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PROPERTIES OF MULTIVARIATE BETA DISTRIBUTIONS

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Abstract

Let $X_1, \ldots, X_{r+1}$ be independent random variables, $X_i \sim \text{Ga}(a_i, \theta_i)$, $i = 1, \ldots, r+1$. Define $Y_i = X_i/(X_i + X_{r+1}), i = 1, \ldots, r$ and $Z_i = X_i/X_{r+1}$, $i = 1, \ldots, r$. Then, $(Y_1, \ldots, Y_r)$ and $(Z_1, \ldots, Z_r)$ follow multivariate beta type 1 and type 2 distributions, respectively. In this article several properties of these distributions and their connections with the multivariate-F and the multivariate-t distributions are discussed.

1. Introduction

The beta (type 1) distribution with parameters (α_1, α_2) is defined by the probability density function (p.d.f.),

$$\text{B1}(u; \alpha_1, \alpha_2) = \{B(\alpha_1, \alpha_2)\}^{-1} u^{\alpha_1 - 1}(1-u)^{\alpha_2 - 1}, \quad 0 < u < 1, \qquad (1)$$

where $\alpha_1 > 0, \alpha_2 > 0$ and $B(\alpha_1, \alpha_2)$ is the beta function defined by

$$B(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)}.$$

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This distribution has two parameters and yet a rich variety of shapes. Because the beta distribution is bounded on both sides, it is often used for representing processes with natural lower and upper bounds. The beta distribution is well known in Bayesian methodology as a prior distribution on the success probability of a binomial distribution. The random variable V with the p.d.f.

$$\mathrm{B2}(v;\, \alpha_1,\, \alpha_2) = \{B(\alpha_1,\, \alpha_2)\}^{-1} v^{\alpha_1 - 1}(1 + v)^{-(\alpha_1 + \alpha_2)}, \quad v > 0, \tag{2}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$, is said to have a beta type 2 distribution with parameters (α_1, α_2). Since (2) can be obtained from (1) by the transformation $v = u/(1 - u)$, some authors call the distribution of v an *inverted beta distribution*. The inverted beta distribution arises from a linear transformation of the F distribution. The beta type 1 and beta type 2 are very flexible distributions for positive random variables and have wide applications in statistical analysis, e.g., see Johnson et al. [7]. Several univariate generalizations of these distributions are given in Gordy [5], Ng and Kotz [12], Nagar and Zarrazola [11], and McDonald and Xu [10]. For an extensive review on matrix variate beta distributions the reader is referred to Gupta and Nagar [6].

It is of interest to note that beta type 1 and type 2 densities can be derived by using independent gamma random variables. The random variable X is said to have a gamma distribution with parameters $\kappa(> 0)$ and $\theta(> 0)$, denoted by $X \sim \mathrm{Ga}(\kappa, \theta)$, if its p.d.f. is given by

$$\mathrm{Ga}(x;\, \kappa,\, \theta) = \{\theta^{\kappa}\Gamma(\kappa)\}^{-1} \exp\left(-\frac{x}{\theta}\right) x^{\kappa - 1}, \quad x > 0. \tag{3}$$

Let X_1 and X_2 be independent random variables, $X_1 \sim \mathrm{Ga}(\alpha_1, \theta)$ and $X_2 \sim \mathrm{Ga}(\alpha_2, \theta)$. Then, it is well known that $X_1/(X_1 + X_2) \sim \mathrm{B1}(\alpha_1, \alpha_2)$ and $X_1/X_2 \sim \mathrm{B2}(\alpha_1, \alpha_2)$. Further, if $X_1 \sim \mathrm{Ga}(\alpha_1, \theta_1)$ and $X_2 \sim \mathrm{Ga}(\alpha_2, \theta_2)$, then the densities of $Y = X_1/(X_1 + X_2)$ and $Z = X_1/X_2$ are given by

$$\mathrm{B1}(y;\, \alpha_1,\, \alpha_2;\, \lambda) = \frac{\Gamma(\alpha_1 + \alpha_2)\lambda^{\alpha_1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{y^{\alpha_1 - 1}(1 - y)^{\alpha_2 - 1}}{[1 - (1 - \lambda)y]^{\alpha_1 + \alpha_2}}, \quad 0 < y < 1 \tag{4}$$

and

$$\mathrm{B2}(z;\, \alpha_1,\, \alpha_2;\, \lambda) = \frac{\Gamma(\alpha_1 + \alpha_2)\lambda^{\alpha_1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \frac{z^{\alpha_1 - 1}}{(1 + \lambda z)^{\alpha_1 + \alpha_2}}, \quad z > 0, \tag{5}$$

respectively, where $\lambda = \theta_2/\theta_1$. The distributions defined by densities (4) and (5) are univariate generalizations of beta type 1 and type 2 distributions, respectively. One can easily see that for $\lambda = 1$ the density (4) reduces to a standard beta density and for $\lambda = 2$ it slides to a beta type 3 density studied by Cardeño et al. [1], and Sánchez and Nagar [15]. The multivariate generalizations of (4) and (5) can be obtained by considering $r + 1$ independent gamma variables. Let $X_i \sim \mathrm{Ga}(\alpha_i,\, \theta_i)$, $i = 1, \ldots, r + 1$, and define

$$Y_i = \frac{X_i}{X_i + X_{r+1}}, \quad i = 1, \ldots, r. \tag{6}$$

Then, the joint distribution of $Y_1, \ldots, Y_r$ is a multivariate beta type 1 distribution with the p.d.f. (Libby and Novic [9]),

$$\mathrm{MB1}(y_1, \ldots, y_r;\, \alpha_1, \ldots, \alpha_r;\, \alpha_{r+1};\, \lambda_1, \ldots, \lambda_r)$$

$$= \frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right) \prod_{i=1}^{r} \lambda_i^{\alpha_i}}{\prod_{i=1}^{r+1} \Gamma(\alpha_i)} \prod_{i=1}^{r} \left[\left(\frac{y_i}{1 - y_i}\right)^{\alpha_i - 1} \left(\frac{1}{1 - y_i}\right)^2\right]$$

$$\times \left[1 + \sum_{i=1}^{r} \lambda_i \left(\frac{y_i}{1 - y_i}\right)\right]^{-\sum_{i=1}^{r+1} \alpha_i}, \quad 0 < y_i < 1,\, i = 1, \ldots, r, \tag{7}$$

where

$$\lambda_i = \theta_{r+1}/\theta_i,\, i = 1, \ldots, r.$$

Further, define

$$Z_i = \frac{X_i}{X_{r+1}}, \quad i = 1, \ldots, r. \tag{8}$$

Then, the joint distribution of $Z_1, \ldots, Z_r$ is a multivariate beta type 2 (or inverted beta) distribution with the p.d.f.

