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ASYMPTOTIC EXPANSION OF THE INVERTED MATRIX VARIATE DIRICHLET DISTRIBUTION

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Abstract

In this paper, the inverted matrix variate Dirichlet distribution for both the real and the complex cases are defined and some of their properties are studied. Also, the asymptotic expansions of the probability density functions of the inverted matrix variate Dirichlet distributions, for real and complex cases, are derived.

1. Introduction

The inverted matrix variate beta distribution with parameters (a, b) is defined by the probability density function (p.d.f.)

$$\frac{\Gamma_m(a+b)}{\Gamma_m(a)\Gamma_m(b)} \det(V)^{a-(m+1)/2} \det(I_m + V)^{-(a+b)}, \quad V > 0, \tag{1}$$

where $a > (m-1)/2$, $b > (m-1)/2$, and the multivariate gamma function

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is defined as

$$\Gamma_m(a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma\left(a - \frac{i-1}{2}\right), \quad \operatorname{Re}(a) > \frac{m-1}{2}. \tag{2}$$

We will denote it by $V \sim B_m^{II}(a, b)$.

As an n matrix variate generalization of the density in (1), we define the inverted matrix variate Dirichlet distribution as follows:

The $m \times m$ random symmetric positive definite matrices $V_1, \ldots, V_n$ are said to have the inverted matrix variate Dirichlet distribution with parameters $(b_1, \ldots, b_{n+1})$ if their joint p.d.f. is given by

$$\frac{\Gamma_m\left(\sum_{i=1}^{n+1} b_i\right)}{\prod_{i=1}^{n+1} \Gamma_m(b_i)} \prod_{i=1}^{n} \det(V_i)^{b_i-(m+1)/2} \det\left(I_m + \sum_{i=1}^{n} V_i\right)^{-\sum_{i=1}^{n+1} b_i}, \tag{3}$$

where $V_i > 0$, $i = 1, \ldots, n$. This distribution is denoted by $(V_1, \ldots, V_n) \sim D_m^{II}(b_1, \ldots, b_n; b_{n+1})$.

It is possible to obtain the inverted matrix variate Dirichlet distribution in the following way. Let the random matrices $X_1, \ldots, X_{n+1}$ be distributed independently as Wishart, $X_i \sim W_m(2b_i, I_m)$, $i = 1, \ldots, n+1$. Define $V_i = X_{n+1}^{-1/2} X_i X_{n+1}^{-1/2}$, $i = 1, \ldots, n$. Then, it can be verified that $(V_1, \ldots, V_n) \sim D_m^{II}(b_1, \ldots, b_n; b_{n+1})$.

The matrix variate Dirichlet distributions have been studied by several authors (see, for example, Olkin and Rubin [4], Tan [6], Javier and Gupta [3], and Gupta and Nagar [1]). An extensive review on the matrix variate Dirichlet distributions is available in Gupta and Nagar [2].

In this article, we will derive certain properties including the asymptotic expansions of the inverted matrix variate Dirichlet distributions in real as well as complex cases.

2. Properties

Let $(V_1, \ldots, V_n) \sim D_m^{II}(b_1, \ldots, b_n; b_{n+1})$. An immediate consequence of (3) is

$$\int_{V_1>0} \cdots \int_{V_n>0} \prod_{i=1}^{n} \det(V_i)^{b_i-(m+1)/2} \det\left(I_m + \sum_{i=1}^{n} V_i\right)^{-\sum_{i=1}^{n+1} b_i} \prod_{i=1}^{n} dV_i$$

$$= \frac{\prod_{i=1}^{n+1} \Gamma(b_i)}{\Gamma\left(\sum_{i=1}^{n+1} b_i\right)}. \tag{4}$$

Applying (4), we have the $(h_1, \ldots, h_n)^{\text{th}}$ mixed *moment*, as

$$E[\det(V_1)^{h_1} \cdots \det(V_n)^{h_n}] = \frac{\Gamma_m\left(b_{n+1} - \sum_{i=1}^{n} h_i\right)}{\Gamma_m(b_{n+1})} \prod_{i=1}^{n} \frac{\Gamma_m(b_i + h_i)}{\Gamma_m(b_i)},$$

if $b_{n+1} > \sum_{i=1}^{n} h_i + (m-1)/2$, $b_i + h_i > (m-1)/2$, $i = 1, \ldots, n$, and does not exist otherwise. The means, variances and the covariances are obtained as

