A BOUNDARY PROPERTY OF SOME SUBCLASSES OF FUNCTIONS OF BOUNDED TYPE IN THE HALF-PLANE

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Abstract

The paper gives the construction of the half-plane analog of the part of the factorization theory of M. M. Djrbashian – V. S. Zakaryan, where Djrbashian's generalized fractional integral was used to establish the descriptive representations and boundary properties of meromorphic in the unit disc functions of the classes $N\{\omega\}$ contained in the Nevanlinna class N of functions of bounded type. Some results of nearly the same type are obtained for several weighted classes of meromorphic in the upper half-plane functions with bounded Tsuji characteristics by application of the Laplace transform along with an Hadamard–Liouville type generalized integro-differential operator with an unbounded integration contour, which becomes the Liouville integro-differentiation in a particular case.

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1. **Introduction**

As it is well-known, the generalized Hadamard operator $L^{(\omega)}$ of M. M. Djrbashian (see [8], pp. xxxi, xxxvi, 344–346, 432, 435) or, as it is called Djrbashian's (Dzherbashyan's) generalized fractional integral, was used to construct the factorization theory of the Nevanlinna type classes $N\{\omega\}$ [1, 2] of functions meromorphic in the unit disc of the complex plane. Some of

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these classes contain Nevanlinna's class N of functions of bounded type and exhaust all functions meromorphic in the unit disc, while the others are contained in N and possess better boundary properties.

This paper gives a half-plane analog of the part of the theory of [1, 2], which relates with the factorization and boundary properties of functions from those classes $N\{\omega\}$ which are contained in Nevanlinna's class N. Along with the Laplace transform, here we use a generalized integro-differential operator with an unbounded integration contour, which becomes the Liouville integro-differentiation in a particular case. As a result, nearly the same type statements, as for $N\{\omega\} \subset N$ in [1, 2], are obtained for several weighted classes of meromorphic in the upper half-plane functions with bounded Tsuji characteristics. The obtained results extend the results of [3], which are on harmonic functions, to meromorphic functions, also they are the extensions of several results of Chapters 3, 5, 8 in [6] to more general weights. The considered in this paper classes of meromorphic functions are defined by the condition that the ω-partial derivatives (see [7]) of the logarithms of their modules satisfy the growth condition of E. D. Solomentsev's class $\mathfrak{N}^m$ [9].

Everywhere below, we assume that $\omega(x)$ is a function of the class Ω, i.e. $\omega(x) > 0$, is nonincreasing in $(0, +\infty)$,

$$\omega(x) \asymp x^\alpha \quad \text{for some} \quad -1 < \alpha < 0 \quad \text{and any} \quad x \geq \Delta_0 > 0$$

($\omega(x) \asymp x^\alpha$ means that $C_1 x^\alpha \leq \omega(x) \leq C_2 x^\alpha$ for some constants $C_{1,2} > 0$)

$$\omega_1(x) := \int_0^x \omega(t) dt < +\infty, \quad 0 < x < +\infty.$$

For $\omega(x) \in \Omega$ and functions $u(z)$ given in the upper half-plane $G^+ = \{z : \text{Im } z > 0\}$, we formally introduce the Hadamard-Liouville type operator

$$L_\omega u(z) := -L_{\omega_1} \frac{\partial}{\partial y} u(z), \quad \text{where} \quad L_{\omega_1} u(z) := \int_0^{+\infty} u(z + i\lambda) d\omega_1(\lambda).$$

Besides, we use the Cauchy type kernel

$$C_\omega(z) := \int_0^{+\infty} e^{izt} \frac{dt}{I_\omega(t)}, \quad I_\omega(t) := t \int_0^{+\infty} e^{-t\lambda} \omega(\lambda) d\lambda,$$

which for power functions $\omega(x) = x^\alpha$ $(-1 < \alpha < +\infty)$ becomes the $1 + \alpha$ order of the Cauchy kernel:

$$C_\omega(z) \Big|_{\omega(x)=x^\alpha} = \frac{1}{(-iz)^{1+\alpha}} := C_\alpha(z), \quad C_\omega(z) \Big|_{\omega(x)=1} = \frac{1}{-iz} = C_0(z)$$

for all $z \in G^+$, and it is easy to verify that

$$L_\omega C_\omega(z) = C_0(z), \quad z \in G^+. \tag{1.1}$$

2. **A boundary property of the ordinary Blaschke product**

It is well known that, if a sequence of numbers $\{\zeta_k\}_1^\infty = \{\xi_k + i\eta_k\}_1^\infty \subset G^+$ satisfies the condition

$$\sum_{k=1}^{\infty} \eta_k < +\infty, \tag{2.1}$$

then, the ordinary Blaschke product

$$B_0(z) := \prod_{k=1}^{\infty} b_0(z, \zeta_k) = \prod_{k=1}^{\infty} \frac{z - \zeta_k}{z - \bar{\zeta}_k}, \quad z \in G^+,$$

uniformly converges everywhere in $\mathbb{C}$, except the closure of the set $\{\bar{\zeta}_k\}_k$, and represents a holomorphic function with zeros $\{\zeta_k\}_k$ and poles $\{\bar{\zeta}_k\}_k$.

