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# A Generalized Matrix Variate Beta Distribution

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**Abstract**

A new distribution which contains the matrix variate beta distribution as a special case is introduced. Several properties of the distribution have been studied.

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## 1. Introduction

A random variable $x$ is said to have the beta distribution with parameters $\alpha$ and $\beta$ if its probability density function (pdf) is given by

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1, \tag{1.1}$$

for $\alpha > 0$ and $\beta > 0$, where

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$$

denotes the beta function. The beta distribution is very versatile and a variety of uncertainties can be usefully modeled by it. Many of the finite range distributions encountered in practice can be easily transformed into the standard beta distribution. Several univariate generalizations of this distribution are given in Gordy [1], McDonald and Xu [11], and Gupta and Nadarajah [6]. Recently, Nadarajah and Kotz [12] introduced a univariate generalization of (1.1) involving the Gauss hypergeometric function. They also studied its particular cases and properties such as cdf, moments and hazard rate function. Their generalization of the beta distribution has the pdf

$$g(x)=\frac{\Gamma(\alpha+\beta+\gamma)\Gamma(\beta+1)}{\Gamma(\alpha+\beta)\Gamma(\beta+\gamma)}x^{\alpha+\beta-1}{}_2F_1\left(1-\gamma,\alpha;\alpha+\beta;x\right),\quad 0<x<1,\tag{1.2}$$

where $\alpha>0$, $\beta\geq 0$, $\beta+\gamma>0$ and ${}_2F_1$ is the Gauss hypergeometric function (Luke [10]).

In this article, we introduce a matrix variate generalization of (1.2) involving Gauss hypergeometric function of matrix argument. Several properties of this distribution such as cdf, moments, invariance, are derived in Section 3. In the end, we give distributions of products of random matrices involving this distribution.

## 2. Preliminaries

We begin with a brief review of some definitions and notations. Let $A=(a_{ij})$ be an $m\times m$ matrix. Then, $A'$ denotes the transpose of $A$; $\mathrm{tr}(A)=a_{11}+\cdots+a_{mm}$; $\mathrm{etr}(A)=\exp(\mathrm{tr}(A))$; $\det(A)=\text{determinant of }A$; $A>0$ means that $A$ is symmetric positive definite and $A^{1/2}$ denotes the unique symmetric positive definite square root of $A>0$. The multivariate gamma function which is frequently used in multivariate statistical analysis is defined by

$$\begin{aligned}\Gamma_m(a)&=\int_{X>0}\mathrm{etr}(-X)\det(X)^{a-(m+1)/2}\,dX\\&=\pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma\left(a-\frac{i-1}{2}\right),\quad \mathrm{Re}(a)>\frac{m-1}{2}.\end{aligned}\tag{2.1}$$

The multivariate generalization of the beta function is given by

$$\begin{aligned}B_m(a,b)&=\int_0^{I_m}\det(X)^{a-(m+1)/2}\det(I_m-X)^{b-(m+1)/2}\,dX\\&=\frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)}=B_m(b,a),\quad \mathrm{Re}(a)>\frac{m-1}{2},\quad \mathrm{Re}(b)>\frac{m-1}{2}.\end{aligned}\tag{2.2}$$

The generalized hypergeometric coefficient is defined by

$$(a)_\rho=\prod_{i=1}^{m}\left(a-\frac{i-1}{2}\right)_{r_i},\tag{2.3}$$

$$(a)_j=a(a+1)\cdots(a+j-1),\quad j=1,2,\ldots\quad\text{and}\quad (a)_0=1\tag{2.4}$$

where $\rho=(r_1,\ldots,r_m), r_1\geq\cdots\geq r_m\geq 0$ and $r_1+\cdots+r_m=r$. The generalized hypergeometric function of one matrix is defined by

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;X)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_p)_\kappa}{(b_1)_\kappa\cdots(b_q)_\kappa}\frac{C_\kappa(X)}{k!}\tag{2.5}$$

where $a_i$, $i = 1, \ldots, p$, $b_j$, $j = 1, \ldots, q$ are arbitrary complex numbers, $X\,(m \times m)$ is a complex symmetric matrix, $C_\kappa(X)$ is the zonal polynomial of $m \times m$ complex symmetric matrix $X$ corresponding to the partition $\kappa$ and $\sum_\kappa$ denotes summation over all partitions $\kappa$. Conditions for convergence of the series in (2.5) are available in the literature. From (2.5) it follows that

$$ {}_0F_0(X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{C_\kappa(X)}{k!} = \sum_{k=0}^{\infty} \frac{(\operatorname{tr} X)^k}{k!} = \operatorname{etr}(X) \tag{2.6} $$

$$ {}_1F_0(a; X) = \sum_{k=0}^{\infty} \sum_{\kappa} (a)_\kappa \frac{C_\kappa(X)}{k!} = \det(I_m - X)^{-a}, \quad \|X\| < 1 \tag{2.7} $$

and

$$ {}_2F_1(a, b; c; X) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(a)_\kappa (b)_\kappa}{(c)_\kappa} \frac{C_\kappa(X)}{k!}, \quad \|X\| < 1. \tag{2.8} $$

The integral representation of the Gauss hypergeometric function ${}_2F_1$ is given by

$$ \begin{aligned} {}_2F_1(a, b; c; X) &= \frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_0^{I_m} \det(R)^{a-(m+1)/2} \\ &\quad \times \det(I_m - R)^{c-a-(m+1)/2} \det(I_m - XR)^{-b}\, dR \end{aligned} \tag{2.9} $$

where $\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$. From above it is easy to see that

$$ {}_2F_1(a, b; c; I_m) = \frac{\Gamma_m(c)\Gamma_m(c-a-b)}{\Gamma_m(c-a)\Gamma_m(c-b)}, \quad \operatorname{Re}(c-a-b) > \frac{m-1}{2}, \tag{2.10} $$

and

$$ \begin{aligned} {}_2F_1(a, b; c; X) &= \det(I_m - X)^{-b}\, {}_2F_1(c-a, b; c; -X(I_m - X)^{-1}) \\ &= \det(I_m - X)^{c-a-b}\, {}_2F_1(c-a, c-b; c; X). \end{aligned} \tag{2.11} $$

