamma)\}^{-1} x^{\nu-\alpha-1}(1 - x)^{\gamma-1}, \quad 0 < x < 1.$$

In this article, we study several properties and stochastic representations of the hypergeometric function type I distribution. We also define the inverted hypergeometric function type I distribution and derive some of its properties.

2. Properties

In this section we will derive several properties of the hypergeometric type I distribution defined in Section 1. Note that $H^I(\nu, \alpha, \beta, \gamma) \equiv H^I(\nu, \beta, \alpha, \gamma)$. Further, using the result

$$_2F_1(a, b; c; z) = (1 - z)^{c-a-b} \, _2F_1(c - a, c - b; c; z), \tag{3}$$

the pdf of X can also be expressed as

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} x^{\nu+\gamma-\alpha-\beta-1}(1 - x)^{\gamma-1}$$

$$\times \, _2F_1(\gamma - \alpha, \gamma - \beta; \gamma; 1 - x), \quad 0 < x < 1. \tag{4}$$

Hence $H^I(\nu, \alpha, \beta, \gamma) \equiv H^I(\nu, \beta, \alpha, \gamma) \equiv H^I(\nu + \gamma - \alpha - \beta, \gamma - \alpha, \gamma - \beta, \gamma)$ $\equiv H^I(\nu + \gamma - \alpha - \beta, \gamma - \beta, \gamma - \alpha, \gamma)$. Further, using series expansion of $_2F_1$ in (2) and (4), series representations of the hypergeometric type I density are obtained as

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\nu + \gamma)\Gamma(\gamma + \nu - \alpha - \beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\nu + \gamma)_r r!}$$

$$\times \{B(\nu, \gamma + r)\}^{-1} x^{\nu-1}(1 - x)^{\gamma+r-1}, \quad 0 < x < 1$$

and

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\nu)\Gamma(2\gamma + \nu - \alpha - \beta)} \sum_{r=0}^{\infty} \frac{(\gamma - \alpha)_r(\gamma - \beta)_r}{(\nu + 2\gamma - \alpha - \beta)_r r!}$$

$$\times \{B(\nu + \gamma - \alpha - \beta, \gamma + r)\}^{-1} x^{\nu+\gamma-\alpha-\beta-1}(1 - x)^{\gamma+r-1}, \quad 0 < x < 1.$$

That is, the pdf of the hypergeometric type I variable is a mixture of beta densities. From (2) and (4) the pdf of $U = 1 - X$ is derived as

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} u^{\gamma-1}(1-u)^{\nu-1} \, {}_2F_1(\alpha, \beta; \gamma; u), \quad 0 < u < 1,$$

and

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} u^{\gamma-1}(1-u)^{\nu+\gamma-\alpha-\beta-1}$$

$$\times \, {}_2F_1(\gamma - \alpha, \gamma - \beta; \gamma; u), \quad 0 < u < 1.$$

The cumulative distribution function (cdf) of X is obtained as

$$P(X \leq x) = \frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)}$$

$$\times \int_0^x z^{\nu-1}(1-z)^{\gamma-1} \, {}_2F_1(\alpha, \beta; \gamma; 1-z)dz.$$

Expanding $ {}_2F_1(\alpha, \beta; \gamma; 1-z)$ in series form and integrating with respect to z, the above expression is rewritten as

$$P(X \leq x) = \frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!} B_x(\nu, \gamma + r), \qquad (5)$$

where the incomplete beta function $B_x(a, b)$ is defined by

$$B_x(a, b) = \int_0^x y^{a-1}(1-y)^{b-1}dy, \quad 0 < x < 1, \quad a > 0, \quad b > 0.$$

Theorem 2.1. *Let* $X \sim H^I(\nu, \alpha, \beta, \gamma)$. *Then, the moment generating function (mgf)* $M_X(t)$ *of X is given by*

$$M_X(t) = \, {}_2F_2(\nu, \nu + \gamma - \alpha - \beta; \nu + \gamma - \alpha, \nu + \gamma - \beta; t), \qquad (6)$$

where $ {}_2F_2$ *is the generalized hypergeometric function (Luke* [4]).

