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ON THE NUMBER OF SECTIONS IN THE AUSLANDER-REITEN QUIVER OF ALGEBRAS OF DYNKIN TYPE

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Abstract

Categorification of integer sequences A083329, A000295 and A049611 in the OEIS is given by using the number of sections in the Auslander-Reiten quiver of path algebras of type $k\Delta$, where Δ is an oriented Dynkin diagram of type A_n, D_n, E_6, E_7 and E_8. In particular, the sequence A000295 counts the number of Dyck paths of semilength n having exactly one long ascent.

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1. Introduction

In [5], Ringel proposed a novel tool to investigate combinatorial properties of representation-finite hereditary artin algebras. Such a tool was named *Dynkin function*. These functions attach an integer (more generally a real number, a set or a sequence of real numbers) to each Dynkin diagram. Thus, a Dynkin function consists of four sequences of numbers namely $f(A_n)$, $f(B_n)$, $f(C_n)$ and $f(D_n)$ and five single values $f(E_6)$, $f(E_7)$, $f(E_8)$, $f(F_4)$ and $f(G_2)$ [5]. Dynkin functions arise from the context of *categorification* of integer sequences, that according to Ringel and Fahr means to consider instead of numbers in a given integer sequence suitable objects in a category. We recall that recently, the Cañadas et al. considered to categorify the sequence A052558 by using 2-Kronecker modules [2] whereas Fahr and Ringel gave a categorification of Fibonacci numbers by using the Auslander-Reiten quiver of the 3-Kronecker quiver [3, 4].

One of the main tools dealing with integer sequences is the On-Line Encyclopedia of Integer Sequences (OEIS), however in [5] Ringel proposed to create a similar tool named On-Line Encyclopedia of Dynkin Functions (OEDF) in order to establish properties of integer sequences arising from Dynkin diagrams. Examples of Dynkin functions are:

(1) $r(\Delta_n)$ the number of indecomposable modules.

(2) $a(\Delta_n)$ the number of antichains in $\operatorname{mod}\Lambda$, where Λ denotes the corresponding hereditary artin algebra associated to the Dynkin diagram Δ_n.

(3) $t_n(\Delta_n)$ the number of tilting modules.

One characteristic of Dynkin functions is that numbers arising from them are independent of orientation, however several interesting combinatorial properties are obtained when dealing with oriented Dynkin diagrams. For instance, the lattice $\mathbf{A}(\operatorname{mod}\Lambda_n)$ may be identified with the lattice of non-crossing partitions, where Λ_n is the path algebra of the linearly oriented

Dynkin diagram A_n. In this way, the module categories $\operatorname{mod}\Lambda_n$ allow to establish a categorification of non-crossing partitions [5].

In this paper, we calculate the number of sections in the Auslander-Reiten quiver of the path algebra of a given oriented Dynkin diagram in order to obtain categorification of several integer sequences.

This paper is organized as follows: In Section 2, we give some notation and definitions to be used throughout the document. Sections 3-5 concern the Auslander-Reiten quiver of algebras of Dynkin type A_n, D_n and E_6, E_7, E_8, respectively. Finally, an appendix is provided where the Auslander-Reiten quiver of some of these algebras have been described.

2. Preliminaries

In this section, we recall notation and definitions to be used throughout the paper [1].

Let $\Sigma = (\Sigma_0, \Sigma_1)$ be a connected and acyclic quiver. An *infinite translation quiver* $(\mathbb{Z}\Sigma, \tau)$ has the set $(\mathbb{Z}\Sigma)_0 = \mathbb{Z} \times \Sigma_0 = \{(n, x) | n \in \mathbb{Z}, x \in \Sigma_0\}$ as its set of vertices, and for each arrow $\alpha : x \to y \in \Sigma_1$ there exist two arrows

$$(n, \alpha) : (n, x) \to (n, y) \quad (n, \alpha') : (n+1, y) \to (n, x) \text{ in } (\mathbb{Z}\Sigma)_1.$$

And these are all the arrows in $(\mathbb{Z}\Sigma)_1$. The *translation* τ on $\mathbb{Z}\Sigma$ is given by the identity $\tau(n, x) = (n+1, x)$. Further, for every $(n, x) \in (\mathbb{Z}\Sigma)_0$, it is defined a bijection between the set of arrows of source $(n+1, x)$ by the formulas [1]:

$$\sigma(n, \alpha) = (n, \alpha') \text{ and } \sigma(n, \alpha') = (n+1, \alpha).$$

Let (Γ, τ) be a connected translation quiver. A connected full subquiver Σ of Γ is a *section* of Γ if the following conditions are satisfied:

S(1) Σ is acyclic.

S(2) For each $x \in \Gamma_0$, there exists a unique $n \in \mathbb{Z}$ such that $\tau^n x \in \Sigma_0$.

S(3) If $x_0 \to x_1 \to \cdots \to x_t$ is a path in Γ with $x_0, x_t \in \Sigma_0$, then $x_i \in \Sigma_0$ for all i such that $0 \le i \le t$.

For a translation quiver (Γ, τ), the τ-*orbit* of a point $x \in \Gamma_0$ is defined to be the set of all points of the form $\tau^n x$ with $n \in \mathbb{Z}$. Thus, any section Σ meets each τ-orbit exactly once.

The following are conditions that a subquiver Σ must satisfy to be a section of a translation quiver (Γ, τ):

(1) If $x \to y$ is an arrow in Γ and $x \in \Sigma_0$, then $y \in \Sigma_0$ or $\tau y \in \Sigma_0$.

(2) If $x \to y$ is an arrow in Γ and $y \in \Gamma_0$, then $x \in \Sigma_0$ or $\tau^{-1} x \in \Sigma_0$.

Henceforth, we let $\mathcal{O}_x$ denote the orbit of a fixed element $x \in \Gamma_0$. In particular, if $\Gamma(\text{Mod}\, A) = (\Gamma_0, \Gamma_1)$ is the Auslander-Reiten quiver of an algebra of Dynkin type Δ, then each element of the τ-orbit of an indecomposable projective module will be denoted by τ_i^n, $i \in \mathbb{N}$.

3. Number of Sections in Algebras of Dynkin Type A_n

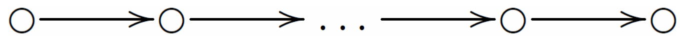
In this section, we calculate the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type A_n.