The generalized binomial expansion is given by

$$ \frac{C_\sigma(I_m + Y)}{C_\sigma(I_m)} = \sum_{k=0}^{s} \sum_{\kappa} \binom{\sigma}{\kappa} \frac{C_\kappa(Y)}{C_\kappa(I_m)}, \tag{2.12} $$

where the inner summation is over all partitions $\kappa$ of the integer $k$. Thus, using the above expansion, ${}_2F_1\,(a, b; c; I_m - X)$ can be expanded as

$$ {}_2F_1\,(a, b; c; I_m - X) = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \sum_{\sigma} \sum_{\kappa} \binom{\sigma}{\kappa} \frac{(a)_\sigma (b)_\sigma}{(c)_\sigma s!} \frac{C_\sigma(I_m) C_\kappa(-X)}{C_\kappa(I_m)}. \tag{2.13} $$

**Theorem 2.1:** Let $X(m \times m)$ be a symmetric matrix, then

$$\int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m - R)^{b-(m+1)/2} C_\kappa(XR)\, dR = \frac{\Gamma_m(a,\kappa)\Gamma_m(b)}{\Gamma_m(a+b,\kappa)} C_\kappa(X) \tag{2.14}$$

and

$$\int_0^{I_m} \det(R)^{a-(m+1)/2} \det(I_m - R)^{b-(m+1)/2} {}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;XR)\, dR = \frac{\Gamma_m(a)\Gamma_m(b)}{\Gamma_m(a+b)} {}_{p+1}F_{q+1}(a_1,\dots,a_p,a;b_1,\dots,b_q,a+b;X). \tag{2.15}$$

The invariant polynomial of $m \times m$ symmetric matrix arguments $X$ and $Y$ will be denoted by $C_\phi^{\kappa,\lambda}(X,Y)$. Let $\kappa$, $\lambda$, $\phi$ and $\rho$ be partitions of the non-negative integers $k$, $\ell$, $f = k + \ell$ and $r$, respectively. Then (Davis [4, 5], Chikuse [2] and Nagar and Gupta [13]),

$$C_\phi^{\kappa,\lambda}(X,X) = \theta_\phi^{\kappa,\lambda} C_\phi(X), \quad \theta_\phi^{\kappa,\lambda} = \frac{C_\phi^{\kappa,\lambda}(I_m,I_m)}{C_\phi(I_m)}, \tag{2.16}$$

$$C_\phi^{\kappa,\lambda}(X,I_m) = \theta_\phi^{\kappa,\lambda} \frac{C_\phi(I_m)C_\kappa(X)}{C_\kappa(I_m)}, \quad C_\phi^{\kappa,\lambda}(I_m,Y) = \theta_\phi^{\kappa,\lambda} \frac{C_\phi(I_m)C_\lambda(Y)}{C_\lambda(I_m)}, \tag{2.17}$$

$$C_\kappa^{\kappa,0}(X,Y) \equiv C_\kappa(X), \quad C_\lambda^{0,\lambda}(X,Y) \equiv C_\lambda(Y),$$

$$C_\kappa(X)C_\lambda(Y) = \sum_{\phi\in\kappa\cdot\lambda} \theta_\phi^{\kappa,\lambda} C_\phi^{\kappa,\lambda}(X,Y), \quad C_\kappa(X)C_\lambda(X) = \sum_{\phi\in\kappa\cdot\lambda} (\theta_\phi^{\kappa,\lambda})^2 C_\phi(X), \tag{2.18}$$

where $\phi \in \kappa \cdot \lambda$ denotes that irreducible representation of $Gl(m,R)$, the group of $m \times m$ real invertible matrices, indexed by $2\phi$, appears in the decomposition of the tensor product $2\kappa \bigotimes 2\lambda$ of the irreducible representation indexed by $2\kappa$ and $2\lambda$, and

$$\int_0^{I_m} \det(R)^{t-(m+1)/2} \det(I_m - R)^{u-(m+1)/2} C_\phi^{\kappa,\lambda}(R, I_m - R)\, dR = \frac{\Gamma_m(t,\kappa)\Gamma_m(u,\lambda)}{\Gamma_m(t+u,\phi)} \theta_\phi^{\kappa,\lambda} C_\phi(I_m). \tag{2.19}$$

In the above expression, $\Gamma_m(a,\rho)$ is defined by

$$\Gamma_m(a,\rho) = (a)_\rho \Gamma_m(a), \quad \Gamma_m(a,0) = \Gamma_m(a). \tag{2.20}$$

Now, consider the following integral involving the Gauss hypergeometric function of matrix argument

$$f(Z) = \int_0^{I_m} \det(X)^{\nu-(m+1)/2} \det(I_m - X)^{\sigma-(m+1)/2} C_\lambda(Z(I_m - X))\, {}_2F_1(a, b; d; X)\, dX \tag{2.21}$$

where $\lambda$ denotes the partition $\lambda = (\ell_1, \ldots, \ell_m)$, $\ell_1 \geq \cdots \geq \ell_m \geq 0$, $\ell_1 + \cdots + \ell_m = \ell$, $C_\lambda(X)$ is the zonal polynomial of the symmetric $m \times m$ matrix $X$ corresponding to the partition $\lambda$.