$$\mathrm{MB2}(z_1, \ldots, z_r; \alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$$

$$= \frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right) \prod_{i=1}^{r} \lambda_i^{\alpha_i}}{\prod_{i=1}^{r+1} \Gamma(\alpha_i)} \prod_{i=1}^{r} z_i^{\alpha_i - 1} \left(1 + \sum_{i=1}^{r} \lambda_i z_i\right)^{-\sum_{i=1}^{r+1} \alpha_i},$$

$$z_i > 0, \; i = 1, \ldots, r. \tag{9}$$

For $\lambda_1 = \cdots = \lambda_r = 1$, the density in (7) slides to

$$\mathrm{MB1}(y_1, \ldots, y_r; \alpha_1, \ldots, \alpha_r; \alpha_{r+1})$$

$$= \frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right)}{\prod_{i=1}^{r+1} \Gamma(\alpha_i)} \prod_{i=1}^{r} \left[\left(\frac{y_i}{1 - y_i}\right)^{\alpha_i - 1} \left(\frac{1}{1 - y_i}\right)^2\right]$$

$$\times \left[1 + \sum_{i=1}^{r} \left(\frac{y_i}{1 - y_i}\right)\right]^{-\sum_{i=1}^{r+1} \alpha_i}, \quad 0 < y_i < 1, \; i = 1, \ldots, r. \tag{10}$$

The density in (10) is named by Chen and Novic [2] the standardized multivariate beta distribution (of the first kind). For $\lambda_1 = \cdots = \lambda_r = 1$, the density (9) slides to the usual Dirichlet type 2 density. For $r = 2$, the density in (10) reduces to a bivariate beta density considered recently by Olkin and Liu [13].

In this article, we study several properties of the multivariate generalized beta distributions defined by the densities (7) and (9). Several other multivariate generalizations of (4) and (5) have been studied by Pham-Gia and Duong [14]. Systematic treatment of multivariate generalizations of the beta type 1 and the beta type 2 distributions is given by Kotz et al. [8].

In Section 2, we discuss several properties of MB1 and MB2 distributions. To establish these properties we use results on statistical distribution theory and stochastic representations of MB1 and MB2 variables in terms independent gamma variables. The method employed here is quite different from the traditional method of Jacobian of transformation. Section 3 and Section 4 deal with limiting forms of MB1 and MB2 distributions.

2. Properties

In this section we study several properties of multivariate beta distributions defined by the densities (7) and (9). First we give definitions and stochastic representations of various multivariate generalizations of beta distributions (Tiao and Guttman [16] and Wilks [17]).

The random variables $U_1, \ldots, U_r$ are said to have a Dirichlet type 1 distribution with parameters $\alpha_1, \ldots, \alpha_r; \alpha_{r+1}$, denoted by $(U_1, \ldots, U_r) \sim \mathrm{D1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1})$, if their joint p.d.f. is given by

$$\frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right)}{\prod_{i=1}^{r+1} \Gamma(\alpha_i)} \prod_{i=1}^{r} u_i^{\alpha_i - 1} \left(1 - \sum_{i=1}^{r} u_i\right)^{\alpha_{r+1} - 1},$$

$$u_i > 0,\ i = 1, \ldots, r,\ \sum_{i=1}^{r} u_i < 1, \tag{11}$$

where

$$\alpha_i > 0,\ i = 1, \ldots, r + 1.$$

The random variables $V_1, \ldots, V_r$ are said to have a Dirichlet type 2 or inverted Dirichlet distribution with parameters $\alpha_1, \ldots, \alpha_r; \alpha_{r+1}$, denoted by $(V_1, \ldots, V_r) \sim \mathrm{D2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1})$, if their joint p.d.f. is given by

$$\frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right)}{\prod_{i=1}^{r+1} \Gamma(\alpha_i)} \prod_{i=1}^{r} v_i^{\alpha_i - 1} \left(1 + \sum_{i=1}^{r} v_i\right)^{-\sum_{i=1}^{r+1} \alpha_i}, \quad v_i > 0,\ i = 1, \ldots, r, \tag{12}$$

where

$$\alpha_i > 0,\ i = 1, \ldots, r + 1.$$

Let $X_1, \ldots, X_{r+1}$ be independent random variables, $X_i \sim \text{Ga}(\alpha_i, \theta)$, $i = 1, \ldots, r+1$. Then, it is well established that

$$(U_1, \ldots, U_r) \overset{d}{=} \left(\frac{X_1}{\sum_{i=1}^{r+1} X_i}, \ldots, \frac{X_r}{\sum_{i=1}^{r+1} X_i} \right) \sim \text{D1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}) \tag{13}$$

and

$$(V_1, \ldots, V_r) \overset{d}{=} \left(\frac{X_1}{X_{r+1}}, \ldots, \frac{X_r}{X_{r+1}} \right) \sim \text{D2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}), \tag{14}$$

where $X \overset{d}{=} Z$ means that X and Z have identical distributions. Further, using transformation of variables, one can easily prove the following result.

Theorem 2.1. *Let the random variables* $X_1, \ldots, X_{r+1}$ *be independent,* $X_i \sim \text{Ga}(\alpha_i, \theta)$, $i = 1, \ldots, r+1$. *Then, for* $1 \le s \le r$,

$$\left(\frac{X_1}{X_{r+1}}, \ldots, \frac{X_s}{X_{r+1}} \right) \sim \text{D2}(\alpha_1, \ldots, \alpha_s; \alpha_{r+1})$$

and

$$\left(\frac{X_{s+1}}{\sum_{i=1}^{s} X_i + X_{r+1}}, \ldots, \frac{X_r}{\sum_{i=1}^{s} X_i + X_{r+1}} \right) \sim \text{D2}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{i=1}^{s} \alpha_i + \alpha_{r+1} \right)$$

are independent.

Let $(Y_1, \ldots, Y_r) \sim \text{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$ and $(Z_1, \ldots, Z_r) \sim \text{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$. Further, let $X_1, \ldots, X_{r+1}$ be independent random variables, $X_i \sim \text{Ga}(\alpha_i, \theta_i)$, $i = 1, \ldots, r+1$. Then

$$(Y_1, \ldots, Y_r) \overset{d}{=} \left(\frac{X_1}{X_1 + X_{r+1}}, \ldots, \frac{X_r}{X_r + X_{r+1}} \right)$$
$$\sim \text{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r) \tag{15}$$

and

$$(Z_1, \ldots, Z_r) \stackrel{d}{=} \left(\frac{X_1}{X_{r+1}}, \ldots, \frac{X_r}{X_{r+1}}\right)$$
$$\sim \text{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r), \tag{16}$$

where $\lambda_i = \theta_{r+1}/\theta_i, \; i = 1, \ldots, r$.

Next, we state a result from Fang et al. [3], and Fang and Zhang [4].