Henceforth, we let $S_{A(n,a)}^{i_j}$ denote the number of sections of an algebra of Dynkin type A_n with a sink at the points i_j with $1 \le j \le a$, n fixed.

Theorem 1. *If A denotes the path algebra associated to the oriented Dynkin diagram A_n with one of the following two orientations*:

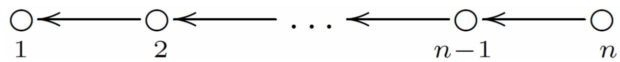
$$\circ \longleftarrow \circ \longleftarrow \cdots \longleftarrow \circ \longleftarrow \circ$$

or

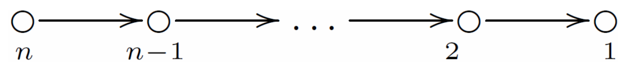


then $S^1_{A(n,1)} = S^n_{A(n,1)} = 2^{n-1}$ *for* $n \geq 1$.

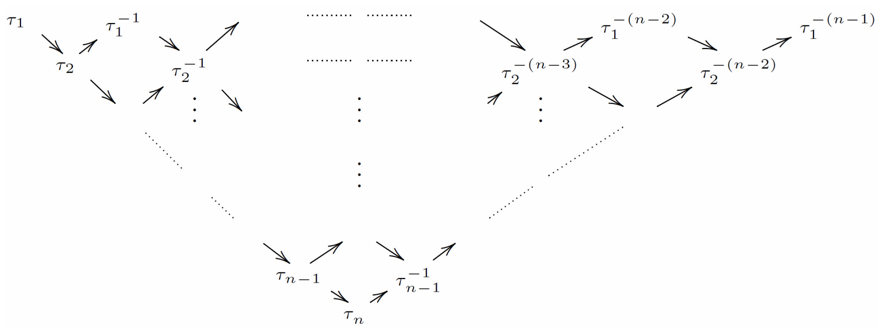
Proof. Consider the following admissible numberings for algebras of type A:



or



then the corresponding Auslander-Reiten quiver $\Gamma(\mathrm{Mod}\, A)$ has the following shape:



Now, we define the map

$$\Psi : \Gamma(\mathrm{Mod}\, A) \to \mathbb{R}^2,$$

$$\tau_r^{-s} \to (s,\, n-r-s)$$

with $1 \leq r \leq n,\ 0 \leq s \leq n-1$, for each irreducible morphism $[\tau_{r_1}^{-s_1}] \to [\tau_{r_2}^{-s_2}]$ in $\Gamma(\mathrm{Mod}\, A)$ it is defined an arrow in $\mathrm{Im}\, \Psi \subseteq \mathbb{R}^2$ as follows:

$$\Psi([\tau_{r_1}^{-s_1}] \to [\tau_{r_2}^{-s_2}])$$

$$= \begin{cases} (s_1,\, n-r_1-s_1) \to (s_2,\, n-r_2-s_2), & \text{if } r_1 > r_2 \text{ and } s_1 < s_2, \\ (s_2,\, n-r_2-s_2) \to (s_1,\, n-r_1-s_1), & \text{if } r_1 < r_2 \text{ and } s_1 = s_2. \end{cases}$$

We let $P_{(0,0)}^{(i,h)}$ denote the set of all paths in $\operatorname{Im}\Psi$ starting at $(0, 0)$ and ending at the point (i, h). Thus, the number of paths $|P_{(0,0)}^{(i,h)}|$ from $(0, 0)$ to (i, h) is given by $\binom{i+h}{h}$. The sets $P_{(0,0)}^{(i,h)}$ with $i + h = n - 1$ satisfy the identity:

$$\sum_{h=0}^{i+h} |P_{(0,0)}^{(i,h)}| = 2^{n-1}$$

for $n \geq 1$ which equals the number of sections of $\Gamma(\operatorname{Mod} A)$. $\square$

Theorem 2. *For $n \geq 2$ fixed and an algebra A associated to a Dynkin diagram A_n oriented as shown below*:

$$\circ \longrightarrow \circ \longleftarrow \cdots \longleftarrow \circ \longleftarrow \circ$$

or

$$\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \longleftarrow \circ$$

it holds that $S_{A(n,1)}^{2} = S_{A(n,1)}^{n-1} = 3(2^{n-2}) - 1.$

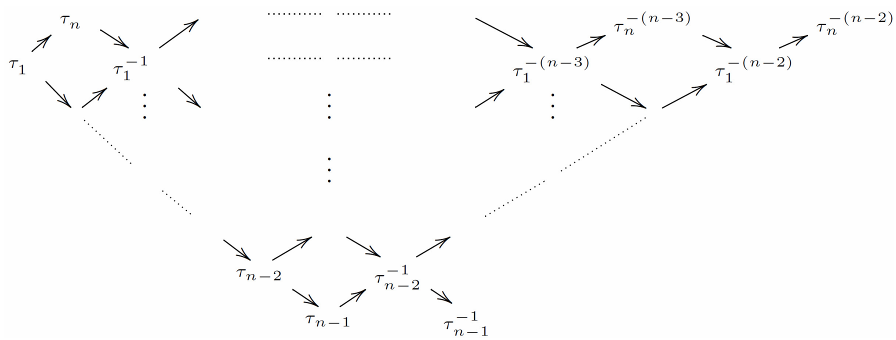
Proof. Consider the following admissible numberings for A_n :

$$\underset{n}{\circ} \longrightarrow \underset{1}{\circ} \longleftarrow \cdots \longleftarrow \underset{n-2}{\circ} \longleftarrow \underset{n-1}{\circ}$$

or

$$\underset{n-1}{\circ} \longrightarrow \underset{n-2}{\circ} \longrightarrow \cdots \longrightarrow \underset{1}{\circ} \longleftarrow \underset{n}{\circ}$$

then the corresponding Auslander-Reiten quiver $\Gamma(\operatorname{Mod} A)$ has the following shape:



We define the map $\Psi : \Gamma(\operatorname{Mod} A) \to \mathbb{R}^2$ in such a way that

$$\tau_r^{-s} \to \begin{cases} (s,\ n-1-(r+s)), & \text{for } 1 \le r \le n-1, \\ (s+1,\ n-2-s), & \text{for } r = n \end{cases}$$

with $0 \le s \le n-2$. Now we use the same arguments as in Theorem 1 to associate a suitable arrow in $\operatorname{Im}\Psi \subseteq \mathbb{R}^2$ to each irreducible morphism of the form $[\tau_{r_1}^{-s_1}] \to [\tau_{r_2}^{-s_2}]$ in $\Gamma(\operatorname{Mod} A)$. The procedure goes as follows:

$$\Psi([\tau_{r_1}^{-s_1}] \to [\tau_{r_2}^{-s_2}])$$

$$= \begin{cases} (s_1, n-1-(r_1+s_1)) & \\ \quad \to (s_2, n-1-(r_2+s_2)), & \text{if } r_1 > r_2,\ s_1 < s_2 \text{ and } 1 \le r_1 \le n-1, \\ (s_2, n-1-(r_2+s_2)) & \\ \quad \to (s_1, n-1-(r_1+s_1)), & \text{if } r_1 < r_2,\ s_1 = s_2 \text{ and } 1 \le r_1 \le n-1, \\ (s_1, n-1-(r_1+s_1)) & \\ \quad \to (s_1+1, n-2-s_1), & \text{if } r_2 = n,\ s_1 = s_2 \text{ and } r_1 = 1, \\ (s_1+1, n-2-s_1) & \\ \quad \to (s_2, n-1-(r_2-s_2)), & \text{if } r_1 = n,\ s_2 = s_1+1 \text{ and } r_2 = 1. \end{cases}$$

Note that, for $i + h = n - 1$,

$$\sum_{h=0}^{i+h} | P_{(0,0)}^{(i,h)} | = 2^{n-1} - 1 \text{ and } | P_{(1,-1)}^{(i,h)} | = | P_{(1,0)}^{(i,h)} |.$$

For $i_1 + h_1 = n - 2,$ we have that

$$\sum_{h=0}^{i+h} | P_{(1,0)}^{(i,h)} | = \sum_{h_1=0}^{i_1+h_1} | P_{(0,0)}^{(i_1,h_1)} | = 2^{n-1}.$$

Therefore, for $n \geq 2$

$$\sum_{h=0}^{i+h} | P_{(0,0)}^{(i,h)} | + \sum_{h_1=0}^{i_1+h_1} | P_{(0,0)}^{(i_1,h_1)} | = 3(2^{n-2}) - 1$$

equals the number of sections in $\Gamma(\mathrm{Mod}\, A)$. □

Corollary 3. $S^2_{A(n,1)} = S^2_{A(n-1,1)} + S^1_{A(n-1,1)} + S^1_{A(n-2,1)}$ *for* $n \geq 3$.

Proof. The arguments used in Theorem 2 allow to define the subquiver $B \subseteq \mathrm{Im}\,\Psi,$ where for $2 \leq i \leq n-1, 0 \leq j \leq n-2,$ the points (i, j) satisfy the condition:

$$\sum_{h=0}^{i+h} | P_{(2,0)}^{(i,h)} | = \sum_{h_1=0}^{i_1+h_i} | P_{(0,0)}^{(i_1,j_1)} | = 2^{n-3} = S^1_{A(n-2,1)}$$

with $i + h = n - 1, i_1 + j_1 = n - 2.$ Further, the paths containing $(0, 0) \to (1, 0) \to (2, 0)$ satisfy the identity:

$$\sum_{h=0}^{i+h} | P_{(0,0)}^{(i,h)} | = \sum_{h=0}^{i+h} | P_{(2,0)}^{(i,h)} |.$$

Let W be a subset of $\mathrm{Im}\,\Psi$ whose points (s, t) with $0 \leq s \leq n-2,$ $1 \leq t \leq n-2.$ Then

$$\sum_{h_2=0}^{i_2+h_2} | P_{(0,1)}^{(i_2,h_2)} | = \sum_{h_3=0}^{i_3+h_3} | P_{(0,0)}^{(i_3,j_3)} | = 2^{n-2} - 1$$

with $i_2 + h_2 = n - 1, \quad i_3 + h_3 = n - 2.$ For all paths containing $(0, 0) \to (0, 1)$ the identity $\sum_{h=0}^{i+h} | P_{(0,0)}^{(i,h)} | = \sum_{h_2=0}^{i_2+h_2} | P_{(1,0)}^{(i_2,h_2)} |$ holds.

Let T be another subset of $\operatorname{Im}\Psi$ whose points (w, x) satisfy the conditions $1 \leq w \leq n-2$, $1 \leq x \leq n-2$. Then

$$\sum_{h_4=0}^{i_4+h_4} |P_{(1,1)}^{(i_4, h_4)}| = \sum_{h_5=0}^{i_5+h_5} |P_{(0,0)}^{(i_5, j_5)}| = 2^{n-3} = S_{A(n-2,1)}^{1}$$

with $i_4 + h_4 = n-1$, $i_5 + h_5 = n-2$. For paths containing $(0, 0) \to (1, 0)$, we have that:

$$\sum_{h=0}^{i+h} |P_{(0,0)}^{(i,h)}| = \sum_{h_4=0}^{i_4+h_4} |P_{(1,1)}^{(i_4,h_4)}|.$$

Finally, we consider a subset $M \subseteq \operatorname{Im}\Psi$, with points (y, z), $1 \leq y \leq n-2$, $0 \leq z \leq n-2$. Thus

$$\sum_{h_6=0}^{i_6+h_6} |P_{(1,0)}^{(i_6, h_6)}| = \sum_{h_7=0}^{i_7+h_7} |P_{(0,0)}^{(i_7, j_7)}| = 2^{n-2} = S_{A(n-1,1)}^{1}$$

with $i_6 + h_6 = n-1$, $i_7 + h_7 = n-2$. Further,

$$\sum_{h=0}^{i+h} |P_{(1,-1)}^{(i,h)}| = \sum_{h_6=0}^{i_6+h_6} |P_{(1,0)}^{(i_6,h_6)}|.$$

Therefore

$$\begin{aligned} S_{A(n,1)}^{2} &= S_{A(n-2,1)}^{1} + 2^{n-2} - 1 + S_{A(n-2,1)}^{1} + S_{A(n-1,1)}^{1} \\ &= S_{A(n-1,1)}^{2} + S_{A(n-2,1)}^{1} + S_{A(n-1,1)}^{1}. \end{aligned}$$ □

For the general case, we have the following result:

Theorem 4. $S_{A(n,1)}^{m} = 2(S_{A(n-1,1)}^{m-1}) + \sum_{i=0}^{m-2} \binom{n-2}{i}$ *for* $n \geq 3$, $1 < m < n$ *with* $S_{A(n,1)}^{1} = S_{A(n,1)}^{n} = 2^{n-1}$.