$$E[\det(V_i)] = \prod_{r=1}^{m} \frac{[b_i - (r-1)/2]}{[b_{n+1} - (r+1)/2]}, \quad i = 1, \ldots, n,$$

$$\text{Var}[\det(V_i)] = \prod_{r=1}^{m} \frac{(b_i + b_{n+1} - r)[b_i - (r-1)/2]}{[b_{n+1} - (r+1)/2]^2 [b_{n+1} - (r+3)/2]}, \quad i = 1, \ldots, n,$$

$$\text{Cov}[\det(V_i), \det(V_j)] = \prod_{r=1}^{m} \frac{[b_i - (r-1)/2][b_j - (r-1)/2]}{[b_{n+1} - (r+1)/2]^2 [b_{n+1} - (r+3)/2]},$$

$$i \neq j, \; i, j = 1, \ldots, n.$$

Next, we generalize a result given in Tan [6]. This result will be used to obtain the distribution of the partial sums of the random matrices $V_1, \ldots, V_n$ which follow an inverted matrix variate Dirichlet distribution.

Theorem 2.1. *Let* $W_1, \ldots, W_n$ *be* $m \times m$ *positive definite matrices jointly distributed as*

$$p(W_1, \ldots, W_n) \propto \prod_{i=1}^{n} \det(W_i)^{\alpha_i - (m+1)/2} g(W_1 + \cdots + W_n),$$

where $g(\cdot)$ *is positive, continuous, supported on the space of* $m \times m$ *positive definite matrices such that* $\int_{W>0} \det(W)^{\sum_{i=1}^{n} \alpha_i - (m+1)/2} g(W) dW < \infty$, *and* $\alpha_i > (m-1)/2$, $i = 1, 2, \ldots, n$. *Further, let* $\alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j$, $n_0^* = 0$, $n_i^* = \sum_{j=1}^{i} n_j$, $i = 1, \ldots, \ell$. *Define* $Z_j = W_{(i)}^{-1/2} W_j W_{(i)}^{-1/2}$, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$ *and* $W_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} W_j$, $i = 1, \ldots, \ell$. *Then,*

(i) $(W_{(1)}, \ldots, W_{(\ell)})$ *and* $(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1})$, $i = 1, \ldots, \ell$ *are independently distributed,*

(ii) $(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1})$ *has a matrix variate Dirichlet type I distribution with the density proportional to*

$$\prod_{j=n_{i-1}^*+1}^{n_i^*-1} \det(Z_j)^{\alpha_j - (m+1)/2} \det\left(I_m - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} Z_j\right)^{\alpha_{n_i^*} - (m+1)/2},$$

where $0 < Z_j < I_m$, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$, $\sum_{j=n_{i-1}^*+1}^{n_i^*-1} Z_j < I_m$, $i = 1, \ldots, \ell$ *and*

(iii) *the density* $p_1(W_{(1)}, \ldots, W_{(\ell)})$ *of* $(W_{(1)}, \ldots, W_{(\ell)})$ *is given by*

$$p_1(W_{(1)}, \ldots, W_{(\ell)}) \propto \prod_{i=1}^{\ell} \det(W_{(i)})^{\alpha_{(i)} - (m+1)/2} g\left(\sum_{i=1}^{\ell} W_{(i)}\right),$$

where $W_{(1)}, \ldots, W_{(\ell)}$ *are* $m \times m$ *symmetric positive definite matrices.*

Proof. Transforming $W_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} W_j$ and $Z_j = W_{(i)}^{-1/2} W_j W_{(i)}^{-1/2}$, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1,\ i = 1, \ldots, \ell$ with the Jacobian

$$J(W_1, \ldots, W_n \to Z_1, \ldots, Z_{n_1-1}, W_{(1)}, \ldots, Z_{n_{\ell-1}^*+1}, \ldots, Z_{n-1}, W_{(\ell)})$$

$$= \prod_{i=1}^{\ell} \det(W_{(i)})^{(m+1)(n_i-1)/2},$$

in the p.d.f. of $(W_1, \ldots, W_n)$, we get the joint p.d.f. of $Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1}$, $W_{(i)}$, $i = 1, \ldots, \ell$ proportional to

$$\prod_{i=1}^{\ell} \det(W_{(i)})^{\alpha_{(i)}-(m+1)/2} g\left(\sum_{i=1}^{\ell} W_{(i)}\right)$$