Theorem 2.2. *Let* $\mathbf{Y}$ *and* $\mathbf{Z}$ *be n-dimensional random vectors. Further, let* $\mathbf{Y} \stackrel{d}{=} \mathbf{Z}$ *and* $f_j(\cdot), \; j = 1, \ldots, m$ *be Borel measurable functions. Then,*

$$\begin{pmatrix} f_1(\mathbf{Y}) \\ \vdots \\ f_m(\mathbf{Y}) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} f_1(\mathbf{Z}) \\ \vdots \\ f_m(\mathbf{Z}) \end{pmatrix}.$$

Next we give a well-known relationship between Dirichlet type 1 and Dirichlet type 2 variables which can easily be established using (13), (14) and the above theorem.

Theorem 2.3. *If* $(U_1, \ldots, U_r) \sim \text{D1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1})$, *then*

$$\left(\frac{U_1}{1 - \sum_{i=1}^r U_i}, \ldots, \frac{U_r}{1 - \sum_{i=1}^r U_i}\right) \sim \text{D2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}).$$

Further, if $(V_1, \ldots, V_r) \sim \text{D2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1})$, *then*

$$\left(\frac{V_1}{1 + \sum_{i=1}^r V_i}, \ldots, \frac{V_r}{1 + \sum_{i=1}^r V_i}\right) \sim \text{D1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}).$$

In the next theorem we give stochastic representations of MB1 and MB2 variables in terms of components of a multivariate normal vector. An $n \times 1$ random vector $\mathbf{X}$ is said to follow a multivariate normal distribution with mean vector $\boldsymbol{\mu}$, and positive definite covariance matrix Σ, denoted by $\mathbf{X} \sim \text{N}_n(\boldsymbol{\mu}, \Sigma)$, if its p.d.f. is given by

$$(2\pi)^{-n/2} \det(\Sigma)^{-1/2} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})' \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu})\right], \quad \mathbf{x} \in \mathbb{R}^n.$$

Theorem 2.4. *Let* $\mathbf{X} \sim \mathrm{N}_n(0, \Sigma)$. *Partition* $\mathbf{X}$ *and* Σ *as*

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}^{(1)} \\ \vdots \\ \mathbf{X}^{(r+1)} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1,r+1} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2,r+1} \\ \vdots & & & \\ \Sigma_{r+1,1} & \Sigma_{r+2,2} & \cdots & \Sigma_{r+r,r+1} \end{pmatrix},$$

where $\mathbf{X}^{(i)}$ *and* Σ_{ij} *are* $n_i \times 1$ *and* $n_i \times n_j$, *respectively, and* $\sum_{i=1}^{r+1} n_i = n(> r)$. *If* $\Sigma_{ij} = 0$, $(i \neq j)$ *and* $\Sigma_{ii} = \sigma_i^2 I_{n_i}$. *Then*

$$\left(\frac{\| \mathbf{X}^{(1)} \|^2}{\| \mathbf{X}^{(1)} \|^2 + \| \mathbf{X}^{(r+1)} \|^2}, \ldots, \frac{\| \mathbf{X}^{(r)} \|^2}{\| \mathbf{X}^{(r)} \|^2 + \| \mathbf{X}^{(r+1)} \|^2} \right)$$

$$\sim \mathrm{MB1}\left(\frac{n_1}{2}, \ldots, \frac{n_r}{2}; \frac{n_{r+1}}{2}; \lambda_1, \ldots, \lambda_r \right)$$

and

$$\left(\frac{\| \mathbf{X}^{(1)} \|^2}{\| \mathbf{X}^{(r+1)} \|^2}, \ldots, \frac{\| \mathbf{X}^{(r)} \|^2}{\| \mathbf{X}^{(r+1)} \|^2} \right) \sim \mathrm{MB2}\left(\frac{n_1}{2}, \ldots, \frac{n_r}{2}; \frac{n_{r+1}}{2}; \lambda_1, \ldots, \lambda_r \right),$$

where $\| \mathbf{z} \|^2 = \mathbf{z}'\mathbf{z}$ *and* $\lambda_i = \sigma_{r+1}^2 / \sigma_i^2$, $i = 1, \ldots, r$.

Proof. Note that the random variables $\| \mathbf{X}^{(1)} \|^2, \ldots, \| \mathbf{X}^{(r+1)} \|^2$ are independently distributed, $\| \mathbf{X}^{(i)} \|^2 \sim \mathrm{Ga}(n_i/2, 2\sigma_i^2)$, $i = 1, \ldots, r+1$. The desired result now follows by using the stochastic representations (15) and (16).

By taking $n_1 = \cdots = n_r = 1$ in the second part of the above theorem, we have

$$\left(\frac{X_1^2}{\| \mathbf{X}^{(r+1)} \|^2}, \ldots, \frac{X_r^2}{\| \mathbf{X}^{(r+1)} \|^2} \right) \sim \mathrm{MB2}\left(\frac{1}{2}, \ldots, \frac{1}{2}; \frac{n-r}{2}; \lambda_1, \ldots, \lambda_r \right),$$

from which the p.d.f. of $\mathbf{T} = (\sqrt{\nu} X_1 / \| \mathbf{X}^{(r+1)} \|, \ldots, \sqrt{\nu} X_r / \| \mathbf{X}^{(r+1)} \|)'$, $\nu = n - r$, is obtained as

$$\frac{\Gamma[(\nu + r)/2] \prod_{i=1}^{r} \lambda_i^{1/2}}{\Gamma(\nu/2)(\nu\pi)^{r/2}} \left(1 + \frac{\sum_{i=1}^{r} \lambda_i t_i^2}{\nu} \right)^{-(\nu+r)/2}, \quad \mathbf{t} \in \mathbb{R}^r \tag{17}$$

which is a multivariate-t density with ν degrees of freedom. For $r = 1$, the density in (17) slides to a generalized t density given by

$$\frac{\Gamma[(\nu+1)/2]\lambda^{1/2}}{\Gamma(\nu/2)(\nu\pi)^{1/2}}\left(1+\frac{\lambda t^2}{\nu}\right)^{-(\nu+1)/2}, \quad t \in \mathbb{R}.$$