Proof. Let A be an algebra associated to a Dynkin diagram A_n and define the following suitable numbering:

$$\underset{n-1}{\circ} \longrightarrow \underset{n-2}{\circ} \longrightarrow \cdots \longrightarrow \underset{n-m+1}{\circ} \longrightarrow \underset{1}{\circ} \longleftarrow \underset{2}{\circ} \longleftarrow \cdots \longleftarrow \underset{n-m}{\circ} \longleftarrow \underset{n}{\circ}$$

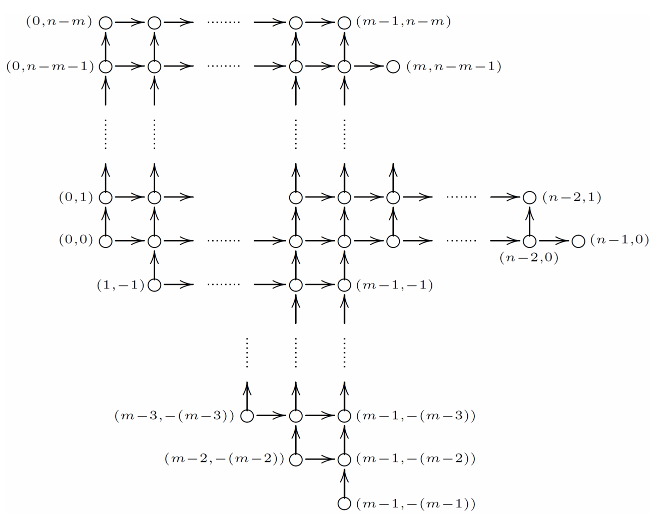
For $\Gamma(\operatorname{mod} A)$, we define the subquiver

$$W_k = \Gamma(\operatorname{mod} A)\backslash(\{\tau_i^0 \mid n-m+1 \leq i \leq n-2\} \cup \tau_1).$$

Let W_{k-1} be the subquiver $W_{k-1} \subset \Gamma(\operatorname{mod} A)$ defined in such a way that

$$W_{k-1} = W_k \backslash \{\tau_{n-1}^{-(j)} \mid 0 \leq j \leq n-m\}.$$

The subquiver W_{k-1} is isomorphic to $\Gamma(\operatorname{mod} A_1)$, where A_1 contains A_{n-1} as subgraph with a sink at the mth position. Thus, sections in W_k are obtained if all sections containing $\tau_{n-2}^{-(j)}$ in W_{k-1} are connected by $\tau_{n-1}^{-(j-1)} \in W_k$ or $\tau_{n-1}^{-(j)} \in W_k$, where $1 \leq j \leq n-m$. Thus, Theorem 1 guarantees the existence of a map φ such that the number of admissible paths in $\varphi(W_k)$ (denoted $|W_k|$) equals $2|W_{k-1}|$. Further, $\Gamma(\operatorname{mod} A)$ is isomorphic to the following quiver:



Therefore, if $|P_{(s,-s)}^{(m-2,n-m)}| = \binom{n-2}{m-2-s}$ with $0 \leq s \leq m-2$, then the number of admissible paths is given by the formula $2|W_{k-1}| + \sum_{i=0}^{m-2}\binom{n-2}{i}$ which equals the number of sections in $\Gamma(\text{mod } A)$. □

Corollary 5. $S_{A(n,1)}^{m} = S_{A(n-1,1)}^{m-1} + S_{A(n-1,1)}^{m} + S_{A(n-2,1)}^{1}$ *for* $n \geq 3$, $1 < m < n$.

Proof. We proceed by induction on *n* taking into account that $S_{A(n,1)}^{1} = S_{A(n,1)}^{n} = 2^{n-1}$. If $n = 3$ and $m = 2$, then we have that

$$S_{A(3,1)}^{2} = 2(S_{A(2,1)}^{2}) + 1 = S_{A(2,1)}^{2} + S_{A(2,1)}^{1} + S_{A(1,1)}^{1}.$$

Suppose that the assertion is true for $3 \leq k \leq n$ and $2 \leq m \leq n-1$. Thus

$$\begin{aligned} S_{A(k+1,1)}^{m} &= 2(S_{A(k,1)}^{m}) + \sum_{i=0}^{m-2}\binom{k-1}{i} \\ &= 2(S_{A(k,1)}^{m}) + \sum_{i=0}^{m-2}\binom{k-2}{i} + \sum_{i=0}^{m-3}\binom{k-2}{i}. \end{aligned}$$

Therefore

$$\begin{aligned} S_{A(k+1,1)}^{m} &= 2(S_{A(k-1,1)}^{m-1} + S_{A(k-1,1)}^{m} + 2^{k-3}) + \sum_{i=0}^{m-2}\binom{k-2}{i} + \sum_{i=0}^{m-3}\binom{k-2}{i} \\ &= S_{A(k,1)}^{m-1} + S_{A(k,1)}^{m} + S_{A(k-1,1)}^{1}. \end{aligned}$$ □

Theorem 4 allows to construct the following triangular table where the rows give the number of sections in the Auslander-Reiten quiver of an algebra *A* associated to a Dynkin graph A_n with a unique sink allocated at the *h*th position, $1 \leq h \leq n$:

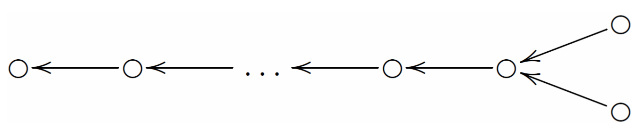
$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 2 & & 2 & & & & & \\
 & & & & 4 & & 5 & & 4 & & & & \\
 & & & 8 & & 11 & & 11 & & 8 & & & \\
 & & 16 & & 23 & & 26 & & 11 & & 16 & & \\
 & 32 & & 47 & & 57 & & 57 & & 47 & & 32 & \\
64 & & 95 & & 120 & & 130 & & 120 & & 95 & & 64 \\
\end{array}$$

Remark 6. We note that the sequences, $a_n = S^2_{A(n,1)}$, $b_n = S^3_{A(n,1)}$ and $c_n = \sum_{h=1}^{n} S^h_{A(n,1)}$ appear in the OEIS as A083329, A000295 and A049611, respectively [9-11].