The below theorem is on a boundary property of the function $B_0(z)$.

THEOREM 2.1. *Let a sequence* $\{\zeta_k\}_{k=1}^\infty \subset G^+$ *satisfy* (2.1) *and*

$$\sum_{k=1}^{\infty} \frac{\eta_k}{|\zeta_k - x|} < +\infty$$

for some point $x \in (-\infty, +\infty)$. *Then, at this point there exists*

$$B_0(x) = \lim_{y \to +0} B_0(x + iy), \quad \textit{and} \quad |B_0(x)| = 1. \tag{2.2}$$

P r o o f. The relations (2.2) are true for all factors of the product $B_0(z)$. For extending (2.2) to an infinite product, observe that

$$\sum_{k=1}^{\infty} \left| \frac{z - \zeta_k}{z - \bar{\zeta}_k} - 1 \right| = 2 \sum_{k=1}^{\infty} \frac{\eta_k}{|z - \bar{\zeta}_k|} \leq 2 \sum_{k=1}^{\infty} \frac{\eta_k}{|x - \zeta_k|} < +\infty,$$

for any $z = x + iy$ with $0 \leq y \leq R_0 < +\infty$, and hence the product $B_0(z)$ is uniformly convergent on the closed interval $\{z = x + iy : 0 \leq y \leq R_0\}$ perpendicular to the real axis at the point x. □

Now, let us prove some lemmas necessary for proving the main theorem of this section. Beforehand, we recall the following definition from [3].

DEFINITION 2.1. Let $\omega \in \Omega$. Then a Borel measurable set (B-set) $E \subseteq (-\infty, +\infty)$ is of positive ω-capacity or $C_\omega(E) > 0$, if there exists a Borel measure (B-measure) $\tau \geq 0$ supported on E ($\tau \prec E$) and such that

$$\int_{-\infty}^{+\infty} d\tau(t) = \int_E d\tau(t) = 1 \tag{2.3}$$

and

$$S := \sup_{z\in G^+} \int_{-\infty}^{+\infty} |C_\omega(z-t)| d\tau(t) < +\infty. \tag{2.4}$$

If there is no such a measure, i.e. $S = +\infty$ for any nonnegative B-measure $\tau \prec E$ satisfying (2.3), then E is of zero ω-capacity, or $C_\omega(E) = 0$.

LEMMA 2.1. *Let $\omega(x) \in \Omega$, and let*

$$\sum_{k=1}^{\infty} \int_0^{\eta_k} \omega(x)dx < +\infty. \tag{2.5}$$

for a sequence $\{\zeta_k\}_{k=1}^{\infty} \subset G^+$. Then

$$\sum_{k=1}^{\infty} \big|C_\omega(\zeta_k - t)\big| \left(\int_0^{\eta_k} \omega(x)dx\right) < +\infty \tag{2.6}$$

for all $t \in (-\infty, +\infty)$ except, perhaps, a set of points E with $C_\omega(E) = 0$.

P r o o f. On the contrary, suppose $C_\omega(E) > 0$ for a set E, where the sum of (2.6) is divergent. Then, the relations (2.3) and (2.4) are true for some nonnegative B-measure $\tau \prec E$, and by (2.5) and (2.4) we come to a contradiction:

$$\begin{aligned} +\infty &= \int_{-\infty}^{+\infty} \left[\sum_{k=1}^{\infty} \big|C_\omega(\zeta_k - t)\big| \left(\int_0^{\eta_k} \omega(x)dx\right)\right] d\tau(t) \\ &= \sum_{k=1}^{\infty} \left[\left(\int_0^{\eta_k} \omega(x)dx\right) \int_{-\infty}^{+\infty} \big|C_\omega(\zeta_k - t)\big| d\tau(t)\right] \\ &\le \sup_{\zeta\in G^+} \left\{\int_{-\infty}^{+\infty} \big|C_\omega(\zeta - t)\big| d\tau(t)\right\} \sum_{k=1}^{\infty} \int_0^{\eta_k} \omega(x)dx < +\infty. \qquad \square \end{aligned}$$

LEMMA 2.2. *Let $\omega \in \Omega$ and let*

$$\lim_{y\to+0} \Phi(y) = +\infty \quad \text{for} \quad \Phi(y) := \int_y^{+\infty} \frac{dh}{h\int_0^h \omega(x)dx}. \tag{2.7}$$

Then

$$\liminf_{y\to+0} \frac{C_\omega(iy)}{\Phi(y)} \ge J := \int_0^{+\infty} \frac{e^{-x}dx}{1 + x^{-1}e^{-x}}. \tag{2.8}$$