It can easily be seen that for any $H \in O(m)$, $f(Z) = f(HZH')$. Thus, integrating $f(HZH')$ over the orthogonal group, $O(m)$, we obtain

$$f(Z) = \frac{f(I_m) C_\lambda(Z)}{C_\lambda(I_m)}. \tag{2.22}$$

Expanding ${}_2F_1$ and using results on invariant polynomials (Nagar and Gupta [14]), we obtain

$$f(I_m) = \frac{\Gamma_m(\nu)\Gamma_m(\sigma, \lambda)}{\Gamma_m(\nu + \sigma)} \sum_{k=0}^{\infty} \sum_{\kappa} \sum_{\phi \in \kappa \cdot \lambda} \frac{(a)_\kappa (b)_\kappa (\nu)_\kappa}{(d)_\kappa (\nu + \sigma)_\phi k!} \left(\theta_\phi^{\kappa,\lambda}\right)^2 C_\phi(I_m). \tag{2.23}$$

For $\nu = d$, a closed form solution is given as (Subrahmaniam [15], Kabe [9]),

$$f(I_m) = \frac{\Gamma_m(d)\Gamma_m(\sigma, \lambda)\Gamma_m(d + \sigma - a - b, \lambda)}{\Gamma_m(d + \sigma - a, \lambda)\Gamma_m(d + \sigma - b, \lambda)} C_\lambda(I_m). \tag{2.24}$$

## 3. The Matrix Variate Hypergeometric Beta Distribution

The matrix variate generalization of (1.1) is given by

$$\{B_m(\alpha, \beta)\}^{-1} \det(X)^{\alpha-(m+1)/2} \det(I_m - X)^{\beta-(m+1)/2}, \quad 0 < X < I_m, \tag{3.1}$$

where $\alpha > (m-1)/2$, $\beta > (m-1)/2$ and $B_m(\alpha, \beta)$ is the multivariate beta function. This distribution is designated by $X \sim B_m^I(\alpha, \beta)$. For properties of this and several other matrix variate distributions the reader is referred to Javier and Gupta [8], Gupta and Nagar [7] and Nagar and Gupta [13]. Next, we define the matrix variate generalization of (1.2).

**Definition 3.1:** An $m \times m$ random symmetric positive definite matrix $X$ is said to have a matrix variate hypergeometric beta distribution with parameters $(\alpha, \beta, \gamma)$, denoted as $X \sim HB_m(\alpha, \beta, \gamma)$, if its p.d.f. is given by

$$K(\alpha, \beta, \gamma) \det(X)^{\alpha+\beta-(m+1)/2}\, {}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha + \beta; X\right), \quad 0 < X < I_m, \tag{3.2}$$

where $\alpha > (m-1)/2$, $\beta \geq (m-1)/2$, $\beta + \gamma > (m-1)/2$ and $K(\alpha, \beta, \gamma)$ is the normalizing constant.

The normalizing constant $K(\alpha,\beta,\gamma)$ is given by

$$\begin{aligned}[K(\alpha,\beta,\gamma)]^{-1} &= \int_0^{I_m} \det(X)^{\alpha+\beta-(m+1)/2}\, {}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta;X\right)dX \\ &= B_m\left(\alpha+\beta,\frac{m+1}{2}\right) {}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta+\frac{m+1}{2};I_m\right)\end{aligned} \tag{3.3}$$

where we have used the result (2.15). Further, using (2.10), the Gauss hypergeometric function in the above expression is simplified as

$${}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta+\frac{m+1}{2};I_m\right) = \frac{\Gamma_m[\alpha+\beta+(m+1)/2]\Gamma_m(\beta+\gamma)}{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]}. \tag{3.4}$$

Now, substituting (3.4) in (3.3), we obtain

$$\begin{aligned}[K(\alpha,\beta,\gamma)]^{-1} &= \frac{B_m(\alpha+\beta,(m+1)/2)B_m(\alpha,\beta+\gamma)}{B_m(\alpha,\beta+(m+1)/2)} \\ &= \frac{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+\gamma)\Gamma_m[(m+1)/2]}{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]}.\end{aligned} \tag{3.5}$$

Using (2.11), the density (3.2) can also be expressed as

$$\begin{aligned}&K(\alpha,\beta,\gamma)\det(X)^{\alpha+\beta-(m+1)/2}\det(I_m-X)^{\beta+\gamma-(m+1)/2} \\ &\times {}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;X\right), \quad 0<X<I_m.\end{aligned} \tag{3.6}$$

In the next theorem we will derive this distribution using independent beta matrices.

**Theorem 3.2:** Let $X_1$ and $X_2$ be independent, $X_1 \sim B_m^I((m+1)/2,\beta)$ and $X_2 \sim B_m^I(\beta+\gamma,\alpha)$. Then $I_m - X_2^{1/2}X_1X_2^{1/2} \sim HB_m(\alpha,\beta,\gamma)$.

*Proof.* The joint pdf of $X_1$ and $X_2$ is given by

$$\begin{aligned}&\frac{\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+\gamma)}{\Gamma_m[(m+1)/2]\Gamma_m(\beta)\Gamma_m(\beta+\gamma)\Gamma_m(\alpha)}\det(I_m-X_1)^{\beta-(m+1)/2} \\ &\times\det(X_2)^{\beta+\gamma-(m+1)/2}\det(I_m-X_2)^{\alpha-(m+1)/2}, \quad 0<X_i<I_m,\ i=1,2.\end{aligned} \tag{3.7}$$