4. Number of Sections in Algebras of Dynkin Type D_n

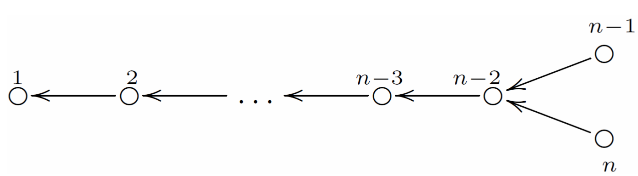
In this section, we calculate the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type D_n with only one sink. Henceforth, we let $S^{i_j}_{D(n,a)}$ denote the number of sections in an algebra of type D_n which have a sink in points i_j, $1 \leq j \leq a$.

Theorem 7. *For an algebra A associated to the following oriented quiver of type D_n:*



it holds that $S^1_{D(n,1)} = 2^{n-3}(2n-1)$ *for* $n \geq 4$.

Proof. For $n \geq 4$ fixed, Consider the following admissible numbering:



with an Auslander-Reiten quiver $\Gamma(\mathrm{Mod}\ A)$ as shown in Figure 1 (see Appendix) and for $i \geq 0$ consider subquivers w_i such that:

$$w_0 = \{\tau_r^{-s} \mid 0 \leq s \leq n-3, 1 \leq r \leq n-2-s\}$$

and

$$w_i = \{\tau_r^{-s} \mid i \leq s \leq n-2, 1 \leq r \leq n-2-(s-i)\}$$

with $1 \leq i \leq n-2$. Now, we can adapt the arguments used in Theorem 1 to define a suitable map φ and conclude that the number of paths W_i associated to $\varphi(w_i)$ is given by the following identities:

$$W_0 = 2^{n-3},$$

$$W_1 = (2^{n-3})2^2,$$

$$W_{n-2} = 2^2,$$

$$W_k = \left(2^{n-3} - \sum_{t=0}^{k-2}\binom{n-3}{t}\right)2^2 \text{ for } 1 \leq k \leq n-4.$$

Thus, the total number of admissible paths is given by

$$\begin{aligned}\sum_{h=0}^{n-2} W_h &= 2^{n-3} + (n-3)2^{n-1} - \left(\sum_{h=2}^{n-3}\sum_{t=0}^{h-2}\binom{n-3}{t}\right)2^2 \\ &= 2^{n-3}(1 + (n-3)2^2 - 2(n-5)) \\ &= 2^{n-3}(2n-1)\end{aligned}$$

which equals the number of sections in the Auslander-Reiten quiver of the algebra A associated to D_n. □

Theorem 8. *For an algebra A associated to the following oriented Dynkin diagram of type* D_n:

$$\begin{tikzcd}
 & & & & & \circ \arrow[dl] \\
\circ \arrow[r] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ & \\
 & & & & & \circ \arrow[ul]
\end{tikzcd}$$

it holds that $S^{n-2}_{D(n,1)} = 2^{n-2}(n+1) - 3$ *for* $n \geq 4$.

Proof. Consider the following admissible numbering:

$$\begin{tikzcd}
 & & & & & \overset{n-1}{\circ} \arrow[dl] \\
\overset{n-2}{\circ} \arrow[r] & \overset{n-3}{\circ} \arrow[r] & \cdots \arrow[r] & \overset{2}{\circ} \arrow[r] & \overset{1}{\circ} & \\
 & & & & & \underset{n}{\circ} \arrow[ul]
\end{tikzcd}$$

with an Auslander-Reiten quiver as shown in Figure 2 (see Appendix).

Now, we adapt arguments used in Theorem 1 to define subquivers $w_i \subseteq \Gamma(\text{Mod } A)$ and a suitable map φ, such that

$$w_i = \{\tau_r^{-s} \,|\, 0 \leq s \leq i,\ i + 1 - s \leq r \leq n - 2\}$$

with $0 \leq i \leq n - 3$ and

$$w_{n-2} = \{\tau_r^{-s} \,|\, 1 \leq s \leq i,\ n - 1 - s \leq r \leq n - 2\}.$$

Thus, the number of admissible paths W_i associated to $\varphi(w_i)$ is given by the following identities:

$$W_0 = 1,$$

$$W_{n-3} = 2^{n-3}(2^2),$$

$$W_k = \left(2^{n-3} - \sum_{t=0}^{n-4-k} \binom{n-3}{t}\right) 2^2 \text{ for } 1 \leq k \leq n - 4.$$

The total number of admissible paths is

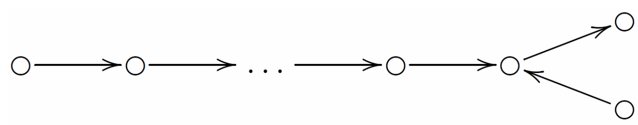
$$\sum_{h=0}^{n-2} W_h = 2^{n-2}(2 + 2 + 2(n-4)) - \left(\sum_{h=1}^{n-5} \sum_{t=0}^{n-4-h} \binom{n-3}{t}\right) 2^2 - 3$$

$$= 2^{n-2}(4 + 2(n - 4) - (n - 5)) - 3$$

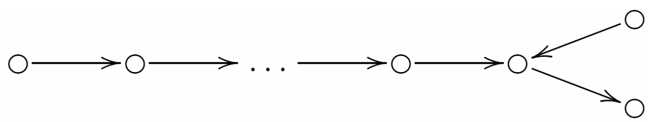
$$= 2^{n-2}(n + 1) - 3$$

which equals the number of sections in the Auslander-Reiten quiver of the algebra A. □

Theorem 9. *For an algebra A associated to a Dynkin diagram of type D_n oriented as follows*:

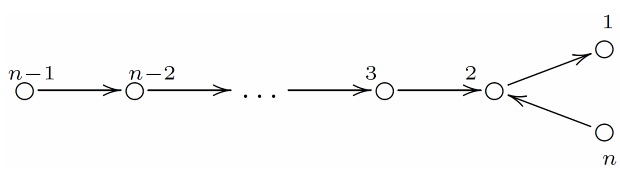


or

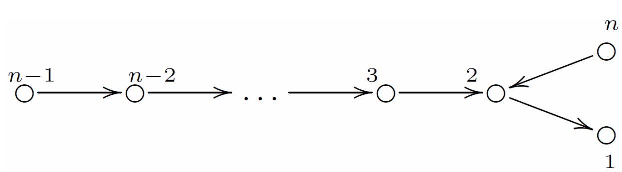


it holds that $S_{D(n,1)}^{n-1} = S_{D(n,1)}^{n} = 2^{n-2}(n) - 2$ *for* $n \geq 4$.