P r o o f. The function $\omega(x)$ is non-increasing on $(0, +\infty)$, and hence

$$\omega(h) \le \frac{1}{h} \int_0^h \omega(x)dx \tag{2.9}$$

for any $h > 0$. Further, for any $t > 0$

$$\int_0^{+\infty} e^{-tx}\omega(x)dx \le \int_0^h \omega(x)dx + \omega(h)\int_h^{+\infty} e^{-tx}dx = \int_0^h \omega(x)dx + \frac{\omega(h)}{t}e^{-th}.$$

Therefore, by (2.9)

$$\int_0^{+\infty} e^{-tx}\omega(x)dx \le \left(1 + \frac{e^{-th}}{th}\right)\int_0^h \omega(x)dx.$$

Hence,

$$-\frac{\partial}{\partial h}C_\omega(ih) = \int_0^{+\infty} \frac{e^{-ht}dt}{\int_0^{+\infty} e^{-tx}\omega(x)dx} \ge \frac{\int_0^{+\infty} \frac{e^{-ht}dt}{1+(ht)^{-1}e^{-ht}}}{\int_0^h \omega(x)dx},$$

i.e.

$$J\left[h\int_0^h \omega(x)dx\right]^{-1} \le -\frac{\partial}{\partial h}C_\omega(ih). \tag{2.10}$$

Now, note that by Lemma 2.1 of [3]

$$C_\omega(iy) = -\int_y^{+\infty} \frac{\partial}{\partial h}C_\omega(ih)dh.$$

Hence, for any fixed $0 < M < +\infty$ and $0 < y < M$

$$C_\omega(iy) = -\int_y^M \frac{\partial}{\partial h}C_\omega(ih)dh + A_0,$$

where $A_0 := -\int_M^{+\infty} \frac{\partial}{\partial h}C_\omega(ih)dh > 0$ is a constant. Therefore, by (2.10)

$$C_\omega(iy) \ge A_0 + J\int_y^M \left[h\int_0^h \omega(x)dx\right]^{-1} dh.$$

Hence, (2.8) holds by (2.7). □

LEMMA 2.3. *Let a function $\omega \in \Omega$ be such that (2.7) is true. Then*

$$\liminf_{y\to+0}\left[C_\omega(iy)\int_0^y \omega(x)dx\right] \ge J > 0. \tag{2.11}$$

P r o o f. Along with $\Phi(y)$ defined by (2.7), we consider the function

$$G(y) = \left[\int_0^y \omega(x)dx\right]^{-1}$$

and observe that $\lim_{y\to+0}\Phi(y) = \lim_{y\to+0}G(y) = +\infty$. Further, applying the Cauchy mean value theorem and (2.9) we conclude that for any fixed $0 <$

$y < y_0 < +\infty$ there is a point $\widetilde{y}$ $(y < \widetilde{y} < y_0)$ such that

$$\frac{\Phi(y_0) - \Phi(y)}{G(y_0) - G(y)} = \frac{\Phi'(\widetilde{y})}{G'(\widetilde{y})} = \frac{\int_0^{\widetilde{y}} \omega(x)dx}{\widetilde{y}\omega(\widetilde{y})} \geq 1.$$

Hence

$$\frac{\Phi(y)}{G(y)} \geq 1 + \frac{\Phi(y_0)}{G(y)} - \frac{G(y_0)}{G(y)} = 1 + o(1) \quad \text{as} \quad y \to +0.$$

Thus, $\liminf_{y\to+0} \left[\frac{\Phi(y)}{G(y)}\right] \geq 1$, and (2.11) follows by (2.8). □

Now, we proceed to the main theorem of this section.

THEOREM 2.2. *Let $\omega(x) \in \Omega$, let (2.7) be true, and let $\{\zeta_k\}_{k=1}^{\infty} \subset G^+$ be sequence satisfying (2.5). Then, the product $B_0(z)$ is convergent in G^+, and the relation (2.2) is true for all $x \in (-\infty, +\infty)$ except, perhaps, a set E with $C_\omega(E) = 0$.*

P r o o f. It is easy to verify that the statement of Lemma 4.1 in [3] is true for any countable unions of zero ω-capacity sets. Hence, it suffices to prove that for any $0 < R < +\infty$ the sets $E_R = E \cap (-R, R)$, where (2.2) is not true, are of zero ω-capacity, i.e. $C_\omega(E_R) = 0$. To this end, we decompose

$$B_0(z) = \left(\prod_{|\xi_k|<R} + \prod_{|\xi_k|\geq R}\right) \frac{z-\zeta_k}{z-\bar{\zeta}_k} := A(z) + B(z)$$

and observe that $B(z)$ is a holomorphic function in the strip $\{z = x + iy : -R \leq x \leq R,\ -\infty < y < +\infty\}$ containing the interval $(-R, R)$ and $|B(x)| \equiv 1$, $-R < x < R$. Thus, it suffices to prove the desired statement for $A(z)$ and the set E_R. To this end, observe that by Lemma 2.1 the sum

$$\sum_{|\xi_k|<R} \left|C_\omega(\xi_k + i\eta_k - t)\right| \left(\int_0^{\eta_k} \omega(x)dx\right)$$