Now, transforming $Z = I_m - X_2^{1/2}X_1X_2^{1/2}$ with the Jacobian $J(X_1 \to Z) = \det(X_2)^{-(m+1)/2}$ in the above density, the joint pdf of $Z$ and $X_2$ is obtained as

$$\begin{aligned}&\frac{\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+\gamma)}{\Gamma_m[(m+1)/2]\Gamma_m(\beta)\Gamma_m(\beta+\gamma)\Gamma_m(\alpha)}\det(X_2-(I_m-Z))^{\beta-(m+1)/2} \\ &\times\det(X_2)^{\gamma-(m+1)/2}\det(I_m-X_2)^{\alpha-(m+1)/2}, \quad 0<I_m-Z<X_2<I_m.\end{aligned} \tag{3.8}$$

Now to obtain the marginal density of $Z$, we need to integrate $X_2$ in the above expression. Using the substitution $W = Z^{-1/2}(I_m - X_2)Z^{-1/2}$ with the Jacobian $J(X_2 \rightarrow W) = \det(Z)^{(m+1)/2}$ in (3.8) and integrating $W$, the marginal density of $Z$ is derived as

$$\begin{aligned}
&\frac{\Gamma_m[\beta + (m+1)/2]\Gamma_m(\alpha+\beta+\gamma)}{\Gamma_m[(m+1)/2]\Gamma_m(\beta)\Gamma_m(\beta+\gamma)\Gamma_m(\alpha)} \det(Z)^{\alpha+\beta-(m+1)/2} \\
&\quad \times \int_0^{I_m} \det(W)^{\alpha-(m+1)/2} \det(I_m - W)^{\beta-(m+1)/2} \det(I_m - ZW)^{\gamma-(m+1)/2}\, dW \\
&\quad = K(\alpha,\beta,\gamma) \det(Z)^{\alpha+\beta-(m+1)/2}\, {}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha+\beta; Z\right)
\end{aligned}$$

where the last line has been obtained by using (2.9). ■

The corresponding cdf is derived as

$$F(X) = K(\alpha,\beta,\gamma) \int_0^X \det(Y)^{\alpha+\beta-(m+1)/2}{}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha+\beta; Y\right) dY. \quad (3.9)$$

Substituting $Z = X^{-1/2}YX^{-1/2}$ with the Jacobian $J(Y \rightarrow Z) = \det(X)^{(m+1)/2}$ and using (2.15), we obtain

$$\begin{aligned}
F(X) &= K(\alpha,\beta,\gamma) \det(X)^{\alpha+\beta} \\
&\quad \times \int_0^{I_m} \det(Z)^{\alpha+\beta-(m+1)/2}{}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha+\beta; XZ\right) dZ \\
&= \frac{B_m(\alpha, \beta + (m+1)/2)}{B_m(\alpha, \beta+\gamma)} \det(X)^{\alpha+\beta}{}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha+\beta+\frac{m+1}{2}; X\right).
\end{aligned}$$

**Theorem 3.3:** Let $X \sim HB_m(\alpha,\beta,\gamma)$ and $A$ be an $m \times m$ constant nonsingular matrix. Then, the density of $Y = AXA'$ is obtained as

$$\begin{aligned}
&K(\alpha,\beta,\gamma) \det(AA')^{-(\alpha+\beta)} \det(Y)^{\alpha+\beta-(m+1)/2} \\
&\times {}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha+\beta; (AA')^{-1}Y\right),
\end{aligned}$$

where $0 < Y < AA'$.

*Proof.* In the p.d.f. (3.2) of $X$, making the transformation $Y = AXA'$ with the Jacobian $J(X \rightarrow Y) = \det(AA')^{-(m+1)/2}$ gives the desired result. ■

We will denote the above density by $Y \sim HB_m(\alpha,\beta,\gamma,AA')$. Note that if $Y \sim HB_m(\alpha,\beta,\gamma,\Omega)$, then $\Omega^{-1/2}Y\Omega^{-1/2} \sim HB_m(\alpha,\beta,\gamma)$. In the next theorem, it is shown that the matrix variate hypergeometric beta distribution is orthogonally invariant.

**Theorem 3.4:** Let $X \sim HB_m(\alpha,\beta,\gamma)$, and $H$ $(m \times m)$ be an orthogonal matrix, whose elements are either constants or random variables distributed independent of $X$. Then, the distribution of $X$ is invariant under the transformation $X \to HXH'$, and is independent of $H$ in the latter case.

*Proof.* First, let $H$ be a constant orthogonal matrix. Then, from Theorem 3.3, $HXH' \sim HB_m(\alpha, \beta, \gamma)$ since $HH' = I_m$. If, however, $H$ is a random orthogonal matrix, then $HXH'|H \sim HB_m(\alpha, \beta, \gamma)$. Since this distribution does not depend on $H$, $HXH' \sim HB_m(\alpha, \beta, \gamma)$. ■

The moment generating functions of $X$ is now obtained in the following theorem.

**Theorem 3.5:** Let $X \sim HB_m(\alpha, \beta, \gamma)$. Then, the moment generating function of $X = (x_{ij})$, i.e., the joint moment generating function of $x_{11}, x_{12}, \ldots, x_{mm}$ is

$$M_X(Z) = \text{etr}(Z)\,{}_2F_2\left(\frac{m+1}{2}, \beta+\gamma; \alpha+\beta+\gamma, \beta+\frac{m+1}{2}; -Z\right)$$

where $Z = Z'\,(m \times m) = ((1+\delta_{ij})z_{ij}/2)$.