Proof. Consider the following admissible numbering:



or



The corresponding Auslander-Reiten quiver $\Gamma(\text{Mod } A)$ is shown in Figure 3 (see Appendix), (if n is even, then $x = n$ and $y = n - 2$; if n is odd, then $x = 1$ and $y = n - 1$). As in Theorems 7 and 8, we define subquivers $w_i \subseteq \Gamma(\text{Mod } A)$ and a suitable map such that:

$$w_i = \{\tau_r^{-s} \mid 0 \leq s \leq i,\ i + 2 - s \leq r \leq n - 1\}$$

with $0 \leq i \leq n - 3$ and

$$w_{n-2} = \{\tau_r^{-s} \mid 1 \leq s \leq i,\ n - s \leq r \leq n - 1\}$$

with the number of admissible paths W_i associated to $\varphi(w_i)$ given by the following identities:

$$W_0 = 2^2 - 2,$$

$$W_{n-3} = 2^{n-3}(2^2),$$

$$W_{n-1} = 2^{n-3}(2)W_k = \left(2^{n-3} - \sum_{t=0}^{n-4-k} \binom{n-3}{t}\right)2^2 \text{ for } 1 \leq k \leq n - 4$$

in this case, the total number of admissible paths is given by

$$\begin{aligned}\sum_{h=0}^{n-2} &= 2^{n-2}(3 + 2(n-4)) - \left(\sum_{h=1}^{n-5}\sum_{t=0}^{n-4-h}\binom{n-3}{t}\right)2^2 - 2 \\ &= 2^{n-2}(3 + 2(n-4) - (n-5)) - 2 \\ &= 2^{n-2}(n) - 2\end{aligned}$$

which equals the number of sections in the Auslander-Reiten quiver of A. $\square$

The following result is a generalization of Theorems 7-9.

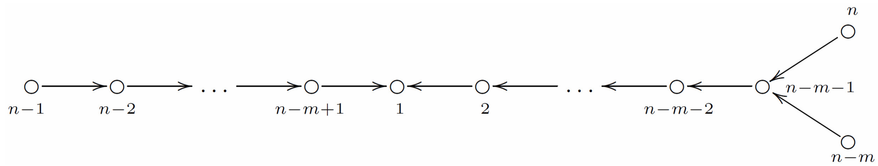
Theorem 10. *The number of sections in the Auslander-Reiten quiver of an algebra associated to a Dynkin diagram* D_n *with a sink at the mth position is given by the following identity*:

$$S_{D(n,1)}^{m} = \begin{cases} 2S_{D(n-1,1)}^{m} + 2^{n-2} + 1, & \text{if } m = 2, \\ 2S_{D(n-1,1)}^{m} + 2^{n-2} + \sum_{i=0}^{m-3} 4\binom{n-4}{i} + \binom{n-4}{m-2}, & \text{if } 3 \leq m \leq n-3, \end{cases}$$

where

$$S^{n-2}_{A(n,1)} = 2^{n-2}(n+1) - 3.$$

Proof. Consider the following admissible numbering:



Now, we define subquivers

$$W_k = \Gamma(\text{mod } A)\backslash(\{\tau_i^0 \mid n-m+1 \le i \le n-2\}$$

$$\cup \{\tau_i^{-(n-2)} \mid i = ny2 \le i \le n-m\}).$$

Let W_{k-1} be the subquiver $W_{k-1} \subset \Gamma(\text{mod } A)$ such that:

$$W_{k-1} = W_k \backslash \{\tau_{n-1}^{-(j)} \mid 0 \le j \le n-2\}.$$

Thus, W_{k-1} is isomorphic to $\Gamma(\text{mod } A_1)$, where A_1 has D_{n-1} as underlying subgraph with a sink at the *m*th position. Since sections in W_k are obtained by connecting sections containing $\tau_{n-2}^{-(j)}$ in W_{k-1} with vertices $\tau_{n-1}^{-(j-1)}$ or $\tau_{n-1}^{-(j)}$ for $1 \le j \le n-2$, the number of sections of W_k equals $2|W_{k-1}|$. Further, the subquiver $W_{k_1} \subseteq \Gamma(\text{mod } A)$

$$W_{k_1} = \{\tau_i^{-j} \mid 1 \le i \le n-1,\ 0 \le j \le n-2\}$$

is isomorphic to the quiver illustrated in Figure I.

We consider the numbers $|P^{(n-(m+1),-1)}_{(s,-s)}| = \binom{n-(m+2)}{s-1}$ for $1 \le s \le n-(m+1)$ and

$$\sum_{i=1}^{n-(m-1)} |P^{(n-(m+1),-1)}_{(s,-s)}| = 2^{n-(m+2)}$$

in the same way, $|P_{(n-(m+1),0)}^{(n-2-j,\ j)}| = \binom{m-1}{j}$ for $0 \le j \le m-1, i = n-2-j$ and

$$\sum_{j=0}^{m-1} |P_{(n-(m+1),0)}^{(n-2-j,\ j)}| = 2^{m-1},$$

then the number of admissible paths containing $(n-(m+1), -1) \to (n-(m+1), 0)$ is $T = |P_{(s,-s)}^{(n-2-j,\ j)}| = 2^{n-3}$.