The p.d.f. of

$$\mathbf{F} = \left(\frac{n_{r+1}\| \mathbf{X}^{(1)} \|^2}{n_1\| \mathbf{X}^{(r+1)} \|^2}, \dots, \frac{n_{r+1}\| \mathbf{X}^{(r)} \|^2}{n_r\| \mathbf{X}^{(r+1)} \|^2}\right)',$$

using the second part of Theorem 2.4, is derived as

$$\frac{\prod_{i=1}^{r}(\lambda_i n_i/n_{r+1})^{n_i/2}\Gamma\left(\sum_{i=1}^{r+1} n_i/2\right)}{\prod_{i=1}^{r+1}\Gamma(n_i/2)} \frac{\prod_{i=1}^{r} f_i^{n_i/2-1}}{\left(1+\sum_{i=1}^{r} n_i\lambda_i f_i/n_{r+1}\right)^{n/2}},$$

where

$$f_i > 0, \; i = 1, \dots, r.$$

The density given above is a multivariate-F density with $(n_1, \dots, n_r, n_{r+1})$ degrees of freedom. Substituting $r = 1$ in the above density, the F density is obtained as

$$\frac{(\lambda n_1/n_2)^{n_1/2}\Gamma[(n_1+n_2)/2]}{\Gamma(n_1/2)\Gamma(n_2/2)} \frac{f^{n_1/2-1}}{(1+\lambda n_1 f/n_2)^{(n_1+n_2)/2}}, \quad f > 0.$$

If $(U_1, \dots, U_n) \sim \text{D1}(a_1, \dots, a_r; a_{r+1})$, then it is well known that for $1 \le s \le r$, $(U_1, \dots, U_s) \sim \text{D1}\left(a_1, \dots, a_s; \sum_{i=s+1}^{r+1} a_i\right)$. In the following theorem we establish similar result for the multivariate generalized beta type I variables.

Theorem 2.5. *If* $(Y_1, \dots, Y_r) \sim \text{MB1}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r)$, *then for* $1 \le s \le r$,

$$(Y_1, \dots, Y_s) \sim \text{MB1}(\alpha_1, \dots, \alpha_s; \alpha_{r+1}; \lambda_1, \dots, \lambda_s)$$

and

$$(Y_{s+1}, \ldots, Y_r) | (y_1, \ldots, y_s)$$

$$\sim \mathrm{MB1}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{i=1}^{s} \alpha_i + \alpha_{r+1}; \frac{\lambda_{s+1}}{1 + \sum_{i=1}^{s} \lambda_i y_i / (1 - y_i)}, \ldots, \frac{\lambda_r}{1 + \sum_{i=1}^{s} \lambda_i y_i / (1 - y_i)}\right).$$

Proof. Let $f_j(y_1, \ldots, y_r) = y_j, 1 \le j \le s$. Then using the stochastic representation (15) and Theorem 2.2, we get

$$(Y_1, \ldots, Y_s) \stackrel{d}{=} \left(\frac{X_1}{X_1 + X_{r+1}}, \ldots, \frac{X_s}{X_s + X_{r+1}}\right)$$

$$\sim \mathrm{MB1}(\alpha_1, \ldots, \alpha_s; \alpha_{r+1}; \lambda_1, \ldots, \lambda_s),$$

where the last line has been obtained by using (15) and the fact that $X_1, \ldots, X_s$ and X_{r+1} are independent, $X_i \sim \mathrm{Ga}(\alpha_i, \theta_i), i = 1, \ldots, s$ and $X_{r+1} \sim \mathrm{Ga}(\alpha_{r+1}, \theta_{r+1})$. The proof of the second part follows from the definition.

Corollary 2.5.1. *If* $(Y_1, \ldots, Y_r) \sim \mathrm{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then* $Y_i \sim \mathrm{B1}(\alpha_i, \alpha_{r+1}; \lambda_i)$ *for* $i = 1, \ldots, r$.

Theorem 2.6. *If* $(Z_1, \ldots, Z_r) \sim \mathrm{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then for* $1 \le s \le r$,

$$(Z_1, \ldots, Z_s) \sim \mathrm{MB2}(\alpha_1, \ldots, \alpha_s; \alpha_{r+1}; \lambda_1, \ldots, \lambda_s)$$

and

$$(Z_{s+1}, \ldots, Z_r) | (z_1, \ldots, z_s)$$

$$\sim \mathrm{MB2}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{i=1}^{s} \alpha_i + \alpha_{r+1}; \frac{\lambda_{s+1}}{1 + \sum_{i=1}^{s} \lambda_i z_i}, \ldots, \frac{\lambda_r}{1 + \sum_{i=1}^{s} \lambda_i z_i}\right).$$

Proof. The proof is similar to the one given for Theorem 2.5.

Corollary 2.6.1. *If* $(Z_1, \dots, Z_r) \sim \text{MB2}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r)$, *then* $Z_i \sim \text{B2}(\alpha_i, \dots, \alpha_{r+1}; \lambda_i)$, $i = 1, \dots, r$.

In the next theorem we give relationship between multivariate beta type 1 and multivariate beta type 2 distributions.

Theorem 2.7. (i) *If* $(Y_1, \dots, Y_r) \sim \text{MB1}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r)$, *then*

$$\left(\frac{Y_1}{1-Y_1}, \dots, \frac{Y_r}{1-Y_r}\right) \sim \text{MB2}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r).$$

(ii) *If* $(Z_1, \dots, Z_r) \sim \text{MB2}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r)$, *then*

$$\left(\frac{Z_1}{1+Z_1}, \dots, \frac{Z_r}{1+Z_r}\right) \sim \text{MB1}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r).$$

Proof. Using the stochastic representation (15) and Theorem 2.2, we have

$$\left(\frac{Y_1}{1-Y_1}, \dots, \frac{Y_r}{1-Y_r}\right) \stackrel{d}{=} \left(\frac{X_1/(X_1+X_{r+1})}{1-X_1/(X_1+X_{r+1})}, \dots, \frac{X_r/(X_r+X_{r+1})}{1-X_r/(X_r+X_{r+1})}\right)$$

$$= \left(\frac{X_1}{X_{r+1}}, \dots, \frac{X_r}{X_{r+1}}\right)$$

$$\sim \text{MB2}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}; \lambda_1, \dots, \lambda_r),$$

where the last step has been obtained by using (16). The proof of the second part follows similarly.

Corollary 2.7.1. (i) *If* $(Y_1, \dots, Y_r) \sim \text{MB1}(\alpha_1, \dots, \alpha_r; \alpha_{r+1})$, *then*

$$\left(\frac{Y_1}{1-Y_1}, \dots, \frac{Y_r}{1-Y_r}\right) \sim \text{D2}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}).$$

(ii) *If* $(Z_1, \dots, Z_r) \sim \text{D2}(\alpha_1, \dots, \alpha_r; \alpha_{r+1})$, *then*

$$\left(\frac{Z_1}{1+Z_1}, \dots, \frac{Z_r}{1+Z_r}\right) \sim \text{MB1}(\alpha_1, \dots, \alpha_r; \alpha_{r+1}).$$