converges for all $|t| < R$, except, perhaps, a set $H \subset [-R, R]$ with $C_\omega(H) = 0$. Using Lemma 4.2 of [3], we obtain that for any $z = x + iy \in G^+$

$$|C_\omega(z)| \geq \text{Re } C_\omega(z) = \text{Re } L_{\widetilde{\omega}}\left(\frac{1}{-iz}\right) = \text{Re } \int_0^{+\infty} \frac{1}{-i(z+i\sigma)} d\widetilde{\omega}(\sigma)$$
$$= \int_0^{+\infty} \frac{y+\sigma}{|z+i\sigma|^2} d\widetilde{\omega}(\sigma), \tag{2.12}$$

where $\widetilde{\omega}(\sigma)$ is a nondecreasing function in $(0,+\infty)$, such that

$$\widetilde{\omega}(0)=0 \quad \text{and} \quad \widetilde{\omega}(t)\le \frac{1}{\omega(t)}, \quad 0<t<+\infty.$$

Further, for $z=x+iy$ with $|x|<R$, $0<y<R_0<+\infty$, and any $\sigma>0$

$$\frac{|z|}{|z+i\sigma|^2}\ge M\frac{y}{(y+\sigma)^2}, \tag{2.13}$$

where $M>0$ is a constant depending only on R and R_0. Indeed, (2.13) is obvious for $\sigma\ge 1$. For the case $0<\sigma<1$, observe that the change the variables $w=e^{iz}$, $\tau=e^{-\sigma}$ in the inequality

$$\frac{|1-w|}{|1-w\tau|^2}\ge\frac{1-|w|}{(1-|w|\tau)^2}, \quad |w|<1, \quad 0\le\tau<1,$$

(see [2], p. 83) gives

$$\frac{|1-e^{iz}|}{|1-e^{i(z+i\sigma)}|^2}\ge\frac{1-e^{-y}}{(1-e^{-(y+\sigma)})^2}, \quad z\in G^+, \quad 0<\sigma<+\infty.$$

For finishing the proof of (2.13), it remains to see that expanding the considered functions in their Taylor series we can find some constants $C_{1,2,3}>0$ depending only on R and R_0 and such that $C_1|z|\ge|1-e^{iz}|\ge 1-e^{-y}\ge C_2y$, $|z+i\sigma|\le C_3\big|1-e^{i(z+i\sigma)}\big|$ and $1-e^{-(y+\sigma)}\le y+\sigma$.

By (2.12) and (2.13) we obtain that for $|x|<R$ and $0<y<R_0$

$$|C_\omega(x+iy)|\ge\frac{My}{|x+iy|}\int_0^{+\infty}\frac{d\widetilde{\omega}(\sigma)}{y+\sigma}=\frac{My}{|x+iy|}C_\omega(iy). \tag{2.14}$$

Further, by (2.11) there exists a number $\delta>0$ such that

$$C_\omega(iy)\int_0^y\omega(x)dx\ge\delta>0, \quad 0<y\le R_0.$$

Consequently, by (2.14) and (2.9) we conclude that

$$M\delta\frac{\eta}{|\zeta-t|}\le|C_\omega(\zeta-t)|\int_0^\eta\omega(x)dx, \quad \zeta=\xi+i\eta\in G^+, \ -R<t<R.$$

Hence, the desired statement for $A(z)$ follows by Theorem 2.1. □

3. A boundary property of subclasses of meromorphic functions of bounded type in the half-plane

We start by a consideration of delta-subharmonic functions which are a generalization of $\log|f(z)|$ of a meromorphic function $f(z)$. Before recalling some necessary definitions from [7], note that under the assumption that $U(z)$ is a delta-subharmonic function and $\omega(t)\in\Omega$ we use the Tsuji characteristics of the form

$$\mathfrak{L}(\rho,\pm L_\omega U)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\big\{\pm L_\omega U(x+i\rho)\big\}^+dx+\iint_{G_\rho^+}\left(\int_0^{\mathrm{Im}\,\zeta}\omega(t)dt\right)d\nu_\mp(\zeta),$$

where $0<\rho<+\infty$ and $a^+=\max\{a,0\}$. Note that this definition is of sense. Indeed, if $U(z)$ has a sufficiently rapid rate of decrease at ∞ and its charge $\nu(\zeta)$ is such that the Green type potential composed by the Blaschke type factors of [7] converges, then the function $u(z)=U(z)-P_\omega(z)$, where

$$P_\omega(z)=\iint_{G^+}\log|b_\omega(z,\zeta)|d\nu(\zeta)$$

is the mentioned potential, is harmonic in G^+, and $L_\omega U(z)=L_\omega u(z)+L_\omega P_\omega(z)$, where $L_\omega u(z)$ and $L_\omega P_\omega(z)$ are well defined in [7].