*Proof.* By definition,

$$\begin{aligned} M_X(Z) &= K(\alpha,\beta,\gamma)\int_0^{I_m} \text{etr}(ZX)\det(X)^{\alpha+\beta-(m+1)/2} \\ &\quad \times {}_2F_1\left(\frac{m+1}{2}-\gamma, \alpha; \alpha+\beta; X\right) dX \end{aligned}$$

Now, writing

$$\text{etr}(ZX) = \text{etr}[Z - Z(I_m - X)] = \text{etr}(Z)\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{(-1)^\ell}{\ell!}C_\lambda(Z(I_m - X))$$

in the above expression, we get

$$\begin{aligned} M_X(Z) &= K(\alpha,\beta,\gamma)\text{etr}(Z)\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{(-1)^\ell}{\ell!}\int_0^{I_m} C_\lambda(Z(I_m-X))\det(X)^{\alpha+\beta-(m+1)/2} \\ &\quad \times {}_2F_1\left(\frac{m+1}{2}-\gamma, \alpha; \alpha+\beta; X\right) dX \\ &= K(\alpha,\beta,\gamma)\text{etr}(Z)\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{(-1)^\ell}{\ell!}\frac{\Gamma_m(\alpha+\beta)\Gamma_m((m+1)/2,\lambda)\Gamma_m(\beta+\gamma,\lambda)}{\Gamma_m(\alpha+\beta+\gamma,\lambda)\Gamma_m(\beta+(m+1)/2,\lambda)}C_\lambda(Z) \end{aligned}$$

where the last step has been obtained by using (2.22) and (2.24). Finally, simplification of the above expression using (2.5) yields the desired result. ■

It may be noted here that if $Y \sim HB_m(\alpha, \beta, \gamma, \Omega)$, then the moment generating function of $Y$ can be obtained from the above theorem. Since $Y = \Omega^{1/2}X\Omega^{1/2}$, where $X \sim HB_m(\alpha, \beta, \gamma)$, we have

$$\begin{aligned} M_Y(Z) &= E[\text{etr}(ZY)] = E[\text{etr}(Z\Omega^{1/2}X\Omega^{1/2})] = M_X(\Omega^{1/2}Z\Omega^{1/2}) \\ &= \text{etr}(\Omega Z)\,{}_2F_2\left(\frac{m+1}{2}, \beta+\gamma; \alpha+\beta+\gamma, \beta+\frac{m+1}{2}; -\Omega Z\right). \end{aligned}$$

Next, we derive moments of some functions of the random matrix $X$ having matrix variate Hypergeometric beta distribution.

**Theorem 3.6:** Let $X \sim HB_m(\alpha, \beta, \gamma)$, then

$$\begin{aligned} E[\det(X)^h] &= \frac{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+h)}{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+\gamma)\Gamma_m[\alpha+\beta+(m+1)/2+h]} \\ &\quad \times {}_3F_2\left(\frac{m+1}{2}-\gamma, \alpha, \alpha+\beta+h; \alpha+\beta, \alpha+\beta+\frac{m+1}{2}+h; I_m\right) \end{aligned}$$

and

$$\begin{aligned} &E[\det(I_m - X)^h] \\ &\quad = \frac{\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[h+(m+1)/2]\Gamma_m(\beta+\gamma+h)}{\Gamma_m[(m+1)/2]\Gamma_m(\beta+\gamma)\Gamma_m(\alpha+\beta+\gamma+h)\Gamma_m[\beta+(m+1)/2+h]}. \end{aligned}$$

*Proof.* The $h$-th moment of $\det(X)$ is obtained by

$$\begin{aligned} E[\det(X)^h] &= K(\alpha,\beta,\gamma)\int_0^{I_m} \det(X)^{\alpha+\beta+h-(m+1)/2} \\ &\qquad\qquad \times {}_2F_1\left(\frac{m+1}{2}-\gamma, \alpha; \alpha+\beta; X\right) dX \\ &= \frac{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+h)}{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+\gamma)\Gamma_m[\alpha+\beta+(m+1)/2+h]} \\ &\quad \times {}_3F_2\left(\frac{m+1}{2}-\gamma, \alpha, \alpha+\beta+h; \alpha+\beta, \alpha+\beta+\frac{m+1}{2}+h; I_m\right) \end{aligned}$$

where the last line has been obtained by using (2.15). Similarly,

$$\begin{aligned} E[\det(I_m - X)^h] &= \frac{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]\Gamma_m[h+(m+1)/2]}{\Gamma_m(\beta+\gamma)\Gamma_m[(m+1)/2]\Gamma_m[\alpha+\beta+(m+1)/2+h]} \\ &\quad \times {}_2F_1\left(\frac{m+1}{2}-\gamma, \alpha; \alpha+\beta+\frac{m+1}{2}+h; I_m\right). \end{aligned} \tag{3.10}$$

Further, using (2.10), the Gauss hypergeometric function in the above expression is simplified as

$$\begin{aligned} &{}_2F_1\left(\frac{m+1}{2}-\gamma, \alpha; \alpha+\beta+h+\frac{m+1}{2}; I_m\right) \\ &\qquad = \frac{\Gamma_m[\alpha+\beta+h+(m+1)/2]\Gamma_m(\beta+\gamma+h)}{\Gamma_m(\alpha+\beta+\gamma+h)\Gamma_m[\beta+h+(m+1)/2]}. \end{aligned} \tag{3.11}$$

Substituting (3.11) in (3.10) and simplifying we finally obtain the desired result. ■

If $X \sim HB_m(\alpha, \beta, \gamma)$, then it is easy to see that for an $m \times m$ constant matrix $B$, we have

$$E[C_\lambda(BX)] = \frac{C_\lambda(B)}{C_\lambda(I_m)} E[C_\lambda(X)] \tag{3.12}$$

and

$$E[C_\lambda(B(I_m - X))] = \frac{C_\lambda(B)}{C_\lambda(I_m)} E[C_\lambda(I_m - X)]. \tag{3.13}$$