Corollary 2.7.2. (i) *If* $(Y_1, ..., Y_r) \sim \text{MB1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then for* $i = 1, ..., r$,

$$\frac{Y_i}{1 - Y_i} \sim \text{B2}(\alpha_i, \alpha_{r+1}; \lambda_i).$$

(ii) *If* $(Z_1, ..., Z_r) \sim \text{MB2}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then*

$$\frac{Z_i}{1 + Z_i} \sim \text{B1}(\alpha_i, \alpha_{r+1}; \lambda_i).$$

Theorem 2.8. (i) *If* $(Z_1, ..., Z_r) \sim \text{MB2}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then*

$$(\lambda_1 Z_1, ..., \lambda_r Z_r) \sim \text{D2}(\alpha_1, ..., \alpha_r; \alpha_{r+1})$$

and

$$\left(\frac{\lambda_1 Z_1}{1 + \sum_{i=1}^r \lambda_i Z_i}, ..., \frac{\lambda_r Z_r}{1 + \sum_{i=1}^r \lambda_i Z_i}\right) \sim \text{D1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}).$$

(ii) *If* $(Y_1, ..., Y_r) \sim \text{MB1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then*

$$\left(\frac{\lambda_1 Y_1}{1 - Y_1}, ..., \frac{\lambda_r Y_r}{1 - Y_r}\right) \sim \text{D2}(\alpha_1, ..., \alpha_r; \alpha_{r+1})$$

and

$$\left(\frac{\lambda_1 Y_1/(1 - Y_1)}{1 + \sum_{i=1}^r \lambda_i Y_i/(1 - Y_i)}, ..., \frac{\lambda_r Y_r/(1 - Y_1)}{1 + \sum_{i=1}^r \lambda_i Y_i/(1 - Y_i)}\right)$$
$$\sim \text{D1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}).$$

Proof. Using the stochastic representation (16) and Theorem 2.2, we have

$$(\lambda_1 Z_1, ..., \lambda_r Z_r) \stackrel{d}{=} \left(\frac{\lambda_1 X_1}{X_{r+1}}, ..., \frac{\lambda_r X_r}{X_{r+1}}\right) = \left(\frac{X_1/\theta_1}{X_{r+1}/\theta_{r+1}}, ..., \frac{X_r/\theta_r}{X_{r+1}/\theta_{r+1}}\right)$$
$$\sim \text{D2}(\alpha_1, ..., \alpha_r; \alpha_{r+1}),$$

where the last step has been obtained by using (16). Now application of Theorem 2.3 yields the second result. The proof of the second part follows from Theorem 2.7(i) and part (i).

Corollary 2.8.1. *If* $(Y_1, ..., Y_r) \sim \text{MB1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$ *and* $(Z_1, ..., Z_r) \sim \text{MB2}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then for* $i = 1, ..., r$,

$$\lambda_i Z_i \sim \text{B2}(\alpha_i, \alpha_{r+1}), \quad \frac{\lambda_i Z_i}{1 + \sum_{i=1}^{r} \lambda_i Z_i} \sim \text{B1}(\alpha_i, \alpha_{r+1}),$$

$$\frac{\lambda_i Y_i}{1 - Y_i} \sim \text{B2}(\alpha_i, \alpha_{r+1}), \quad \frac{\lambda_i Y_i/(1 - Y_i)}{1 + \sum_{i=1}^{r} \lambda_i Y_i/(1 - Y_i)} \sim \text{B1}(\alpha_i, \alpha_{r+1}).$$

Theorem 2.9. *If* $(Y_1, ..., Y_r) \sim \text{MB1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then*

$$\left(\frac{\lambda_1 Y_1}{1 - (1 - \lambda_1) Y_1}, ..., \frac{\lambda_r Y_r}{1 - (1 - \lambda_r) Y_r}\right) \sim \text{MB1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}).$$

Proof. Using Theorem 2.8(ii) and Theorem 2.2, we have

$$\left(\frac{\lambda_1 Y_1}{1 - (1 - \lambda_1) Y_1}, ..., \frac{\lambda_r Y_r}{1 - (1 - \lambda_r) Y_r}\right)$$

$$= \left(\frac{\lambda_1 Y_1}{\lambda_1 Y_1 + 1 - Y_1}, ..., \frac{\lambda_r Y_r}{\lambda_r Y_r + 1 - Y_r}\right)$$

$$= \left(\frac{\lambda_1 Y_1/(1 - Y_1)}{1 + \lambda_1 Y_1/(1 - Y_1)}, ..., \frac{\lambda_r Y_r/(1 - Y_r)}{1 + \lambda_r Y_r/(1 - Y_r)}\right)$$

$$\stackrel{d}{=} \left(\frac{Z_1}{1 + Z_1}, ..., \frac{Z_r}{1 + Z_r}\right),$$

where $(Z_1, ..., Z_r) \sim \text{D2}(\alpha_1, ..., \alpha_r; \alpha_{r+1})$. Now the result follows using Corollary 2.7.1(ii).

Corollary 2.9.1. *If* $(Y_1, ..., Y_r) \sim \text{MB1}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then for* $i = 1, ..., r$,

$$\frac{\lambda_i Y_i}{1 - (1 - \lambda_i) Y_i} \sim \text{B1}(\alpha_i, \alpha_{r+1}).$$

Theorem 2.10. *If* $(Z_1, ..., Z_r) \sim \text{MB2}(\alpha_1, ..., \alpha_r; \alpha_{r+1}; \lambda_1, ..., \lambda_r)$, *then*

$$\left(\frac{Z_1}{Z_i}, ..., \frac{Z_{i-1}}{Z_i}, \frac{1}{Z_i}, \frac{Z_{i+1}}{Z_i}, ..., \frac{Z_r}{Z_i}\right)$$

$$\sim \text{MB2}\left(\alpha_1, ..., \alpha_{i-1}, \alpha_{r+1}, \alpha_{i+1}, ..., \alpha_r; \alpha_i; \frac{\lambda_1}{\lambda_i}, ..., \frac{\lambda_{i-1}}{\lambda_i}, \frac{1}{\lambda_i}, \frac{\lambda_{i+1}}{\lambda_i}, ..., \frac{\lambda_r}{\lambda_i}\right).$$

Proof. Using the stochastic representation (16) and Theorem 2.2, we have

$$\left(\frac{Z_1}{Z_i}, \ldots, \frac{Z_{i-1}}{Z_i}, \frac{1}{Z_i}, \frac{Z_{i+1}}{Z_i}, \ldots, \frac{Z_r}{Z_i}\right)$$

$$\stackrel{d}{=} \left(\frac{X_1/X_{r+1}}{X_i/X_{r+1}}, \ldots, \frac{X_{i-1}/X_{r+1}}{X_i/X_{r+1}}, \frac{1}{X_i/X_{r+1}}, \frac{X_{i+1}/X_{r+1}}{X_i/X_{r+1}}, \ldots, \frac{X_r/X_{r+1}}{X_i/X_{r+1}}\right)$$

$$= \left(\frac{X_1}{X_i}, \ldots, \frac{X_{i-1}}{X_i}, \frac{X_{r+1}}{X_i}, \frac{X_{i+1}}{X_i}, \ldots, \frac{X_r}{X_i}\right).$$

Now the result follows by using (16).