$$\times \prod_{i=1}^{\ell} \left\{ \prod_{j=n_{i-1}^*+1}^{n_i^*-1} \det(Z_j)^{\alpha_j-(m+1)/2} \det\left(I_m - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} Z_j\right)^{\alpha_{n_i^*}-(m+1)/2} \right\},$$

where $W_{(i)} > 0,\ i = 1, \ldots, \ell,\ 0 < Z_j < I_m,\ j = n_{i-1}^* + 1, \ldots, n_i^* - 1,\ \sum_{j=n_{i-1}^*+1}^{n_i^*-1} Z_j < I_m$, $i = 1, \ldots, \ell$. Now, from the above factorization, the desired result follows.

Corollary 2.1.1. *Let* $(V_1, \ldots, V_n) \sim D_m^{II}(b_1, \ldots, b_n; b_{n+1})$. *Further, let* $b_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} b_j$, $n_0^* = 0$, $n_i^* = \sum_{j=1}^{i} n_j$, $i = 1, \ldots, \ell$. *Define the random matrix* $V_{(i)}$ *as* $V_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} V_j$, $i = 1, \ldots, \ell$. *Then,*

$$(V_{(1)}, \ldots, V_{(\ell)}) \sim D_m^{II}(b_{(1)}, \ldots, b_{(\ell)}; b_{n+1}).$$

Proof. The density of $(V_1, \ldots, V_n)$ is given by (3). Application of Theorem 2.1 with the function g given by $g = \det(I_m + V)^{-\sum_{i=1}^{n+1} b_i}$ yields the desired result.

By substituting $\ell = 1$ in the above corollary we observe that $V = \sum_{i=1}^{n} V_i \sim B_m^{II}\left(\sum_{i=1}^{n} b_i,\, b_{n+1}\right)$.

3. Asymptotic Expansion

In this section we derive the asymptotic expansion of the probability density function of the inverted Dirichlet random matrices. We first give three lemmas which are needed to derive the final result.

Lemma 3.1. *For* $|\arg(z)| \leq \pi - \varepsilon,\ \varepsilon > 0,$ *the logarithm of* $\Gamma(z + a)$ *can be expanded as*

$$\ln \Gamma(z + a) = (z + a - .5)\ln z - z + \ln \sqrt{2\pi} + \sum_{s=1}^{r} \frac{(-1)^{s+1} B_{s+1}(a)}{s(s+1)} z^{-s} + O(z^{-r-1}),$$

where $B_k(x)$ *is the Bernoulli polynomial of degree k and order unity.*

Lemma 3.2. *For* a_1, a_2 *scalars, we have*

$$\ln\left[\frac{\Gamma_m(z + a_1)}{\Gamma_m(z + a_2)}\right] = (a_1 - a_2)m \ln z + \sum_{i=1}^{m}\sum_{s=1}^{r} \frac{(-1)^{s+1}}{s(s+1)}\left[B_{s+1}\left(a_1 - \frac{i-1}{2}\right) - B_{s+1}\left(a_2 - \frac{i-1}{2}\right)\right] z^{-s} + O(z^{-r-1}),\quad |\arg(z)| \leq \pi - \varepsilon,\ \varepsilon > 0$$

where $B_k(x)$ *is the Bernoulli polynomial of degree k and order unity.*

Proof. Writing multivariate gamma functions in terms of ordinary gamma functions using (2), one obtains

$$\frac{\Gamma_m(z + a_1)}{\Gamma_m(z + a_2)} = \prod_{i=1}^{m} \frac{\Gamma[z + a_1 - (i-1)/2]}{\Gamma[z + a_2 - (i-1)/2]}. \tag{5}$$

Now, taking logarithm of the above expression and using Lemma 3.1, one gets the desired result.