Further, using the density of $X$, $E[C_\lambda(I_m - X)]$ is derived as

$$\begin{aligned} E[C_\lambda(I_m - X)] &= K(\alpha, \beta, \gamma) \int_0^{I_m} C_\lambda(I_m - X) \det(X)^{\alpha+\beta-(m+1)/2} \\ &\quad \times {}_2F_1\left(\frac{m+1}{2} - \gamma, \alpha; \alpha + \beta; X\right) dX \qquad (3.14) \\ &= \frac{((m+1)/2)_\lambda (\beta+\gamma)_\lambda}{(\alpha+\beta+\gamma)_\lambda (\beta + (m+1)/2)_\lambda} C_\lambda(I_m), \qquad (3.15) \end{aligned}$$

where the last line has been obtained using (2.24). Using results on zonal polynomials, it is easy to see that

$$E[C_{(1)}(I_m - X)] = \frac{m(m+1)(\beta+\gamma)}{(\alpha+\beta+\gamma)(2\beta+m+1)},$$

$$E[\mathrm{tr}(I_m - X)] = \frac{m(m+1)(\beta+\gamma)}{(\alpha+\beta+\gamma)(2\beta+m+1)},$$

$$E[C_{(2)}(I_m - X)] = \frac{m(m+1)(m+2)(m+3)(\beta+\gamma)(\beta+\gamma+1)}{3(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+1)(2\beta+m+1)(2\beta+m+3)},$$

$$E[C_{(1^2)}(I_m - X)] = \frac{2m^2(m^2-1)(\beta+\gamma)(\beta+\gamma-1/2)}{3(\alpha+\beta+\gamma)(\alpha+\beta+\gamma-1/2)(2\beta+m)(2\beta+m+1)},$$

and

$$\begin{aligned} E[\mathrm{tr}(I_m - X)^2] &= E[C_{(2)}(I_m - X)] + E[C_{(1^2)}(I_m - X)] \\ &= \frac{m(m+1)(\beta+\gamma)}{3(\alpha+\beta+\gamma)(2\beta+m+1)} \left[ \frac{(m+2)(m+3)(\beta+\gamma+1)}{(\alpha+\beta+\gamma+1)(2\beta+m+3)} \right. \\ &\quad \left. + \frac{2m(m-1)(\beta+\gamma-1/2)}{(\alpha+\beta+\gamma-1/2)(2\beta+m)} \right]. \end{aligned}$$

Further, using the invariance of the distribution of $I_m - X$ and above results, one obtains

$$\begin{aligned} E(I_m - X) &= \frac{(m+1)(\beta+\gamma)}{(\alpha+\beta+\gamma)(2\beta+m+1)} I_m, \\ E[(I_m - X)^2] &= \frac{(m+1)(\beta+\gamma)}{3(\alpha+\beta+\gamma)(2\beta+m+1)} \left[\frac{(m+2)(m+3)(\beta+\gamma+1)}{(\alpha+\beta+\gamma+1)(2\beta+m+3)} \right. \\ & \qquad \left. + \frac{2m(m-1)(\beta+\gamma-1/2)}{(\alpha+\beta+\gamma-1/2)(2\beta+m)}\right] I_m. \end{aligned}$$

By writing $C_\lambda(X)$ as

$$\begin{aligned} C_\lambda(X) &= C_\lambda(I_m - (I_m - X)) \\ &= C_\lambda(I_m) \sum_{k=0}^{\ell} \sum_{\kappa} \binom{\lambda}{\kappa} \frac{(-1)^k C_\kappa(I_m - X)}{C_\kappa(I_m)} \end{aligned}$$

we get

$$E[C_\lambda(X)] = C_\lambda(I_m) \sum_{k=0}^{\ell} \sum_{\kappa} \binom{\lambda}{\kappa} \frac{(-1)^k E[C_\kappa(I_m - X)]}{C_\kappa(I_m)},$$

where, using (3.15), one obtains

$$E[C_\kappa(I_m - X)] = \frac{((m+1)/2)_\kappa (\beta+\gamma)_\kappa}{(\alpha+\beta+\gamma)_\kappa (\beta + (m+1)/2)_\kappa} C_\kappa(I_m).$$

## 4. Distributions of Random Quadratic Forms

First we give definitions of several matrix variate distributions.

**Definition 4.1:** The random symmetric positive definite matrix $W$ is said to follow a matrix variate Gamma distribution, denoted as $W \sim G_m(a, \Xi)$, if its p.d.f. is

$$\{\Gamma_m(a)\}^{-1} \det(\Xi)^a \operatorname{etr}(-\Xi W) \det(W)^{a-(m+1)/2}, \quad W > 0 \tag{4.1}$$

where $a > (m-1)/2$ and $\Xi$ is am $m \times m$ symmetric positive definite non-random matrix.

**Definition 4.2:** The $m \times m$ random symmetric positive definite matrix $Y$ is said to have a matrix variate beta type II distribution with parameters $(a, b)$, denoted as $Y \sim B_m^{II}(a, b)$, if its p.d.f. is given by

$$\{B_m(a,b)\}^{-1} \det(Y)^{a-(m+1)/2} \det(I_m + Y)^{-(a+b)}, \quad Y > 0, \tag{4.2}$$

where $a > (m-1)/2$, $b > (m-1)/2$, and $B_m(a, b)$ is the multivariate beta function.

In the rest of the section we will derive density functions of certain random quadratic forms.