Theorem 2.11. *If* $(Z_1, \ldots, Z_r) \sim \mathrm{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then* $\left(Z_{s+1}/\left(1 + \sum_{i=1}^{s} \lambda_i Z_i\right), \ldots, Z_r/\left(1 + \sum_{i=1}^{s} \lambda_i Z_i\right)\right)$ *and* $(Z_1, \ldots, Z_s)$ *are independent. Further*,

$$\left(\frac{Z_{s+1}}{1 + \sum_{i=1}^{s} \lambda_i Z_i}, \ldots, \frac{Z_r}{1 + \sum_{i=1}^{s} \lambda_i Z_i}\right)$$

$$\sim \mathrm{MB2}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{i=1}^{s} \alpha_i + \alpha_{r+1}; \lambda_{s+1}, \ldots, \lambda_r\right).$$

Proof. Take $f_i(z_1, \ldots, z_r) = z_i$, $i = 1, \ldots, s$ and $f_i(z_1, \ldots, z_r) = \left(1 + \sum_{i=1}^{s} \lambda_i z_i\right)^{-1} z_i$, $i = s + 1, \ldots, r$. Then, using the stochastic representation (16) and Theorem 2.2, we get

$$\left(Z_1, \ldots, Z_s, \frac{Z_{s+1}}{1 + \sum_{i=1}^{s} \lambda_i Z_i}, \ldots, \frac{Z_r}{1 + \sum_{i=1}^{s} \lambda_i Z_i}\right)$$

$$\stackrel{d}{=} \left(\frac{X_1}{X_{r+1}}, \ldots, \frac{X_s}{X_{r+1}}, \frac{X_{s+1}}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}}, \ldots, \frac{X_r}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}}\right),$$

where $X_1, \ldots, X_{r+1}$ are independent, $\lambda_i X_i \sim (\alpha_i, \theta_{r+1})$, $i = 1, \ldots, r+1$. Since, from Theorem 2.1, $(\lambda_1 X_1 / X_{r+1}, \ldots, \lambda_s X_s / X_{r+1})$ and

$$\left(\frac{\lambda_{s+1} X_{s+1}}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}}, \ldots, \frac{\lambda_r X_r}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}} \right)$$

are independent, the independence of $(X_1 / X_{r+1}, \ldots, X_s / X_{r+1})$ and

$$\left(\frac{X_{s+1}}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}}, \ldots, \frac{X_r}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}} \right)$$

is obvious. Further

$$\left(\frac{X_{s+1}}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}}, \ldots, \frac{X_r}{\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}} \right)$$

$$\sim \text{MB2}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{i=1}^{s} \alpha_i + \alpha_{r+1}; \lambda_{s+1}, \ldots, \lambda_r \right),$$

where the last line has been obtained by using the fact that $X_{s+1}, \ldots, X_r$ and $\sum_{i=1}^{s} \lambda_i X_i + X_{r+1}$ are independent, $X_i \sim \text{Ga}(\alpha_i, \theta_i)$, $i = s+1, \ldots, r$ and $\sum_{i=1}^{s} \lambda_i X_i + X_{r+1} \sim \text{Ga}\left(\sum_{i=1}^{s} \alpha_i + \alpha_{r+1}, \theta_{r+1} \right)$.

Corollary 2.11.1. *If* $(Y_1, \ldots, Y_r) \sim \text{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then* $(Y_1, \ldots, Y_s)$ *and*

$$\left(\frac{Y_{s+1}/(1 - Y_{s+1})}{1 + \sum_{i=1}^{s} \lambda_i Y_i/(1 - Y_i)}, \ldots, \frac{Y_r/(1 - Y_r)}{1 + \sum_{i=1}^{s} \lambda_i Y_i/(1 - Y_i)} \right)$$

$$\sim \text{MB2}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{i=1}^{s} \alpha_i + \alpha_{r+1}; \lambda_{s+1}, \ldots, \lambda_r \right)$$

are independent.

Corollary 2.11.2. *If* $(Y_1, \ldots, Y_r) \sim \text{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then for* $s + 1 \leq i \leq r$

$$\frac{Y_i/(1 - Y_i)}{1 + \sum_{i=1}^{s} \lambda_i Y_i/(1 - Y_i)} \sim \text{B2}\left(\alpha_i, \sum_{i=1}^{s} \alpha_i + \alpha_{r+1}; \lambda_i\right).$$

Corollary 2.11.3. *If* $(Z_1, \ldots, Z_r) \sim \text{D2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1})$, *then* $(Z_1, \ldots, Z_s)$ *and* $\left(Z_{s+1}\Big/\left(1 + \sum_{i=1}^{s} Z_i\right), \ldots, Z_r\Big/\left(1 + \sum_{i=1}^{s} Z_i\right)\right)$ *are independent. Further*

$$\left(\frac{Z_{s+1}}{1 + \sum_{i=1}^{s} Z_i}, \ldots, \frac{Z_r}{1 + \sum_{i=1}^{s} Z_i}\right) \sim \text{D2}\left(\alpha_{s+1}, \ldots, \alpha_r; \sum_{k=1}^{s} \alpha_k + \alpha_{r+1}\right).$$

Theorem 2.12. *If* $(Z_1, \ldots, Z_r) \sim \text{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then the random variables* $Z_1, Z_2/(1 + \lambda_1 Z_1), \ldots, Z_{r-1}\Big/\left(1 + \sum_{i=1}^{r-2} \lambda_i Z_i\right)$ *and* $Z_r\Big/\left(1 + \sum_{i=1}^{r-1} \lambda_i Z_i\right)$ *are independent,* $Z_1 \sim \text{B2}(\alpha_1, \alpha_{r+1}; \lambda_1)$ *and*

$$\frac{Z_i}{1 + \sum_{k=1}^{i-1} \lambda_k Z_k} \sim \text{B2}\left(\alpha_i, \sum_{k=1}^{i-1} \alpha_k + \alpha_{r+1}; \lambda_i\right), \quad i = 2, \ldots, r.$$

Proof. Theorem 2.11 for $s = r - 1$ gives independence of $(Z_1, \ldots, Z_{r-1})$ and $Z_r\Big/\left(1 + \sum_{i=1}^{r-1} \lambda_i Z_i\right)$, where $(Z_1, \ldots, Z_{r-1}) \sim \text{MB2}(\alpha_1, \ldots, \alpha_{r-1}; \alpha_{r+1}; \lambda_1, \ldots, \lambda_{r-1})$ and

$$\frac{Z_r}{1 + \sum_{k=1}^{r-1} \lambda_k Z_k} \sim \text{B2}\left(\alpha_r, \sum_{i=1}^{r-1} \alpha_i + \alpha_{r+1}; \lambda_r\right).$$

Further application of Theorem 2.11 gives independence of $(Z_1, \ldots, Z_{r-2})$ and $Z_{r-1}\Big/\left(1 + \sum_{k=1}^{r-2} \lambda_k Z_k\right)$, where

$$(Z_1, \ldots, Z_{r-2}) \sim \text{MB2}(\alpha_1, \ldots, \alpha_{r-2}; \alpha_{r+1}; \lambda_1, \ldots, \lambda_{r-2})$$

and

$$\frac{Z_{r-1}}{1+\sum_{k=1}^{r-2}\lambda_k Z_k} \sim \text{B2}\left(\alpha_{r-1},\ \sum_{i=1}^{r-2}\alpha_i + \alpha_{r+1};\ \lambda_{r-1}\right).$$

Repetition of this procedure finally yields the desired result.