Lemma 3.3. *For* $\| Z/n \| < 1$,

$$- \ln \det\left(I_m - \frac{Z}{n} \right) = \sum_{s=1}^{r} \frac{n^{-s}\mathrm{tr}(Z^s)}{s} + O(n^{-r-1}).$$

Theorem 3.1. *Let* $(V_1, \ldots, V_n) \sim D_m^{II}(b_1, \ldots, b_n; b_{n+1})$. *Define* $Y_i = b_{n+1}V_i$, $i = 1, 2, \ldots, n$. *Then, the p.d.f. of* $(Y_1, \ldots, Y_n)$ *can be expressed as*

$$\left[\prod_{i=1}^{n} \frac{\det(Y_i)^{b_i - (m+1)/2}}{\Gamma_m(b_i)} \right] \mathrm{etr}\left(- \sum_{i=1}^{n} Y_i \right)$$

$$\times \left[1 + \frac{a_1}{2b_{n+1}} + \frac{3a_1^2 + 4a_2}{24b_{n+1}^2} + O(b_{n+1}^{-3}) \right], \quad Y_i > 0, \quad i = 1, \ldots, n, \tag{6}$$

where

$$a_1 = \mathrm{tr}\left(- \sum_{i=1}^{n} Y_i \right)^2 + 2b\,\mathrm{tr}\left(- \sum_{i=1}^{n} Y_i \right) + b^2 m - (1/2) bm(m+1),$$

$$a_2 = 2\mathrm{tr}\left(- \sum_{i=1}^{n} Y_i \right)^3 + 3b\,\mathrm{tr}\left(- \sum_{i=1}^{n} Y_i \right)^2$$

$$- b^3 m + (3/4) b^2 m(m+1) - (1/8) bm(2m^2 + 3m - 1)$$

and

$$b = \sum_{i=1}^{n} b_i.$$

Proof. Substituting $Y_i = b_{n+1}V_i$, $i = 1, 2, \ldots, n$, with $J(V_1, \ldots, V_n \to Y_1, \ldots, Y_n) = b_{n+1}^{-nm(m+1)/2}$ in (3), we obtain the p.d.f. of $(Y_1, \ldots, Y_n)$ as

$$\left[\prod_{i=1}^{n} \frac{\det(Y_i)^{b_i - (m+1)/2}}{\Gamma_m(b_i)} \right] I_1 I_2, \quad Y_i > 0, \quad i = 1, \ldots, n, \tag{7}$$

where

$$I_1 = \frac{\Gamma_m\left(\sum_{i=1}^{n+1} b_i \right)}{\Gamma_m(b_{n+1})} b_{n+1}^{-m\Sigma_{i=1}^{n} b_i},$$

$$I_2 = \det\left(I_m + \frac{Y}{b_{n+1}}\right)^{-\sum_{i=1}^{n+1} b_i} \quad \text{with} \quad Y = \sum_{i=1}^{n} Y_i.$$

Now, using Lemma 3.2 with $r = 2$, $z = b_{n+1}$, $a_1 = b$ and $a_2 = 0$, we obtain

$$\ln I_1 = \frac{1}{2b_{n+1}} \sum_{i=1}^{m} \left[B_2\left(b - \frac{i-1}{2}\right) - B_2\left(\frac{1-i}{2}\right)\right]$$

$$- \frac{1}{6b_{n+1}^2} \sum_{i=1}^{m} \left[B_3\left(b - \frac{i-1}{2}\right) - B_3\left(\frac{1-i}{2}\right)\right] + O(b_{n+1}^{-3}),$$

where $B_2(x) = x^2 - x + 1/6$ and $B_3(x) = x^3 - 3x^2/2 + x/2$. Now, substituting for $B_2(\cdot)$ and $B_3(\cdot)$ in the above expression and simplifying, the above expression is re-written as

$$\ln I_1 = \frac{1}{2b_{n+1}}\left[b^2 m - \frac{1}{2} bm(m+1)\right] - \frac{1}{6b_{n+1}^2}\left[b^3 m - \frac{3}{4} b^2 m(m+1)\right.$$

$$\left. + \frac{1}{8} bm(2m^2 + 3m - 1)\right] + O(b_{n+1}^{-3}). \tag{8}$$

Further, the application of Lemma 3.3 yields

$$\ln I_2 = \operatorname{tr}(-Y) + \frac{1}{2b_{n+1}}[2b\operatorname{tr}(-Y) + \operatorname{tr}(-Y)^2]$$

$$+ \frac{1}{6b_{n+1}^2}[3b\operatorname{tr}(-Y)^2 + 2\operatorname{tr}(-Y)^3] + O(b_{n+1}^2). \tag{9}$$

Therefore, using (8) and (9) we get

$$\ln I_1 + \ln I_2 = \operatorname{tr}(-Y) + \frac{a_1}{2b_{n+1}} + \frac{a_2}{6b_{n+1}^2} + O(b_{n+1}^{-3}), \tag{10}$$

where a_1 and a_2 are given in Theorem 3.1. Hence we get

$$I_1 I_2 = \operatorname{etr}(-Y)\left[1 + \frac{a_1}{2b_{n+1}} + \frac{3a_1^2 + 4a_2}{24b_{n+1}^2} + O(b_{n+1}^{-3})\right]. \tag{11}$$

Finally, substituting from (11) in (7) we get the desired result.