Definition 3.1. For $\omega(t)\in\Omega$ satisfying Hölder's condition on $(0,R_0]$ $(0<R_0<+\infty)$, the class $\mathfrak{N}_\omega^m$ is the set of all delta-subharmonic in G^+ functions $U(z)$ such that:

(i) The associated with $U(z)$ charge ν is such that $\mathrm{Im}\,\{\mathrm{supp}\,\nu\}\le R_0<+\infty$, and for any $\rho>0$ the closure of the set $\mathrm{Re}\,\{(\mathrm{supp}\,\nu)\cap G_\rho^+\}$ is of zero Lebesgue measure.

(ii) $U(z)\in M_\omega$, i.e. there exists an angular domain $\Delta(\delta_0,R_0)=\{z:|\pi/2-\arg z|<\delta_0,|z|\ge R_0\}$ with some $0<\delta_0\le\pi/2$ and $0<R_0<+\infty$, such that

$$\sup_{z\in\mathcal{K}}\int_0^{+\infty}\left|\frac{\partial}{\partial y}U(z+i\sigma)\right|\omega(\sigma)d\sigma<+\infty$$

for any compact $\mathcal{K}\subset G\cap\Delta(\delta_0,R_0)$.

(iii) The Tsuji characteristics of $L_\omega U(z)$ is such that

$$\sup_{\rho>0}\big[\mathfrak{L}(\rho,L_\omega U)+\mathfrak{L}(\rho,-L_\omega U)\big]:=S<+\infty.$$

Remark 3.1. The class $\mathfrak{N}_\omega^m$ is the same as that of Definition 4.1 in [7]. For harmonic in G^+ functions $U(z)$, Definition 3.1 differs from Definition 3.1 of [3] only in Hölder's condition for $\omega(x)$, which provides some properties of the functions $L_\omega P_\omega(z)$ and $L_\omega P_0(z)$.

Before proving a theorem on the difference of the Green type potential $P_\omega(z)$ and the ordinary Green potential $P_0(z)$, note the following formula

$$\log b_0(z,\zeta)=\log\frac{z-\zeta}{z-\overline{\zeta}}=\int_{-\eta}^{\eta}\frac{dt}{t+i(z-\xi)},\quad z\neq\zeta,\tag{3.1}$$

for the logarithm of the ordinary Blaschke factor and recall that the Green potential

$$P_0(z)=\iint_{G^+}\log|b_0(z,\zeta)|d\nu(\zeta),\quad z\in G^+,$$

is convergent when the Borel measure $\nu(\zeta)\geq 0$ satisfies the Blaschke condition

$$\iint_{G^+}\frac{\eta}{1+|\zeta|^2}\,d\nu(\zeta)<+\infty,\quad \zeta=\xi+i\eta.$$

THEOREM 3.1. *Let $\omega(x)\in\Omega$, let the Borel measure $\nu(\zeta)\geq 0$ of a Green type potential $P_\omega(z)$ satisfy the condition $\iint_{G^+}\left(\int_0^\eta\omega(x)dx\right)d\nu(\zeta)<+\infty$ and $\sup\{\mathrm{Im}(\mathrm{supp}\ \nu)\}=R_0<+\infty$. Further, let for any $\rho>0$ the set $\mathrm{Re}\left\{(\mathrm{supp}\ \nu)\bigcap G^+_\rho\right\}$ be nowhere dense in $(-\infty,+\infty)$. Then, the function*

$$\Phi_\omega(z):=\iint_{G^+}L_\omega\log\frac{b_\omega(z,\zeta)}{b_0(z,\zeta)}d\nu(\zeta)\tag{3.2}$$

is holomorphic in G^+, and for any $z\in G^+$

$$\mathrm{Re}\,L_{\widetilde{\omega}}\Phi_\omega(z)=P_\omega(z)-P_0(z)=a_0+a_1x-\frac{1}{\pi}\int_{-\infty}^{+\infty}\mathrm{Re}\,C_\omega(z-t)d\sigma(t),\tag{3.3}$$

where $L_{\widetilde{\omega}}$ is the operator of (2.12), $a_0,a_1\in(-\infty,+\infty)$ are some numbers and $\sigma(t)$ is of bounded variation on $(-\infty,+\infty)$.

P r o o f. Note that by Lemma 2.5 of [7] the integrand $\varphi_\omega(z,\zeta):=L_\omega\log\frac{b_\omega(z,\zeta)}{b_0(z,\zeta)}$ in (3.2) is holomorphic in G^+. Besides, by formulas (2.13) of [7] and (2.9) we obtain hat for any $0<\rho_0<+\infty$