Corollary 2.12.1. *If* $(Y_1, \ldots, Y_r) \sim \text{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then the random variables*

$$Y_1,\ \frac{Y_2/(1-Y_2)}{1+\lambda_1 Y_1/(1-Y_1)},\ \ldots,\ \frac{Y_{r-1}/(1-Y_{r-1})}{1+\sum_{k=1}^{r-2}\lambda_k Y_k/(1-Y_k)}$$

and

$$\frac{Y_r/(1-Y_r)}{1+\sum_{k=1}^{r-1}\lambda_k Y_k/(1-Y_k)}$$

are mutually independent. Further, $Y_1 \sim \text{B1}(\alpha_1, \alpha_{r+1}; \lambda_1)$ *and*

$$\frac{Y_i/(1-Y_i)}{1+\sum_{k=1}^{i-1}\lambda_k Y_k/(1-Y_k)} \sim \text{B2}\left(\alpha_i,\ \sum_{k=1}^{i-1}\alpha_k + \alpha_{r+1};\ \lambda_i\right), \quad 2 \le i \le r.$$

In the next theorem, we derive the joint p.d.f.'s of partial sums of random variables distributed as multivariate beta type 1 or 2.

Theorem 2.13. *Let* $r_1, \ldots, r_\ell$ *be positive integers such that* $\sum_{i=1}^{\ell} r_i = r$ *and define* $r_i^* = \sum_{j=1}^{i} r_j$, $r_0^* = 0$, $\alpha_{(i)} = \sum_{j=r_{i-1}^*+1}^{r_i^*}\alpha_j$, $\lambda_{(i)} = \sum_{j=r_{i-1}^*+1}^{r_i^*}\lambda_j$, $W_j = Z_j/Z_{(i)}$, $j = r_{i-1}^* + 1, \ldots, r_i^* - 1$ *and* $Z_{(i)} = \sum_{j=r_{i-1}^*+1}^{r_i^*}\lambda_j Z_j/\lambda_{(i)}$, $i = 1, \ldots, \ell$. *If* $(Z_1, \ldots, Z_r) \sim \text{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then* $(W_{r_{i-1}^*+1}, \ldots, W_{r_i^*-1})$, $i = 1, \ldots, \ell$ *and* $(Z_{(1)}, \ldots, Z_{(\ell)})$ *are independently distributed,*

$$(Z_{(1)}, \ldots, Z_{(\ell)}) \sim \text{MB2}(\alpha_{(1)}, \ldots, \alpha_{(\ell)}; \alpha_{r+1}; \lambda_{(1)}, \ldots, \lambda_{(\ell)})$$

and for $i = 1, \ldots, \ell$,

$$\left(\frac{\lambda_{r_{i-1}^*+1} W_{r_{i-1}^*+1}}{\lambda_{(i)}}, \ldots, \frac{\lambda_{r_i^*-1} W_{r_i^*-1}}{\lambda_{(i)}}\right) \sim \text{D1}(\alpha_{r_{i-1}^*+1}, \ldots, \alpha_{r_i^*-1}; \alpha_{r_i^*}).$$

Proof. Transforming $z_{(i)} = \sum_{j=r_{i-1}^*+1}^{r_i^*} \lambda_j z_j / \lambda_{(i)}$ and $w_j = z_j / z_{(i)}$, $j = r_{i-1}^* + 1, \ldots, r_i^* - 1$, $i = 1, \ldots, \ell$ with the Jacobian

$$J(z_1, \ldots, z_r \to w_1, \ldots, w_{r_1-1}, z_{(1)}, \ldots., w_{r_{\ell-1}^*+1}, \ldots, w_{r-1}, z_{(\ell)})$$

$$= \prod_{i=1}^{\ell} J(z_{r_{i-1}^*+1}, \ldots, z_{r_i^*} \to w_{r_{i-1}^*+1}, \ldots, w_{r_i^*-1}, z_{(i)})$$

$$= \prod_{i=1}^{\ell} (z_{(i)}^{r_i-1} \lambda_{(i)} \lambda_{r_i^*}^{-1})$$

in the joint density of $(Z_1, \ldots, Z_r)$ given by (9), we get the joint density of $W_{r_{i-1}^*+1}, \ldots, W_{r_i^*-1}, Z_{(i)}$, $i = 1, \ldots, \ell$ as

$$\frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right) \prod_{i=1}^{\ell} \lambda_{(i)}^{\alpha_{r_i^*}} \prod_{j=r_{i-1}^*+1}^{r_i^*-1} \lambda_j^{\alpha_j}}{\prod_{i=1}^{r+1} \Gamma(\alpha_i)} \frac{\prod_{i=1}^{\ell} z_{(i)}^{\alpha_{(i)}-1}}{\left(1 + \sum_{i=1}^{\ell} \lambda_{(i)} z_{(i)}\right)^{\sum_{i=1}^{r+1} \alpha_i}}$$

$$\times \prod_{i=1}^{\ell} \left[\prod_{j=r_{i-1}^*+1}^{r_i^*-1} w_j^{\alpha_j - 1} \left(1 - \sum_{j=r_{i-1}^*+1}^{r_i^*-1} \frac{\lambda_j w_j}{\lambda_{(i)}}\right)^{\alpha_{r_i^*}-1}\right], \tag{18}$$

where

$$z_{(i)} > 0,\ w_j > 0,\ j = r_{i-1}^* + 1, \ldots, r_i^* - 1,\ \sum_{j=r_{i-1}^*+1}^{r_i^*-1} \lambda_j w_j / \lambda_{(i)} < 1,$$

$i = 1, \ldots, \ell$. Now, from the factorization in (18), the desired result follows.