Proof. By definition

$$M_X(t) = \frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)}$$

$$\times \int_0^1 \exp(tx)x^{\nu-1}(1-x)^{\gamma-1} \, {}_2F_1(\alpha, \beta; \gamma; 1-x)dx. \qquad (7)$$

Expanding $\exp(tx)$ in power series and using the results

$$\int_0^1 x^{a-1}(1-x)^{b-1} {}_2F_1(\alpha, \beta; \gamma; 1-x)dx$$

$$= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} {}_3F_2(b, \alpha, \beta; a+b, \gamma; 1), \tag{8}$$

and

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0, \tag{9}$$

$c \neq 0, -1, -2, \ldots,$ the above integral is evaluated as

$$\int_0^1 \exp(tx) x^{\nu-1}(1-x)^{\gamma-1} {}_2F_1(\alpha, \beta; \gamma; 1-x)dx$$

$$= \frac{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)}{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}$$

$$\times {}_2F_2(\nu, \nu+\gamma-\alpha-\beta; \nu+\gamma-\alpha, \nu+\gamma-\beta; t). \tag{10}$$

Now, substitution of (10) in (7) yields the desired result.

Theorem 2.2. *Let* $X \sim H^I(\nu, \alpha, \beta, \gamma)$. *Then*

$$E[X^r(1-X)^s] = \frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)} \frac{\Gamma(\gamma+s)\Gamma(\nu+r)}{\Gamma(\nu+\gamma+r+s)}$$

$$\times {}_3F_2(\alpha, \beta, \gamma+s; \gamma, \nu+\gamma+r+s; 1), \tag{11}$$

where ${}_3F_2$ *is the generalized hypergeometric function* (*Luke* [4]).

Proof. From the density of *X*, we have

$$E[X^r(1-X)^s] = \frac{\Gamma(\gamma+\nu-\alpha)\Gamma(\gamma+\nu-\beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma+\nu-\alpha-\beta)}$$

$$\times \int_0^1 x^{\nu+r-1}(1-x)^{\gamma+s-1} {}_2F_1(\alpha, \beta; \gamma; 1-x)dx.$$

Now, using (8) and simplifying the resulting expression we get the desired result.

Substituting $s = 0$ in (11) and using (9), the r-th moment of X is obtained as

$$E(X^r) = \frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} \frac{\Gamma(\nu + r)\Gamma(\nu + \gamma + r - \alpha - \beta)}{\Gamma(\nu + \gamma + r - \alpha)\Gamma(\nu + \gamma + r - \beta)}, \quad (12)$$

where $\text{Re}(\nu + \gamma + r) > \alpha + \beta$. Finally, using the above expression, the mean and variance of X are derived as

$$E(X) = \frac{\nu(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)(\nu + \gamma - \beta)},$$

and

$$\text{Var}(X) = \frac{\nu(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)(\nu + \gamma - \beta)}\left[\frac{(\nu + 1)(\nu + \gamma - \alpha - \beta + 1)}{(\nu + \gamma - \alpha + 1)(\nu + \gamma - \beta + 1)}\right.$$

$$\left. - \frac{\nu(\nu + \gamma - \alpha - \beta)}{(\nu + \gamma - \alpha)(\nu + \gamma - \beta)}\right].$$

3. Stochastic Representations

In this section we obtain stochastic representations of the hypergeometric type I variable in terms of beta and inverted beta random variables. First we define inverted beta distribution.