$$\begin{aligned}|\varphi_\omega(z,\zeta)|&\leq\int_\eta^{+\infty}\frac{2\eta\omega(t)dt}{|z-\zeta+it||z-\overline{\zeta}+it|}+\int_0^\eta\frac{\omega(t)dt}{|z-\overline{\zeta}-it|}+\int_0^\eta\frac{\omega(t)dt}{|z-\overline{\zeta}+it|}\\&\leq 2\eta\omega(\eta)\int_\eta^{+\infty}\frac{dt}{(y-\eta+t)(y+\eta+t)}+\int_0^\eta\frac{\omega(t)dt}{y+\eta-t}+\int_0^\eta\frac{\omega(t)dt}{y+\eta+t}\\&\leq 2\eta\omega(\eta)\int_\eta^{+\infty}\frac{dt}{(y-\eta+t)^2}+\frac{2}{y}\int_0^\eta\omega(t)dt=\frac{2\eta\omega(\eta)}{y}+\frac{2}{y}\int_0^\eta\omega(t)dt\\&\leq\frac{4}{y}\int_0^\eta\omega(t)dt\leq\frac{4}{\rho_0}\int_0^\eta\omega(t)dt,\quad z=x+iy\in G^+_{\rho_0}.\end{aligned}\tag{3.4}$$

Hence, by (2.5) we conclude that for any $z\in G^+_{\rho_0}$

$$|\Phi_\omega(z)|\leq\iint_{G^+}\left|L_\omega\frac{b_\omega(z,\zeta)}{b_0(z,\zeta)}\right|d\nu(\zeta)\leq\frac{4}{\rho_0}\iint_{G^+}\left(\int_0^\eta\omega(t)dt\right)d\nu(\zeta)<+\infty.$$

Thus, the integral of $\Phi_\omega(z)$ is uniformly convergent in any half-plane $G^+_{\rho_0}$, $\rho_0>0$, and $\Phi_\omega(z)$ is holomorphic in G^+. Besides, $\operatorname{Re}\Phi_\omega(z)\le 0$, $z\in G^+$, by Lemma 2.6 of [7], and hence by the Herglotz-Riesz theorem

$$\operatorname{Re}\Phi_\omega(z)=py-\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\sigma(t)}{(x-t)^2+y^2},\quad z=x+iy\in G^+,$$

where $p=\lim\limits_{y\to+\infty} y^{-1}\operatorname{Re}\Phi_\omega(iy)\le 0$, while $\sigma(t)$ is nondecreasing and $\int_{-\infty}^{+\infty}\frac{d\sigma(t)}{1+t^2}<+\infty$. Further, $\sup\limits_{y>0} y\,|\Phi_\omega(iy)|<+\infty$ by (3.4) and (2.5), and hence

$$\begin{aligned}\int_{-\infty}^{+\infty}d\sigma(t)&\le\liminf_{y\to+\infty}\int_{-\infty}^{+\infty}\frac{y^2}{t^2+y^2}|d\sigma(t)|\\&=\liminf_{y\to+\infty} y\,|\mathrm{Re}\ \Phi_\omega(iy)|\le\sup_{y>0} y\,|\Phi(iy)|<+\infty.\end{aligned}$$

Consequently, using (1.1) we obtain that for any $z=x+iy\in G^+$

$$\begin{aligned}\operatorname{Re}\Phi_\omega(z)&=-\frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{d\sigma(t)}{(x-t)^2+y^2}\\&=-L_\omega\left(\operatorname{Re}\frac{1}{\pi}\int_{-\infty}^{+\infty}C_\omega(z-t)d\sigma(t)\right),\end{aligned}\tag{3.5}$$

where $\sigma(t)$ is nondecreasing and bounded. On the other hand, $\operatorname{Re}\Phi_\omega(z)=L_\omega P_\omega(z)-L_\omega P_0(z)$ in the whole G^+. To prove this, first observe that

$$L_\omega P_0(z)=\iint_{G^+}L_\omega\log|b_0(z,\zeta)|d\nu(\zeta),\quad y\ge 2\eta.$$

Indeed, by (3.1)

$$\begin{aligned}L_\omega\log|b_0(z,\zeta)|&=\operatorname{Re}\int_0^{+\infty}\omega(\sigma)d\sigma\int_{-\eta}^{\eta}\frac{dt}{[z+i\sigma-\xi-it]^2}\\&=\operatorname{Re}\left(\int_0^\eta+\int_\eta^{\Delta_0}+\int_{\Delta_0}^{+\infty}\right)\omega(\sigma)d\sigma\int_{-\eta}^{\eta}\frac{dt}{[z+i\sigma-\xi-it]^2}\\&:=K_1+K_2+K_3.\end{aligned}$$

Evidently, for any $z=x+iy\in G^+_{2R_0}$, $\eta\le R_0$ and $y+\sigma-t\ge y/2$

$$\begin{aligned}|K_1|&\le\int_0^\eta\omega(\sigma)d\sigma\int_{-\eta}^{\eta}\frac{dt}{|z+i\sigma-\xi-it|^2}\le\int_0^\eta\omega(\sigma)d\sigma\int_{-\eta}^{\eta}\frac{dt}{(y+\sigma-t)^2}\\&\le\frac{4}{R_0^2}\int_0^\eta\omega(\sigma)d\sigma\int_{-\eta}^{\eta}dt\le M_{R_0}\int_0^\eta\omega(\sigma)d\sigma.\end{aligned}$$