4. Complex Matrix Variate Dirichlet Distribution

Let $V_1, \ldots, V_n$ be $m \times m$ random Hermitian positive definite (hpd) matrices. Then $(V_1, \ldots, V_n)$ is said to have the complex inverted matrix variate Dirichlet distribution (Tan [5]) with parameters $b_1, \ldots, b_{n+1}$, if its p.d.f. is given by

$$\frac{\tilde{\Gamma}_m\left(\sum_{i=1}^{n+1} b_i\right)}{\prod_{i=1}^{n+1} \tilde{\Gamma}_m(b_i)} \prod_{i=1}^{n} \det(V_i)^{b_i - m} \det\left(I_m + \sum_{i=1}^{n} V_i\right)^{-\sum_{i=1}^{n+1} b_i},$$

$$V_i \text{ hpd}, \ i = 1, \ldots, n, \tag{12}$$

where the complex multivariate gamma function is defined as

$$\tilde{\Gamma}_m(a) = \pi^{m(m-1)/2} \prod_{i=1}^{m} \Gamma(a - i + 1), \quad \mathrm{Re}(a) > m - 1.$$

We denote it by $(V_1, \ldots, V_n) \sim CD_m^{II}(b_1, \ldots, b_n; b_{n+1})$. For $n = 1$, (12) reduces to the p.d.f. of a complex inverted beta matrix given by

$$\frac{\tilde{\Gamma}_m(a + b)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(b)} \det(V)^{a-m} \det(I_m + V)^{-(a+b)}, \quad V \text{ hpd} \tag{13}$$

which will be denoted by $V \sim CB_m^{II}(a, b)$.

From (12), we have

$$\int_{V_1 \text{ hpd}} \cdots \int_{V_n \text{ hpd}} \prod_{i=1}^{n} \det(V_i)^{b_i - m} \det\left(I_m + \sum_{i=1}^{n} V_i\right)^{-\sum_{i=1}^{n+1} b_i} dV_1 \cdots dV_n$$

$$= \frac{\prod_{i=1}^{n+1} \tilde{\Gamma}(b_i)}{\tilde{\Gamma}\left(\sum_{i=1}^{n+1} b_i\right)}. \tag{14}$$

Applying (14), the $(h_1, \ldots, h_n)^{\text{th}}$ mixed *moment*, is derived as

$$E[\det(V_1)^{h_1} \cdots \det(V_n)^{h_n}] = \frac{\tilde{\Gamma}_m\left(b_{n+1} - \sum_{i=1}^{n} h_i\right)}{\tilde{\Gamma}_m(b_{n+1})} \prod_{i=1}^{n} \frac{\tilde{\Gamma}_m(b_i + h_i)}{\tilde{\Gamma}_m(b_i)},$$

if $b_{n+1} > \sum_{i=1}^{n} h_i + m - 1$, $b_i + h_i > m - 1$, $i = 1, \ldots, n$, and does not exist otherwise.

The means, variances and the covariances are

$$E[\det(V_i)] = \prod_{r=1}^{m} \frac{(b_i - r + 1)}{(b_{n+1} - r)}, \quad i = 1, \ldots, n,$$

$$\mathrm{Var}[\det(V_i)] = \prod_{r=1}^{m} \frac{(b_i - r + 1)(b_i + b_{n+1} - 2r + 1)}{(b_{n+1} - r)^2 (b_{n+1} - r - 1)}, \quad i = 1, \ldots, n,$$

$$\mathrm{Cov}[\det(V_i), \det(V_j)] = \prod_{r=1}^{m} \frac{(b_i - r + 1)(b_j - r + 1)}{(b_{n+1} - r)^2 (b_{n+1} - r - 1)}, \quad i \neq j, i, j = 1, \ldots, n.$$