**Theorem 4.3:** Let $X \sim B_m^I(a,b)$ and $U \sim HB_m(\alpha,\beta,\gamma)$ be independent. Then, the density of $Z = U^{1/2}X(U^{1/2})'$ is given by

$$\frac{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]\Gamma_m(a+b)}{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+\gamma)\Gamma_m[(m+1)/2]\Gamma_m(a)\Gamma_m(b)} \det(Z)^{a-(m+1)/2}$$
$$\times \det(I_m - Z)^{b+\gamma+\beta-(m+1)/2} \sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{\lambda}\sum_{\sigma}\sum_{\kappa} \binom{\sigma}{\kappa}\frac{C_\sigma(I_m)}{C_\kappa(I_m)}$$
$$\times \frac{(a+b-\alpha-\beta)_\lambda}{\ell!}\frac{(-1)^k(\alpha+\beta+\gamma-(m+1)/2)_\sigma(\beta)_\sigma}{(\alpha+\beta)_\sigma s!}$$
$$\times \sum_{\phi\in\kappa\cdot\lambda}(\theta_\phi^{\kappa,\lambda})^2\frac{\Gamma_m(\gamma+\beta,\phi)\Gamma_m(b)}{\Gamma_m(\gamma+\beta+b,\phi)}C_\phi(I_m-Z), \quad 0<Z<I_m. \tag{4.3}$$

*Proof.* Using (3.1) and (3.6), the joint density of $X$ and $U$ is given as

$$K(\alpha,\beta,\gamma)\{B_m(a,b)\}^{-1}\det(X)^{a-(m+1)/2}\det(I_m-X)^{b-(m+1)/2}\det(U)^{\alpha+\beta-(m+1)/2}$$
$$\times \det(I_m-U)^{\gamma+\beta-(m+1)/2}{}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;U\right), \tag{4.4}$$

where $0<U<I_m$ and $0<X<I_m$. Making the transformation $Z=U^{1/2}X(U^{1/2})'$ with the Jacobian $J(X,U\to Z,U)=\det(U)^{-(m+1)/2}$ in (4.4) we get the joint density of $Z$ and $U$ as

$$K(\alpha,\beta,\gamma)\{B_m(a,b)\}^{-1}\det(Z)^{a-(m+1)/2}\det(U)^{-(a+b-\alpha-\beta)}\det(I_m-U)^{\gamma+\beta-(m+1)/2}$$
$$\times \det(U-Z)^{b-(m+1)/2}{}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;U\right), \tag{4.5}$$

where $0<Z<U<I_m$. Now to obtain the marginal density of $Z$, we need to integrate $U$ in (4.5). Collecting terms containing $U$ and using the substitution $W=(I_m-Z)^{-1/2}(I_m-U)(I_m-Z)^{-1/2}$ with the Jacobian $J(U\to W)=\det(I_m-Z)^{(m+1)/2}$, we get

$$\int_Z^{I_m}\det(U)^{-(a+b-\alpha-\beta)}\det(I_m-U)^{\gamma+\beta-(m+1)/2}\det(U-Z)^{b-(m+1)/2}$$
$$\times{}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;U\right)dU$$
$$=\det(I_m-Z)^{b+\gamma+\beta-(m+1)/2}\int_0^{I_m}\frac{\det(W)^{\gamma+\beta-(m+1)/2}\det(I_m-W)^{b-(m+1)/2}}{\det(I_m-(I_m-Z)W)^{a+b-\alpha-\beta}}$$
$$\times{}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;I_m-(I_m-Z)W\right)dW. \tag{4.6}$$

Expanding $\det(I_m-(I_m-Z)W)^{-(a+b-\alpha-\beta)}$ and ${}_2F_1(\alpha+\beta+\gamma-(m+1)/2,\beta;\alpha+\beta;$ $I_m-(I_m-Z)W)$ using (2.13) and (2.7) and applying (2.18), we obtain

$$\begin{aligned}
&{}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;I_m-(I_m-Z)W\right)\\
&\quad\times\det(I_m-(I_m-Z)W)^{-(a+b-\alpha-\beta)}\\
&\quad=\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{\lambda}\sum_{\sigma}\sum_{\kappa}\binom{\sigma}{\kappa}\frac{C_\sigma(I_m)}{C_\kappa(I_m)}\frac{(a+b-\alpha-\beta)_\lambda}{\ell!}\\
&\qquad\times\frac{(-1)^k(\alpha+\beta+\gamma-(m+1)/2)_\sigma(\beta)_\sigma}{(\alpha+\beta)_\sigma s!}\sum_{\phi\in\kappa\cdot\lambda}(\theta_\phi^{\kappa,\lambda})^2C_\phi((I_m-Z)W).
\end{aligned}\tag{4.7}$$

Substituting (4.7) in (4.6) and integrating $W$ using (2.14), we obtain

$$\begin{aligned}
&\int_Z^{I_m}\det(U)^{-(a+b-\alpha-\beta)}\det(I_m-U)^{\gamma+\beta-(m+1)/2}\det(U-Z)^{b-(m+1)/2}\\
&\qquad\times{}_2F_1\left(\alpha+\beta+\gamma-\frac{m+1}{2},\beta;\alpha+\beta;U\right)dU\\
&=\det(I_m-Z)^{b+\gamma+\beta-(m+1)/2}\sum_{\ell=0}^{\infty}\sum_{s=0}^{\infty}\sum_{k=0}^{s}\sum_{\lambda}\sum_{\sigma}\sum_{\kappa}\binom{\sigma}{\kappa}\frac{C_\sigma(I_m)}{C_\kappa(I_m)}\\
&\qquad\times\frac{(a+b-\alpha-\beta)_\lambda}{\ell!}\frac{(-1)^k(\alpha+\beta+\gamma-(m+1)/2)_\sigma(\beta)_\sigma}{(\alpha+\beta)_\sigma s!}\\
&\qquad\times\sum_{\phi\in\kappa\cdot\lambda}(\theta_\phi^{\kappa,\lambda})^2\frac{\Gamma_m(\gamma+\beta,\phi)\Gamma_m(b)}{\Gamma_m(\gamma+\beta+b,\phi)}C_\phi(I_m-Z).
\end{aligned}\tag{4.8}$$

Integrating $U$ in (4.5) using (4.8), and simplifying the resulting expression we get the desired result. ∎