Corollary 2.13.1. *If* $(Z_1, \ldots, Z_r) \sim \text{MB2}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then*

$$\frac{\sum_{i=1}^{r} \lambda_i Z_i}{\sum_{i=1}^{r} \lambda_i} \sim \text{B2}\left(\sum_{i=1}^{r} \alpha_i, \alpha_{r+1}; \sum_{i=1}^{r} \lambda_i\right)$$

and

$$\left(\frac{\lambda_1 Z_1}{\sum_{i=1}^{r}\lambda_i Z_i}, \ldots, \frac{\lambda_{r-1}Z_{r-1}}{\sum_{i=1}^{r}\lambda_i Z_i}\right) \sim \mathrm{D1}(\alpha_1, \ldots, \alpha_{r-1}; \alpha_r)$$

are independent. Further,

$$\frac{\sum_{i=1}^{s}\lambda_i Z_i}{\sum_{i=1}^{r}\lambda_i Z_i} \sim \mathrm{B1}\left(\sum_{i=1}^{s}\alpha_i, \alpha_r\right), \quad 1 \le s \le r-1$$

is independent of $\sum_{i=1}^{r}\lambda_i Z_i$.

Corollary 2.13.2. *If* $(Y_1, \ldots, Y_r) \sim \mathrm{MB1}(\alpha_1, \ldots, \alpha_r; \alpha_{r+1}; \lambda_1, \ldots, \lambda_r)$, *then*

$$\frac{1}{\sum_{i=1}^{r}\lambda_i}\sum_{i=1}^{r}\frac{\lambda_i Y_i}{1-Y_i} \sim \mathrm{B2}\left(\sum_{i=1}^{r}\alpha_i, \alpha_{r+1}; \sum_{i=1}^{r}\lambda_i\right)$$

and

$$\left(\frac{\lambda_1 Y_1/(1-Y_1)}{\sum_{i=1}^{r}\lambda_i Y_i/(1-Y_i)}, \ldots, \frac{\lambda_{r-1}Y_{r-1}/(1-Y_{r-1})}{\sum_{i=1}^{r}\lambda_i Y_i/(1-Y_i)}\right)$$

$$\sim \mathrm{D1}(\alpha_1, \ldots, \alpha_{r-1}; \alpha_r)$$

are independent. Further,

$$\frac{\sum_{i=1}^{s}\lambda_i Y_i/(1-Y_i)}{\sum_{i=1}^{r}\lambda_i Y_i/(1-Y_i)} \sim \mathrm{B1}\left(\sum_{i=1}^{s}\alpha_i, \alpha_r\right), \quad 1 \le s \le r-1$$

is independent of $\sum_{i=1}^{r}\lambda_i Y_i/(1-Y_i)$.

Theorem 2.5, Theorem 2.7(i), Theorem 2.8(ii) and Theorem 2.9 were also obtained by Libby and Novic [9], and Chen and Novic [2].

3. Limiting Behavior of MB1 Distribution

Making the transformation $W_i = \alpha_{r+1} Y_i, i = 1, ..., r,$ with the Jacobian $J(y_1, ..., y_r \to w_1, ..., w_r) = \alpha_{r+1}^{-r}$ in the joint density of $(Y_1, ..., Y_r)$ given in (7), the density $f(w_1, ..., w_r)$ of $W = (W_1, ..., W_r)$ is obtained as

$$f(w_1, ..., w_r)$$

$$= \left[\prod_{i=1}^{r} \frac{\lambda_i^{\alpha_i} w_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}\right] \frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right)}{\alpha_{r+1}^{\sum_{i=1}^{r} \alpha_i} \Gamma(\alpha_{r+1})} \left[\prod_{i=1}^{r} \left(1 - \frac{w_i}{\alpha_{r+1}}\right)^{-(\alpha_i + 1)}\right]$$

$$\times \left[1 + \frac{1}{\alpha_{r+1}} \sum_{i=1}^{r} \frac{\lambda_i w_i}{1 - w_i / \alpha_{r+1}}\right]^{-\sum_{i=1}^{r+1} \alpha_i}.$$

It is easy to see that

$$\lim_{\alpha_{r+1} \to \infty} \frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right)}{\Gamma(\alpha_{r+1})} \left(\frac{1}{\alpha_{r+1}}\right)^{\sum_{i=1}^{r} \alpha_i} = 1, \tag{19}$$

$$\lim_{\alpha_{r+1} \to \infty} \left(1 - \frac{w_i}{\alpha_{r+1}}\right)^{-(\alpha_i + 1)} = 1 \tag{20}$$

and

$$\lim_{\alpha_{r+1} \to \infty} \left[1 + \frac{1}{\alpha_{r+1}} \sum_{i=1}^{r} \frac{\lambda_i w_i}{1 - w_i / \alpha_{r+1}}\right]^{-\sum_{i=1}^{r+1} \alpha_i} = \exp\left(-\sum_{i=1}^{r} \lambda_i w_i\right). \tag{21}$$

Hence, using (19), (20) and (21), one can see that

$$\lim_{\alpha_{r+1} \to \infty} f(w_1, ..., w_r) = \prod_{i=1}^{r} \left[\frac{\lambda_i^{\alpha_i} w_i^{\alpha_i - 1} \exp(-\lambda_i w_i)}{\Gamma(\alpha_i)}\right],$$

where

$$w_i > 0, \; i = 1, ..., r.$$

4. Limiting Behavior of MB2 Distribution

Making the transformation $W_i = \alpha_{r+1} Z_i,\ i = 1, \ldots, r,$ with the Jacobian $J(z_1, \ldots, z_r \to w_1, \ldots, w_r) = \alpha_{r+1}^{-r}$ in (9), the density of $W = (W_1, \ldots, W_r)$ is derived as

$$g(w_1, \ldots, w_r) = \left[\prod_{i=1}^{r} \frac{\lambda_i^{\alpha_i} w_i^{\alpha_i - 1}}{\Gamma(\alpha_i)}\right] \frac{\Gamma\left(\sum_{i=1}^{r+1} \alpha_i\right)}{\alpha_{r+1}^{\sum_{i=1}^{r} \alpha_i} \Gamma(\alpha_{r+1})}$$

$$\times \left[1 + \frac{1}{\alpha_{r+1}} \sum_{i=1}^{r} \lambda_i w_i\right]^{-\sum_{i=1}^{r+1} \alpha_i}.$$

Now, using (19) and

$$\lim_{\alpha_{r+1} \to \infty} \left[1 + \frac{1}{\alpha_{r+1}} \sum_{i=1}^{r} \lambda_i w_i\right]^{-\sum_{i=1}^{r+1} \alpha_i} = \exp\left(-\sum_{i=1}^{r} \lambda_i w_i\right), \tag{22}$$

we obtain

$$\lim_{\alpha_{r+1} \to \infty} g(w_1, \ldots, w_r) = \prod_{i=1}^{r} \left[\frac{\lambda_i^{\alpha_i} w_i^{\alpha_i - 1} \exp(-\lambda_i w_i)}{\Gamma(\alpha_i)}\right],$$

where $w_i > 0,\ i = 1, \ldots, r.$

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