Besides, by (2.9)

$$|K_2| \leq \int_{\eta}^{\Delta_0} \omega(\sigma)d\sigma \int_{-\eta}^{\eta} \frac{dt}{|z+i\sigma-\xi-it|^2} \leq \frac{8\eta\omega(\eta)\Delta_0}{y^2}$$
$$\leq M'_{\Delta_0,R_0} \int_0^{\eta} \omega(\sigma)d\sigma,$$

and by the inequalities $y+\sigma-t \geq \sigma$ and $\omega(\eta) \geq \omega(R_0)$ we get

$$|K_3| \leq \int_{\Delta_0}^{+\infty} \sigma^{\alpha} d\sigma \int_{-\eta}^{\eta} \frac{dt}{|z+i\sigma-\xi-it|^2} \leq \int_{\Delta_0}^{+\infty} \sigma^{\alpha-2} d\sigma \int_{-\eta}^{\eta} dt$$
$$\leq \frac{2\eta\omega(\eta)\Delta_0^{\alpha-1}}{(1-\alpha)\omega(R_0)} \leq M''_{\Delta_0,R_0,\alpha} \int_0^{\eta} \omega(\sigma)d\sigma,$$

where $M_{R_0}, M'_{\Delta_0,R_0}, M''_{\Delta_0,R_0,\alpha} > 0$ depend only on α, Δ_0 and R_0. Thus,

$$|L_\omega \log|b_0(z,\zeta)|| \leq M'''_{\Delta_0,R_0,\alpha} \int_0^{\eta} \omega(\sigma)d\sigma$$

with $M'''_{\Delta_0,R_0,\alpha} = M_{R_0} + M'_{\Delta_0,R_0} + M''_{\Delta_0,R_0,\alpha}$, and by (2.5)

$$|L_\omega P_0(z)| \leq \iint_{G^+} |L_\omega \log|b_0(z,\zeta)||\, d\nu(\zeta)$$
$$\leq M'''_{\Delta_0,R_0,\alpha} \iint_{G^+} \left(\int_0^{\eta} \omega(\sigma)d\sigma\right) d\nu(\zeta). \tag{3.6}$$

So, the integrand of $L_\omega P_0(z)$ has an independent of $z \in G^+_{2R_0}$, integrable majorant, and hence $L_\omega P_0(z)$ is harmonic in $G^+_{2R_0}$. Moreover, it is easy to verify that

$$L_\omega P_\omega(z) = \iint_{G^+} L_\omega \log|b_\omega(z,\zeta)| d\nu(\zeta)$$

is harmonic in the domain $\mathcal{D} = \left\{ z \in G^+ : z \notin \bigcup_{\zeta \in \operatorname{supp}\nu} [\zeta, \operatorname{Re}\zeta] \right\}$ of Theorem 3.2 of [7]. Thus, $\operatorname{Re}\Phi_\omega(z) = L_\omega P_\omega(z) - L_\omega P_0(z)$ in the whole G^+ by the uniqueness of harmonic function.

Now observe that by (3.6) $P_0(z) \in M_\omega$ with the domain $\Delta(\pi/2, 2R_0)$, while $P_\omega(z) \in M_\omega$ for $\Delta(\pi/2, R_0+1)$ by (3.8) in [7]. Therefore, by (2.5)

$$\int_{-\infty}^{+\infty} \left|\frac{\partial}{\partial y} P_0(x+iy)\right| dx \leq \int_{-\infty}^{+\infty} dx \iint_{G^+} d\nu(\zeta) \int_{-\eta}^{\eta} \frac{dt}{|z-\xi-it|^2}$$
$$= \iint_{G^+} d\nu(\zeta) \int_{-\eta}^{\eta} dt \int_{-\infty}^{+\infty} \frac{dx}{(x-\xi)^2+(y-t)^2} = \frac{\pi}{y-t} \iint_{G^+} d\nu(\zeta) \int_{-\eta}^{\eta} dt$$
$$\leq \frac{4\pi}{y} \iint_{G^+} \eta\, d\nu(\zeta) < +\infty, \quad y \geq 2R_0.$$

Thus $P_0(z)$ satisfies the condition (2.3) of [3], and by the representation (2.1) of [7] we obtain that for any $z = x + iy$ with $y \geq 2R_0$

$$P_\omega(z) = \mathrm{Re} \iint_{G^+} \left(\int_{-\eta}^{\eta} C_\omega(z - \xi - it)\omega(\eta - |t|)dt \right) d\nu(\zeta).$$