**Theorem 4.4:** Let $V\sim B_m^{II}(a,b)$ and $U\sim HB_m(\alpha,\beta,\gamma)$ be independent. Then, the density of $Z=U^{1/2}VU^{1/2}$ is given by

$$\begin{aligned}
&\frac{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+b)\Gamma_m(a+b)}{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+\gamma)\Gamma_m[\alpha+\beta+b+(m+1)/2]\Gamma_m(a)\Gamma_m(b)}\\
&\times\frac{\det(Z)^{a-(m+1)/2}}{\det(I_m+Z)^{a+b}}\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{(a+b)_\lambda((m+1)/2)_\lambda}{\ell!}\frac{C_\lambda((I_m+Z)^{-1})}{C_\lambda(I_m)}\\
&\times\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{\phi\in\kappa\cdot\lambda}\frac{(\alpha+\beta+b)_\kappa((m+1)/2-\gamma)_\kappa(\alpha)_\kappa}{(\alpha+\beta)_\kappa(\alpha+\beta+b+(m+1)/2)_\phi\,k!}(\theta_\phi^{\kappa,\lambda})^2C_\phi(I_m),\quad Z>0.
\end{aligned}$$

*Proof.* The joint density of $U$ and $V$ is given by

$$K(\alpha,\beta,\gamma)\{B_m(a,b)\}^{-1}\det(U)^{\alpha+\beta-(m+1)/2}{}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta;U\right)$$
$$\times\det(V)^{a-(m+1)/2}\det(I_m+V)^{-(a+b)},\quad 0<U<I_m,\ V>0.$$

Transforming $Z=U^{1/2}VU^{1/2}$ with $J(V\to Z)=\det(U)^{-(m+1)/2}$ in above, we get the joint density of $U$ and $Z$ as

$$K(\alpha,\beta,\gamma)\{B_m(a,b)\}^{-1}\det(U)^{\alpha+\beta+b-(m+1)/2}{}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta;U\right)$$
$$\times\det(Z)^{a-(m+1)/2}\det(U+Z)^{-(a+b)},\quad 0<U<I_m,\ Z>0.$$

Now, writing

$$\begin{aligned}\det(U+Z)^{-(a+b)} &= \det(I_m+Z)^{-(a+b)}\det(I_m-(I_m+Z)^{-1}(I_m-U))^{-(a+b)}\\ &= \det(I_m+Z)^{-(a+b)}\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{(a+b)_\lambda}{\ell!}C_\lambda((I_m+Z)^{-1}(I_m-U))\end{aligned}$$

and integrating out $U$, the density of $Z$ is obtained as

$$\frac{K(\alpha,\beta,\gamma)}{B_m(a,b)}\frac{\det(Z)^{a-(m+1)/2}}{\det(I_m+Z)^{a+b}}\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{(a+b)_\lambda}{\ell!}\int_0^{I_m}\det(U)^{\alpha+\beta+b-(m+1)/2}$$
$$\times C_\lambda((I_m+Z)^{-1}(I_m-U))\,{}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta;U\right)dU.$$

Finally, evaluating the above integral using (2.21), (2.22) and (2.23), we get the desired result. ■

**Theorem 4.5:** Let $S$ and $X$ be independent, $S\sim G_m(\mu,I_m)$ and $X\sim HB_m(\alpha,\beta,\gamma)$. Define $Z=X^{-1/2}SX^{-1/2}$. Then the density of $Z$ is given by

$$\frac{\Gamma_m(\alpha+\beta+\gamma)\Gamma_m[\beta+(m+1)/2]\Gamma_m(\alpha+\beta+\mu)}{\Gamma_m(\alpha+\beta)\Gamma_m(\beta+\gamma)\Gamma_m[\alpha+\beta+\mu+(m+1)/2]}$$
$$\times\frac{\operatorname{etr}(-Z)\det(Z)^{\mu-(m+1)/2}}{\Gamma_m(\mu)}\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{((m+1)/2)_\lambda}{\ell!}\frac{C_\lambda(Z)}{C_\lambda(I_m)}$$
$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\sum_{\phi\in\kappa\cdot\lambda}(\theta_\phi^{\kappa,\lambda})^2\frac{(\alpha+\beta+\mu)_\kappa((m+1)/2-\gamma)_\kappa(\alpha)_\kappa}{(\alpha+\beta+\mu+(m+1)/2)_\phi(\alpha+\beta)_\kappa k!}C_\phi(I_m),\quad Z>0.$$

*Proof.* The joint density of $S$ and $X$ is given by

$$\{\Gamma_m(\mu)\}^{-1}K(\alpha,\beta,\gamma)\operatorname{etr}(-S)\det(S)^{\mu-(m+1)/2}\det(X)^{\alpha+\beta-(m+1)/2}$$
$$\times{}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta;X\right),S>0,0<X<I_m. \tag{4.9}$$

Now, use of the transformation $Z = X^{-1/2}SX^{-1/2}$ with the Jacobian $J(S \to Z) = \det(X)^{(m+1)/2}$ yields the joint density of $Z$ and $X$ as

$$\{\Gamma_m(\mu)\}^{-1}K(\alpha,\beta,\gamma)\operatorname{etr}(-ZX)\det(Z)^{\mu-(m+1)/2}\det(X)^{\alpha+\beta+\mu-(m+1)/2} \times {}_2F_1\left(\frac{m+1}{2}-\gamma,\alpha;\alpha+\beta;X\right), \quad 0 < X < I_m,\ Z > 0. \tag{4.10}$$

Now, writing

$$\operatorname{etr}(-ZX) = \operatorname{etr}(-Z)\sum_{\ell=0}^{\infty}\sum_{\lambda}\frac{1}{\ell!}C_\lambda(Z(I_m - X))$$

and integrating $X$ using using (2.21), (2.22) and (2.23), we get the desired result. ■

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