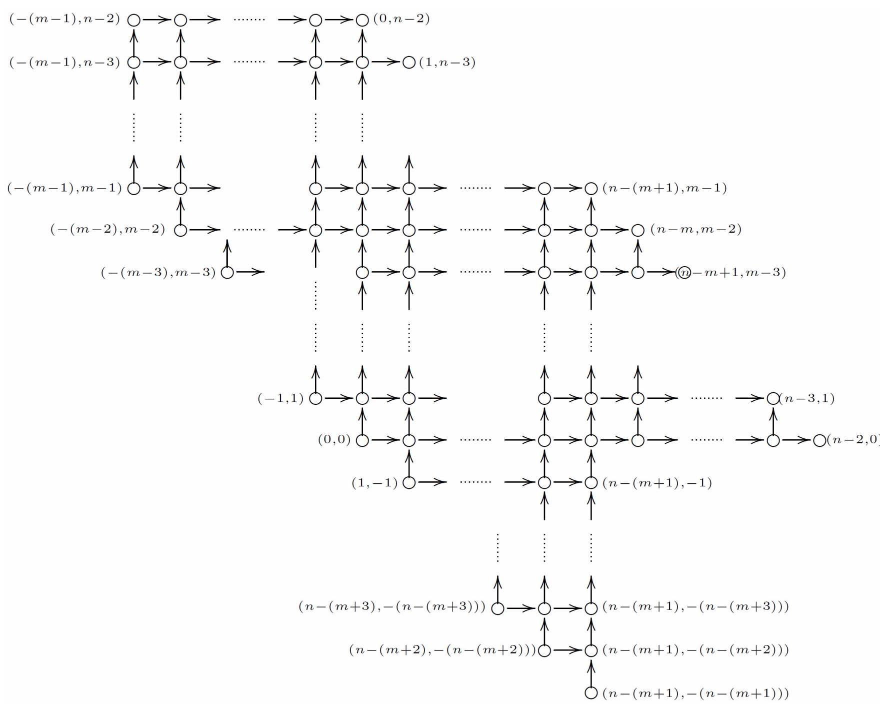


Figure I

Since, vertices $\tau_n^{-j} \in \Gamma(\mathrm{mod}\, A)$ are excluded for $m \le j \le n-2$ and vertices $\tau_{n-(m+1)}^{-r}$ can be connected with the vertices $\tau_n^{-(r-1)}$ or τ_n^{-r} for $m+1 \le r \le n-2$ in $\Gamma(\mathrm{mod}\, A)$, it is possible to define the following sets:

$$w_{p_1} = \{\tau_i^{-j} \mid 2 \le i \le n-m,\ n-i \le j \le n-2\},$$

$$w_{p_2} = \{\tau_n^{-j} \mid m \leq j \leq n-2\},$$

$$w_{p_3} = \{\tau_i^{-j} \mid n-m-1 \leq j \leq n-3,\ 2n-m-2-j \leq i \leq n-1\},$$

$$w_{p_4} = \{\tau_i^{-(n-2)} \mid i = 1 \text{ and } n-m+1 \leq i \leq n-1\}$$

such that the number of sections of $W_p = w_{p_1} \cup w_{p_2} \cup w_{p_3} \cup w_{p_4}$ equals $2T = 2^{n-2}$. Let W_t be the number:

$$W_t = \left| P_{(-s,\ s+1)}^{(-1,\, n-2)} \right| = \binom{n-4}{s-1}$$

for $1 \leq s \leq m-1$.

If $m = 2$, then $s = 1$ and

$$\left| P_{(-1,\ 2)}^{(-1,\, n-2)} \right| = \binom{n-4}{0} = \left| \{\tau_i \mid 1 \leq i \leq n\} \right|.$$

Thus, the number of sections in the Auslander-Reiten quiver of the algebra A associated to D_n with a sink at the 2nd position is given by:

$$2|w_{k-1}| + 2T + \binom{n-4}{0} = 2S_{D(n-1,1)}^{m} + 2^{n-2} + 1.$$

If $3 \leq m \leq n-3$, then

$$\sum_{i=1}^{m-1} \left| P_{(-s,\ s+1)}^{(-1,\, n-2)} \right| = \sum_{i=1}^{m-1} \binom{n-4}{s-1} = \sum_{i=0}^{m-2} \binom{n-4}{i}.$$

If the sets w_{p_5} and w_{p_6} are defined in such a way that

$$w_{p_5} = \{\tau_i^{-j} \mid n-m+1 \leq i \leq n-2,\ 0 \leq j \leq n-2-i\},$$

$$w_{p_6} = \{\tau_i^{-j} \mid 1 \leq i \leq n-m \text{ and } i = n,\ 0 \leq j \leq m-2\},$$

then the number of sections in $w_{p_5} \cup w_{p_6}$ is given by $4W_t - 3\binom{n-4}{m-2}$, provided that the vertex τ^{-j}_{n-m-1} can be connected to τ^{-j}_{n} or $\tau^{-(j-1)}_{n}$ or with τ^{-j}_{n-m} or $\tau^{-(j-1)}_{n-m}$ for $1 \leq j \leq m-2$.

Sets W_k, W_p and the number $4W_t - 3\binom{n-4}{m-2}$ allow us to give the number of sections in the Auslander-Reiten quiver of A with a sink in the mth position with $3 \leq m \leq n-3$ in the following way:

$$2|w_{k-1}| + 2T + 4W_t - 3\binom{n-4}{m-2}$$

$$= 2S^{m}_{D(n-1,1)} + 2^{n-2} + 4\sum_{i=0}^{m-3}\binom{n-4}{i} + \binom{n-4}{m-2}. \qquad \square$$

Corollary 11. $S^{m}_{D(n,1)} = S^{m-1}_{D(n-1,1)} + S^{m}_{D(n-1,1)} + 3(2^{n-3})$ *for* $n \geq 5$, $1 \leq m < n-2$ *with* $S^{1}_{D(n,1)} = 2^{n-3}(2n-1)$ *and* $S^{n-2}_{A(n,1)} = 2^{n-2}(n+1) - 3$.

Proof. By induction on n, if $n = 5$ and $m = 2$, then we have that

$$S^{2}_{D(5,1)} = 2(S^{2}_{D(4,1)}) + 2^{3} + 1$$

$$= S^{2}_{D(4,1)} + 3(2^{2}) + 2^{4}(4) - 2$$

$$= S^{2}_{D(4,1)} + (2(4) - 1) + 3(2^{2})$$

$$= S^{2}_{D(4,1)} + S^{1}_{D(4,1)} + 3(2^{2}).$$

Suppose that the assertion is true for $n \leq k$ and $2 \leq m \leq n-3$. Then for $n = k+1$, we consider three cases:

First, if $m = 2,$ then

$$S^2_{D(k+1,1)} = 2(S^2_{D(k,1)}) + 2^{k-1} + 1.$$

By induction, we have that

$$\begin{aligned} S^2_{D(k+1,1)} &= 2(S^2_{D(k-1,1)} + S^1_{D(k-1,1)} + 3(2^{k-3})) + 2^{k-1} + 1 \\ &= 2S^2_{D(k-1,1)} + (k-1)2^{k-2} - 2^{k-3} + 3(2^{k-2}) + 2^{k-2} + 1 \\ &= 2S^2_{D(k-1,1)} + 2^{k-3}(2k-1) + 2^{k-2} + 2(2^{k-2}) + 2^{k-2} + 1 \\ &= 2S^2_{D(k-1,1)} + 2^{k-2} + 1 + 2^{k-3}(2k-1) + 3(2^{k-2}) \\ &= S^2_{D(k,1)} + S^1_{D(k,1)} + 3(2^{k-2}). \end{aligned}$$

Second, if $m = 3,$ then

$$S^3_{D(k+1,1)} = 2(S^3_{D(k,1)}) + 2^{k-1} + 4\binom{k-3}{0} + \binom{k-3}{1}.$$

By induction, we have that

$$\begin{aligned} S^3_{D(k+1,1)} &= 2(S^3_{D(k-1,1)} + S^2_{D(k-1,1)} + 3(2^{k-3})) + 2^{k-1} \\ &\quad + 4\binom{k-3}{0} + \binom{k-3}{1} \\ &= 2S^3_{D(k-1,1)} + 2^{k-2} + 4\binom{k-4}{0} + \binom{k-4}{1} + 2S^2_{D(k-1,1)} \\ &\quad + 2^{k-2} + 1 + 3(2^{k-2}) \\ &= S^3_{D(k,1)} + S^2_{D(k,1)} + 3(2^{k-2}). \end{aligned}$$

Third, if $m > 3$, then

$$S^m_{D(k+1,1)} = 2(S^m_{D(k,1)}) + 2^{k-1} + \sum_{i=0}^{m-3} 4\binom{k-3}{i} + \binom{k-3}{m-2}.$$

By induction, we have that

$$\begin{aligned}
S^m_{D(k+1,1)} &= 2(S^m_{D(k-1,1)} + S^{m-1}_{D(k-1,1)} + 3(2^{k-3})) + 2^{k-1} \\
&\quad + 4\sum_{i=0}^{m-3}\binom{k-3}{i} + \binom{k-3}{m-2} \\
&= 2S^m_{D(k-1,1)} + 2S^{m-1}_{D(k-1,1)} + 3(2^{k-2}) + 2^{k-2} + 2^{k-1} \\
&\quad + \sum_{i=0}^{m-4} 4\binom{k-4}{i} + \sum_{i=0}^{m-3} 4\binom{k-4}{i} + \binom{k-4}{m-3} + \binom{k-4}{m-2} \\
&= 2S^m_{D(k-1,1)} + 2^{k-2} + \sum_{i=0}^{m-3} 4\binom{k-4}{i} + \binom{k-4}{m-2} + 2S^{m-1}_{D(k-1,1)} \\
&\quad + 2^{k-2} + \sum_{i=0}^{m-4} 4\binom{k-4}{i}\binom{k-4}{m-3} + 3(2^{k-2}) \\
&= S^m_{D(k,1)} + S^{m-1}_{D(k,1)} + 3(2^{k-2}).
\end{aligned}$$

□

Therefore, the following triangular table is built whose arrows are giving the number of sections in the Auslander-Reiten quiver of an algebra A associated to a Dynkin diagram of type D_n with a sink at the hth position with $1 \leq h \leq n-2$:

					14		17					
				36		43		45				
			88		103		112		109			
		208		239		263		269		253		
	480		543		598		628		618		573	
1088		1215		133		1418		1438		1383		1277

Remark 12. Sequence $a_n = S^1_{D(n,1)}$ for $n \geq 4$ appears in the OEIS as A052951.

5. Number of Sections in Algebras of Dynkin Type E_6, E_7 and E_8

In order to give the number of sections in the Auslander-Reiten quiver of algebras of Dynkin type E_6, E_7 and E_8.

We let $S_{E(i,1)}$ denote a vector such that

$$S_{E(i,1)} = (a_1, a_2, \ldots, a_m, \ldots, a_i)$$

which gives the number of sections in the Auslander-Reiten quiver of an algebra associated to a Dynkin diagram of type E_i for each sink allocated at the mth position, in this case, $6 \leq i \leq 8$ and $1 \leq m \leq i$, i is the position of the higher vertex.

For instance the number of sections in the Auslander-Reiten quiver of an algebra associated to E_6 at the position $m = 6$ is $a_6 = 132$. The following are the vectors $S_{E_{(i,1)}}$ defined by the Dynkin diagrams, E_6, E_7 and E_8:

$$S_{E_{(6,1)}} = (124, 139, 147, 139, 124, 132),$$

$$S_{E_{(7,1)}} = (412, 443, 465, 462, 439, 408, 434),$$

$$S_{E_{(8,1)}} = (1532, 1595, 1647, 1663, 1637, 1583, 1520, 1584).$$

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Appendix

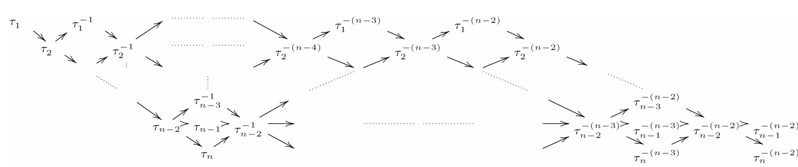

Figure 1

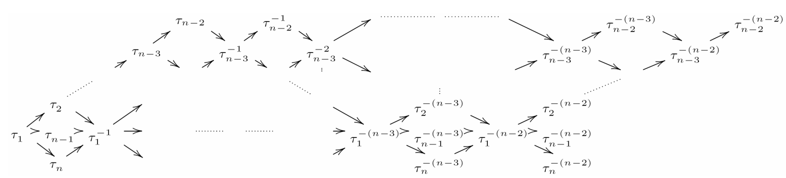

Figure 2

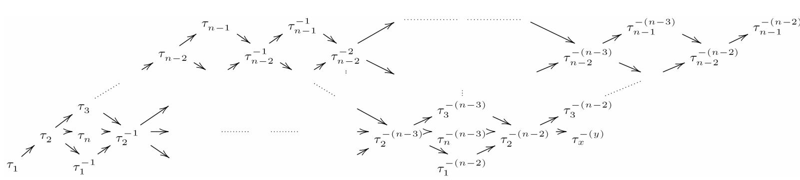

Figure 3