Hence, using the estimate (3.2) of [5] with $\varepsilon \in (0, 1+\alpha)$ and $y \geq 2R_0$ (then $y - t \geq y/2$) and the two-sided inequality $(a+b)^\lambda \asymp a^\lambda + b^\lambda$ $(a, b, \lambda \geq 0)$, we get

$$\begin{aligned}
\int_{-\infty}^{+\infty} &\left| \frac{\partial}{\partial y} P_\omega(x+iy) \right| dx \\
&\leq \int_{-\infty}^{+\infty} |C_{\omega_1}(z - \xi - it)| dx \iint_{G^+} d\nu(\zeta) \int_{-\eta}^{\eta} \omega(\eta - |t|)dt \\
&\leq C_{R_0,\varepsilon} \int_{-\infty}^{+\infty} \frac{dx}{|z - \xi - it|^{1+\varepsilon}} \iint_{G^+} d\nu(\zeta) \int_0^{\eta} \omega(\eta - t)dt \\
&\leq C'_{R_0,\varepsilon} \int_{-\infty}^{+\infty} \frac{ds}{|s|^{1+\varepsilon} + {R_0}^{1+\varepsilon}} \iint_{G^+} d\nu(\zeta) \int_0^{\eta} \omega(t)dt \\
&\leq C''_{R_0,\varepsilon} \int_{-\infty}^{+\infty} \frac{ds}{(|s|+1)^{1+\varepsilon}} \iint_{G^+} \left(\int_0^{\eta} \omega(t)dt \right) d\nu(\zeta) < +\infty,
\end{aligned}$$

where the constants $C_{R_0,\varepsilon}, C'_{R_0,\varepsilon}, C''_{R_0,\varepsilon} > 0$ depend only on R_0 and ε. Thus, $P_\omega(z)$ satisfies the condition (2.3) of [3], along with $P_0(z)$. Finally, by (3.5)

$$\mathrm{Re}\, \Phi_\omega(z) = L_\omega(P_\omega(z) - P_0(z)) = -L_\omega \left(\frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{Re}\, C_\omega(z - t)d\sigma(t) \right)$$

for all $z \in G^+$, and by Lemma 2.3 of [3] we come to formula (3.3). □

Now, we prove the main theorem of this paper describing the boundary behavior of meromorphic functions $f(z)$ for which $\log|f(z)| \in \mathfrak{N}_\omega^m$ (see Definition 3.1).

Theorem 3.2. *Let $\omega(x) \in \Omega$ satisfy Hölder's condition on $(0, R_0]$ $(0 < R_0 < +\infty)$ and be such that (2.5) and (2.7) are true. Then any meromorphic in G^+ function $f(z)$, such that $\log|f(z)| \in \mathfrak{N}_\omega^m$, has non-zero, finite non-tangential boundary values at all points $x \in (-\infty, +\infty)$, except, perhaps, a set of zero ω-capacity.*

P r o o f. If $\log|f(z)| \in \mathfrak{N}_\omega^m$, then by Theorem 4.1 of [7] and Remark 3.1 for any $z \in G^+$

$$\log|f(z)| = a_0 + a_1 x + \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{Re}\big\{C_\omega(z-t)\big\} d\mu(t) + P_0(z) + (P_\omega(z) - P_0(z)),$$

where the potentials $P_\omega(z)$ and $P_0(z)$ are formed by the discrete charge of the delta-subharmonic function $\log|f(z)|$, while a_0, a_1 are some real numbers. Hence, the following factorization holds by Theorem 3.1:

$$f(z) = \frac{B_0(z,\{a_k\})}{B_0(z,\{b_k\})} \times \exp\left\{c_0 + c_1 z + \frac{1}{\pi}\int_{-\infty}^{+\infty} C_\omega(z-t)d(\mu(t)-\sigma(t)) + iC\right\} \quad (3.7)$$

for $z \in G^+$, where c_0, c_1, C are real numbers, $\mu(t)$ and $\sigma(t)$ are some functions of bounded variation on $(-\infty,+\infty)$ and $\{a_k\},\{b_n\} \subset G^+$ are the sequences of zeros and poles of the function $f(z)$. Further, by Theorem 2.2 the limits of the Blaschke products in the numerator and denominator of the factorization (3.7) by perpendiculars, and hence also by any nontangential paths, to the real axis exist, are finite and non-zero everywhere, except some sets $E_{1,2}$ with $C_\omega(E_{1,2}) = 0$. By Lemma 4.4 of [3], the exponential factor in the factorization (3.7) has the same property everywhere, except a set E_3 with $C_\omega(E_3) = 0$. Hence, the function $f(z)$ has the same property everywhere, except the set $E_4 = E_1 \cup E_2 \cup E_3$, and $C_\omega(E_4) = 0$ by Lemma 4.1 of [3]. □

Corollary 3.1. *The uniqueness sets of the class of meromorphic in G^+ functions possessing the factorization* (3.7) *are the exceptional sets of positive ω-capacity on the real axis, where the concision of the nontangential boundary values of two functions imply the concision of these functions in the whole G^+.*

Remark 3.2. If in the factorization (3.7)

$$c_1 = \lim_{y\to+\infty} y^{-1}\log|f(x+iy)| = 0$$

for some $x \in (-\infty,+\infty)$, then the function $f(z)$ is of bounded type in G^+. Thus, in particular Theorem 3.2 and its Corollary 3.1 establish a boundary property of a subclass of functions of bounded type in G^+.

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