Theorem 4.1. *Let $Z_1, \ldots, Z_n$ be $m \times m$ Hermitian positive definite matrices jointly distributed as*

$$p(Z_1, \ldots, Z_n) \propto \det(Z_1)^{b_1 - m} \cdots \det(X_n)^{b_n - m} \tilde{g}(Z_1 + \cdots + Z_n),$$

where $\tilde{g}(\cdot)$ is positive, continuous, supported on the space of $m \times m$ Hermitian positive definite matrices such that

$$\int_{Z \text{ hpd}} \det(Z)^{\sum_{i=1}^{n} b_i - m} g(Z) dZ < \infty,$$

and $b_i > m - 1$, $i = 1, 2, \ldots, n$. Further, let $\alpha_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} \alpha_j$, $n_0^* = 0$, $n_i^* = \sum_{j=1}^{i} n_j$, $i = 1, \ldots, \ell$. Define $Z_j = W_{(i)}^{-1/2} W_j W_{(i)}^{-1/2}$, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$ and $W_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} W_j$, $i = 1, \ldots, \ell$. Then,

(i) $(W_{(1)}, \ldots, W_{(\ell)})$ *and* $(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1})$, $i = 1, \ldots, \ell$, *are independently distributed,*

(ii) $(Z_{n_{i-1}^*+1}, \ldots, Z_{n_i^*-1})$ *has a complex matrix variate Dirichlet type I distribution with the density proportional to*

$$\prod_{j=n_{i-1}^*+1}^{n_i^*-1} \det(Z_j)^{\alpha_j - m} \det\left(I_m - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} Z_j\right)^{\alpha_{n_i^*} - m},$$

where Z_j, $j = n_{i-1}^* + 1, \ldots, n_i^* - 1$, *and* $I_m - \sum_{j=n_{i-1}^*+1}^{n_i^*-1} Z_j$ *are Hermitian positive definite*, $i = 1, \ldots, \ell$, *and*

(iii) *the density* $p_1(W_{(1)}, \ldots, W_{(\ell)})$ *of* $(W_{(1)}, \ldots, W_{(\ell)})$ *is given by*

$$p_1(W_{(1)}, \ldots, W_{(\ell)}) \propto \prod_{i=1}^{\ell} \det(W_{(i)})^{\alpha_i - m} \tilde{g}\left(\sum_{i=1}^{\ell} W_{(i)}\right),$$

where $W_{(1)}, \ldots, W_{(\ell)}$ *are* $m \times m$ *Hermitian positive definite matrices.*

Proof. Similar to the proof of Theorem 2.1.

Corollary 4.1.1. *Let* $(V_1, \ldots, V_n) \sim CD_m^{II}(b_1, \ldots, b_n; b_{n+1})$. *Further, let* $b_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} b_j$, $n_0^* = 0$, $n_i^* = \sum_{j=1}^{i} n_j$, $i = 1, \ldots, \ell$. *Define the complex random matrix* $V_{(i)}$ *as* $V_{(i)} = \sum_{j=n_{i-1}^*+1}^{n_i^*} V_j$, $i = 1, \ldots, \ell$. *Then,*

$$(V_{(1)}, \ldots, V_{(\ell)}) \sim CD_m^{II}(b_{(1)}, \ldots, b_{(\ell)}; b_{n+1}).$$

Note that for $\ell = 1$, $V = \sum_{i=1}^{n} V_i \sim CB_m^{II}\left(\sum_{i=1}^{n} b_i, b_{n+1}\right)$.

Lemma 4.1. *For* a_1, a_2 *scalars, we have*

$$\ln\left[\frac{\tilde{\Gamma}_m(z + a_1)}{\tilde{\Gamma}_m(z + a_2)}\right] = (a_1 - a_2) m \ln z$$

$$+ \sum_{i=1}^{m} \sum_{s=1}^{r} \frac{(-1)^{s+1}}{s(s+1)} [B_{s+1}(a_1 - i + 1) - B_{s+1}(a_2 - i + 1)] z^{-s}$$

$$+ O(z^{-r-1}), \quad |\arg(z)| \le \pi - \varepsilon, \ \varepsilon > 0$$

where $B_k(x)$ *is the Bernoulli polynomial of degree k and order unity.*

Proof. Similar to the proof of Lemma 3.2.