Definition 3.1. The random variable X is said to have an inverted beta distribution with parameters (a, b), denoted as $X \sim IB(a, b)$, $a > 0,\ b > 0$, if its pdf is given by

$$\{B(a, b)\}^{-1} x^{a-1}(1 + x)^{-(a+b)}, \quad x > 0.$$

Theorem 3.1. *Let* X_1 *and* X_2 *be independent,* $X_1 \sim B(a, b)$ *and* $X_2 \sim B(c, d)$. *Then* $X_1 X_2 \sim H^I(c, b, c + d - a, b + d)$ *with the pdf*

$$\frac{\Gamma(a + b)\Gamma(c + d)}{\Gamma(a)\Gamma(c)\Gamma(b + d)} x^{c-1}(1 - x)^{b+d-1}$$

$$\times\ {}_2F_1(b, c + d - a; b + d; 1 - x), \quad 0 < x < 1.$$

Proof. The r-th moment of $X_1 X_2$, $0 < X_1 X_2 < 1$, is given by

$$E(X_1^r X_2^r) = \frac{\Gamma(a+b)\Gamma(c+d)\Gamma(a+r)\Gamma(c+r)}{\Gamma(a)\Gamma(c)\Gamma(a+b+r)\Gamma(c+d+r)},$$

where $\mathrm{Re}(r) > 0$. Now by comparing the above moment expression with (12) it is easy to see that $\nu = c$, $\alpha = b$, $\beta = c + d - a$ and $\gamma = b + d$.

Corollary 3.1.1. *Let $X_1 \sim B(a, b)$ and $X_2 \sim B(c, d)$ be independent. Then, $X_1 X_2 \sim B(a, b + d)$ if $c = a + b$ and $X_1 X_2 \sim B(c, b + d)$ if $a = c + d$.*

An alternative proof of the above theorem using transformation of variables is given in Gupta and Nagar [2, p. 299]. Next, in Theorems 3.2-3.4, we give stochastic representations of the hypergeometric type I variable in terms of beta and inverted beta random variables.

Theorem 3.2. *Let X_1 and X_2 be independent, $X_1 \sim B(a, b)$ and $X_2 \sim B(c, d)$. Then, $(1 - X_1)X_2 \sim H^I(c, a, c + d - b, a + d)$, $X_1(1 - X_2) \sim H^I(d, b, c + d - a, b + c)$ and $(1 - X_1)(1 - X_2) \sim H^I(d, a, c + d - b, a + c)$.*

Proof. The result follows from Theorem 3.1 by noting that $1 - X_1 \sim B(b, a)$ and $1 - X_2 \sim B(d, c)$.

Corollary 3.2.1. *Let X_1 and X_2 be independent, $X_1 \sim B(a, b)$ and $X_2 \sim B(c, d)$. Then, $(1 - X_1)X_2 \sim B(c, a + d)$ if $b = c + d$, $(1 - X_1)X_2 \sim B(b, a + d)$ if $c = a + b$, $X_1(1 - X_2) \sim B(d, b + c)$ if $a = c + d$, $X_1(1 - X_2) \sim B(a, b + c)$ if $d = a + b$, $(1 - X_1)(1 - X_2) \sim B(d, a + c)$ if $b = c + d$ and $(1 - X_1)(1 - X_2) \sim B(b, a + c)$ if $d = a + b$.*

Theorem 3.3. *Let X_1 and X_2 be independent, $X_1 \sim B(a, b)$ and $X_2 \sim IB(c, d)$. Then, $X_1 X_2/(1 + X_2) \sim H^I(c, b, c + d - a, b + d)$, $(1 - X_1)X_2/(1 + X_2) \sim H^I(c, a, c + d - b, a + d)$, $X_1/(1 + X_2) \sim H^I(d, b, c + d - a, b + c)$ and $(1 - X_1)/(1 + X_2) \sim H^I(d, a, c + d - b, a + c)$.*

Proof. In this case observe that $1 - X_1 \sim B(b, a)$, $X_2/(1 + X_2) \sim B(c, d)$, $1/(1 + X_2) \sim B(d, c)$ and apply Theorem 3.1.

Corollary 3.3.1. *Let* X_1 *and* X_2 *be independent,* $X_1 \sim B(a, b)$ *and* $X_2 \sim IB(c, d)$. *Then,* $X_1 X_2 / (1 + X_2) \sim B(a, b + d)$ *if* $c = a + b$ *and* $X_1 X_2 / (1 + X_2) \sim B(c, b + d)$ *if* $a = c + d$, $(1 - X_1) X_2 / (1 + X_2) \sim B(c, a + d)$ *if* $b = c + d$, $(1 - X_1) X_2 / (1 + X_2) \sim B(b, a + d)$ *if* $c = a + b$, $X_1 / (1 + X_2)$ $\sim B(d, b + c)$ *if* $a = c + d$, $X_1 / (1 + X_2) \sim B(a, b + c)$ *if* $d = a + b$, $(1 - X_1) / (1 + X_2) \sim B(d, a + c)$ *if* $b = c + d$ *and* $(1 - X_1) / (1 + X_2) \sim B(b, a + c)$ *if* $d = a + b$.

Theorem 3.4. *Let* X_1 *and* X_2 *be independent,* $X_1 \sim IB(a, b)$ *and* $X_2 \sim IB(c, d)$. *Then,* $X_1 X_2 / (1 + X_1)(1 + X_2) \sim H^I(c, b, c + d - a, b + d)$, $X_2 / (1 + X_1)(1 + X_2) \sim H^I(c, a, c + d - b, a + d)$, $X_1 / (1 + X_1)(1 + X_2) \sim H^I(d, b, c + d - a, b + c)$ *and* $1 / (1 + X_1)(1 + X_2) \sim H^I(d, a, c + d - b, a + c)$.

Proof. Similar to the proof of Theorem 3.3.

Corollary 3.4.1. *Let* X_1 *and* X_2 *be independent,* $X_1 \sim IB(a, b)$ *and* $X_2 \sim IB(c, d)$. *Then,* $X_1 X_2 / (1 + X_1)(1 + X_2) \sim B(a, b + d)$ *if* $c = a + b$ *and* $X_1 X_2 / (1 + X_1)(1 + X_2) \sim B(c, b + d)$ *if* $a = c + d$, $X_2 / (1 + X_1)(1 + X_2)$ $\sim B(c, a + d)$ *if* $b = c + d$, $X_2 / (1 + X_1)(1 + X_2) \sim B(b, a + d)$ *if* $c = a + b$, $X_1 / (1 + X_1)(1 + X_2) \sim B(a, b + c)$ *if* $d = a + b$, $X_1 / (1 + X_1)(1 + X_2) \sim B(d, a + c)$ *if* $a = c + d$, $1 / (1 + X_1)(1 + X_2) \sim B(d, a + c)$ *if* $b = c + d$ *and* $1 / (1 + X_1)(1 + X_2) \sim B(b, a + c)$ *if* $d = a + b$.

4. Inverted Hypergeometric Function Type I Distribution

In this section we will define the inverted hypergeometric function type I distribution, study its properties and relationship to the hypergeometric function type I distribution. First we define the inverted hypergeometric function type I distribution.

Definition 4.1. The random variable Y is said to have an inverted hypergeometric function type I distribution with parameters $(\nu, \alpha, \beta, \gamma)$, denoted as $Y \sim IH^I(\nu, \alpha, \beta, \gamma)$, if its pdf is given by

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} \, y^{\nu-1}(1 + y)^{-(\gamma+\nu)}$$

$$\times \, {}_2F_1\!\left(\alpha, \, \beta; \, \gamma; \, \frac{1}{1 + y}\right), \quad y > 0, \tag{13}$$

where $\nu > 0$, $\gamma > 0$, $\gamma + \nu > \alpha + \beta$, and ${}_2F_1$ is the Gauss hypergeometric function.

From the above definition it is easy to see that $IH^I(\nu, \, \alpha, \, \gamma, \, \gamma) \equiv IB(\nu - \alpha, \, \gamma)$ and $IH^I(\nu, \, \gamma, \, \beta, \, \gamma) \equiv IB(\nu - \beta, \, \gamma)$.