Theorem 4.2. *Let* $(V_1, \ldots, V_n) \sim CD_m^{II}(b_1, \ldots, b_n; b_{n+1})$. *Define* $Y_i = b_{n+1}V_i$, $i = 1, \ldots, n$. *Then, the p.d.f. of* $(Y_1, \ldots, Y_n)$ *can be expressed as*

$$\left[\prod_{i=1}^{n} \frac{\det(Y_i)^{b_i - m}}{\tilde{\Gamma}_m(b_i)}\right] \exp\left(-\sum_{i=1}^{n} Y_i\right)\left[1 + \frac{\tilde{a}_1}{2b_{n+1}} + \frac{3\tilde{a}_1^2 + 4\tilde{a}_2}{24b_{n+1}^2} + O(b_{n+1}^{-3})\right],$$

where $Y_1, \ldots, Y_n$ *are Hermitian positive definite matrices,*

$$\tilde{a}_1 = \operatorname{tr}\left(\sum_{i=1}^{n} Y_i\right)^2 + 2b\operatorname{tr}\left(-\sum_{i=1}^{n} Y_i\right) + b^2 m - bm^2,$$

$$\tilde{a}_2 = 2\operatorname{tr}\left(-\sum_{i=1}^{n} Y_i\right)^3 + 3b\operatorname{tr}\left(\sum_{i=1}^{n} Y_i\right)^2 - b^3 m + (3/2)b^2 m^2$$

$$- bm^3 + (1/2)bm$$

and

$$b = \sum_{i=1}^{n} b_i.$$

Proof. Similar to the proof of Theorem 3.1.

5. Remarks

The expression (6) may be used to yield a corresponding asymptotic formula for the c.d.f. of $(V_1, \ldots, V_n)$, i.e.,

$$P_n(A_1, \ldots, A_n; b_1, \ldots, b_n; b_{n+1}) = P_n(0 < V_1 < A_1, \ldots, 0 < V_n < A_n).$$

Writing $B_i = b_{n+1}A_i$, $i = 1, 2, \ldots, n$ we have

$$P_n(A_1, \ldots, A_n; b_1, \ldots, b_n; b_{n+1})$$

$$= P_n(0 < Y_1 < B_1, \ldots, 0 < Y_n < B_n)$$

$$= \int_{0 < Y_1 < B_n} \cdots \int_{0 < Y_1 < B_n} \left[\prod_{i=1}^{n} \frac{\det(Y_i)^{b_i - (m+1)/2}}{\Gamma_m(b_i)}\right] \exp\left(-\sum_{i=1}^{n} Y_i\right)$$

$$\times\left[1+\frac{a_1}{2b_{n+1}}+\frac{3a_1^2+4a_2}{24b_{n+1}^2}+O(b_{n+1}^{-3})\right]dY_1\cdots dY_n. \tag{15}$$

It is seen that each term in (15) is a combination of the functions

$$G_{\alpha, K_1, K_2}(B_1, \ldots, B_n)$$

$$= \int_{0<Y_1<B_n}\cdots\int_{0<Y_1<B_n}\left[\prod_{i=1}^{n}\det(Y_i)^{b_i-(m+1)/2}\right]\exp\left(-\sum_{i=1}^{n}Y_i\right)$$

$$\times\left[\operatorname{tr}\left(-\sum_{i=1}^{n}Y_i\right)^{\alpha}\right]^{K_1}\left[\operatorname{tr}\left(-\sum_{i=1}^{n}Y_i\right)\right]^{K_2}dY_1\cdots dY_n. \tag{16}$$

The integral on the right-hand side of (16) does not seem to be easy to evaluate. Further work on this will be reported elsewhere.

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References

[1] A. K. Gupta and D. K. Nagar, Distribution of the product of determinants of random matrices connected with noncentral multivariate Dirichlet distribution, S. Afr. Stat. J. 21 (1987), 141-153.

[2] A. K. Gupta and D. K. Nagar, Matrix Variate Distributions, Chapman & Hall/CRC, Boca Raton, 2000.

[3] W. R. Javier and A. K. Gupta, On generalized matric variate beta distributions, Math. Oper. Statist. 16 (1985), 549-558.

[4] I. Olkin and H. Rubin, Multivariate beta distributions and independence properties of the Wishart distribution, Ann. Math. Statist. 35 (1964), 261-269.

[5] W. Y. Tan, Some distribution theory associated with complex Gaussian distribution, Tamkang J. 7 (1968), 263-301.

[6] W. Y. Tan, Note on the multivariate and the generalized multivariate beta distributions, J. Amer. Statist. Assoc. 64 (1969), 230-241.

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