In the following theorem we give relationship between hypergeometric function type I and inverted hypergeometric function type I distributions. The proof is straightforward and is left to the reader.

Theorem 4.1. *If $X \sim H^I(\nu, \, \alpha, \, \beta, \, \gamma)$, then $X/(1 - X) \sim IH^I(\nu, \, \alpha, \, \beta, \, \gamma)$. Similarly, if $Y \sim IH^I(\nu, \, \alpha, \, \beta, \, \gamma)$, then $Y/(1 + Y) \sim H^I(\nu, \, \alpha, \, \beta, \, \gamma)$.*

Using (3), the inverted hypergeometric function type I density can also be written as

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} \, y^{\nu+\gamma-\alpha-\beta-1}(1 + y)^{-(2\gamma+\nu-\alpha-\beta)}$$

$$\times \, {}_2F_1\!\left(\gamma - \alpha, \, \gamma - \beta; \, \gamma; \, \frac{1}{1 + y}\right), \quad y > 0. \tag{14}$$

Thus, $IH^I(\nu, \, \alpha, \, \beta, \, \gamma) \equiv IH^I(\nu, \, \beta, \, \alpha, \, \gamma) \equiv IH^I(\nu + \gamma - \alpha - \beta, \, \gamma - \alpha, \, \gamma - \beta, \, \gamma)$ $\equiv IH^I(\nu + \gamma - \alpha - \beta, \, \gamma - \beta, \, \gamma - \alpha, \, \gamma)$. Using series expansion of ${}_2F_1$ in (13) and (14), the pdf of the inverted hypergeometric type I variable can be expressed as a mixture of inverted beta densities:

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\nu + \gamma)\Gamma(\gamma + \nu - \alpha - \beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r(\beta)_r}{(\nu + \gamma)_r \, r!}$$

$$\times \left\{B(\nu, \, \gamma + r)\right\}^{-1} y^{\nu-1}(1 + y)^{-(\gamma+\nu+r)}, \quad y > 0,$$

and

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\nu)\Gamma(2\gamma + \nu - \alpha - \beta)} \sum_{r=0}^{\infty} \frac{(\gamma - \alpha)_r (\gamma - \beta)_r}{(\nu + 2\gamma - \alpha - \beta)_r\, r!}$$

$$\times \{B(\nu + \gamma - \alpha - \beta,\, \gamma + r)\}^{-1}\, y^{\nu + \gamma - \alpha - \beta - 1}(1 + y)^{-(\nu + 2\gamma - \alpha - \beta + r)}, \quad y > 0.$$

From (13) and (14) the pdf of $V = 1/Y$ is derived as

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)}\, v^{\gamma - 1}(1 + v)^{-(\nu + \gamma)}\, {}_2F_1\!\left(\alpha,\, \beta;\, \gamma;\, \frac{v}{1 + v}\right), \quad v > 0,$$

and

$$\frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)}\, v^{\gamma - 1}(1 + v)^{-(\nu + 2\gamma - \alpha - \beta)}$$

$$\times {}_2F_1\!\left(\gamma - \alpha,\, \gamma - \beta;\, \gamma;\, \frac{v}{1 + v}\right), \quad v > 0.$$

The cumulative distribution function (cdf) of Y is obtained as

$$P(Y \le y) = P\!\left(X \le \frac{y}{1 + y}\right),$$

where $X \sim H^I(\nu,\, \alpha,\, \beta,\, \gamma)$. Now, from (5), the cdf of Y is obtained as

$$P(Y \le y) = \frac{\Gamma(\gamma + \nu - \alpha)\Gamma(\gamma + \nu - \beta)}{\Gamma(\gamma)\Gamma(\nu)\Gamma(\gamma + \nu - \alpha - \beta)} \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r\, r!} B_{y/(y+1)}(\nu,\, \gamma + r).$$

Acknowledgement

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