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CATEGORIFICATION OF SOME INTEGER SEQUENCES AND HIGHER DIMENSIONAL PARTITIONS

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Abstract

In this paper, we prove that the number of some 3-dimensional partitions is given by the number of indecomposable objects of some posets of finite representation type.

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1. Introduction

Recently, several efforts have been made in order to describe some integer sequences as invariants of objects in some additive categories, such an identification is said to be a *categorification* of the sequence. For instance, Ringel and Fahr interpreted pairs of Fibonacci numbers as vector dimensions of indecomposable objects in the preprojective, preinjective and regular components of the category of representations of the 3-Kronecker quiver Q. Such identification is a consequence of the Gabriel's covering theory. According to this theory, there exists an equivalence $\overline{F} : \text{rep}\, Q \to \text{rep}\, \overline{Q}$ between the category of representations of the 3-Kronecker quiver and the category of representations of its corresponding covering $\overline{Q}$. The relationship between Fibonacci numbers and vector dimensions of objects in the Auslander-Reiten quiver can be obtained provided that $\overline{F}$ preserves indecomposability and satisfies $\dim \overline{V} = \overline{\dim V}$ for each object $V \in \text{rep}\ Q$ [6, 7, 11].

Worth noting that Cañadas, Giraldo et al. found out a relationship between the number of indecomposable of some posets of finite representation type and elements of the sequence A002662 in OEIS (the On-Line Encyclopedia of Integer Sequences) via the algorithm of differentiation with respect to a maximal point introduced by Nazarova and Roiter in 1972 to classify different types of posets [5, 9]. We recall that the category whose objects are k-linear representations of a given poset $\mathcal{P}$ is an additive category. In this case, a morphism $\varphi : U \to V$ between representations $U = (U_0; U_x \mid x \in \mathcal{P})$ and $V = (V_0; V_x \mid x \in \mathcal{P})$ is a k-linear map such that $\varphi(U_x) \subset V_x$ for all $x \in \mathcal{P}$.

The algorithm of differentiation with respect to a maximal point allowed to Kleiner establishes a criterion for posets of finite representation type in 1972. We quote that, classification problems for additive categories having only finitely many isomorphism classes of indecomposable objects are called *problems of finite representation type*. The problems admitting a classification of their indecomposable objects are called of *tame representation type*. There are also problems for which the classification seems to be impossible; they are called of *wild representation type* [2, 3, 9].

The algorithm with respect to a maximal point was the first tool to classify posets in the ways described above. Such an algorithm can be applied to posets $\mathcal{P}$ of the form $\mathcal{P} = N + b_{\Delta}$, where the subset N consists of all points incomparable with b has width $w(N) \leq 2$. The derived poset $\mathcal{P}'_b$ with respect to the maximal point b is defined in such a way that

$$\mathcal{P}'_b = (\mathcal{P} \backslash b) \bigcup \hat{N}$$

with a partial order induced by $\mathcal{P} \backslash b$ and $\hat{N} = N \bigcup \{x + y \,|\, x, y \in N, x \not\leq y, y \not\leq x\}$.

In this case, a representation $U'_b = (U'_0; U'_x \,|\, x \in \mathcal{P})$ of a derived poset $\mathcal{P}'_b$ arises from a representation $U = (U_0; U_x \,|\, x \in \mathcal{P})$ of $\mathcal{P}$ by restricting space U_0 to U_b and defining spaces $U_{x+y} = (U_x + U_y) \bigcap U_b$ for points $x + y \in \hat{N}$, $U'_x = U_x \bigcap U_b$ for all points $x \in \mathcal{P}'_b \backslash \{\hat{N}\}$.

In 1973, Gabriel proved the following relation between indecomposable representations of posets $\mathcal{P}$ and $\mathcal{P}'_b$ [8, 9]:

$$|\operatorname{Ind} \mathcal{P}| = |\operatorname{Ind} \mathcal{P}'_b| + |\hat{N}| + 1.$$

A k-linear representation U is said to be *decomposable* if there exist not null representations U_1 and U_2 non-isomorphic to U such that $U \cong U_1 \oplus U_2$. U is said to be *indecomposable* if U is not null and it is not decomposable. Ind $\mathcal{P}$ denotes the complete set of representatives of all classes of indecomposable representations of the poset $\mathcal{P}$.

In this paper, we use again the algorithm of Nazarova and Roiter to prove that sequence A002662 also gives the number of partitions of type $\mathcal{H}$ (introduced by Cañadas et al. in [4]) of a positive integer n. In fact, relationship between this sequence, the number of indecomposable of some posets introduced in [5] and its dimension vectors allows us to describe variations of the so called partitions or compositions of type $\mathcal{H}$ in a very simple way. We recall that these partitions are 3-dimensional partitions whose parts are matrices. Entries in these parts satisfy some constraints which make them a kind of mathematical objects with a very tricky description. Regarding this problem, we quote that the problem of giving a description of higher dimensional partitions and its generating functions is a cumbersome problem in the theory of partitions [1]. In this work, we shall prove that these kinds of problems can be tackled by using poset representation theory.

This paper is organized as follows. Some of the basic definitions and notations concerning partitions are included in Section 2. In Section 3, we recall some recent results regarding the categorification of the sequence A002662. Finally, in Section 4, we describe partitions of type $\mathcal{H}$ and prove that elements in such sequence give the number of partitions of type $\mathcal{H}$ of some positive numbers.

2. Preliminaries

In this section, some basic definitions regarding partitions are introduced. We refer to the interested reader to [2-5] for more precise definitions.

2.1. Higher dimensional partitions

A *partition* of a positive integer *n* is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The λ_i are called the *parts* of the partition [1]. A *composition* is a partition in which the order of the summands is considered.

Partitions of positive numbers may be treated as a linear array whose sum is prescribed

$$n = n_1 + n_2 + \cdots + n_s = \sum_{i=1}^{s} n_i, \quad n_i \geq n_{i+1},$$

higher-dimensional partitions are arrays whose sum is n. In this case;

$$n = \sum_{i_1, \ldots, i_r \geq 0} n_{i_1 i_2 \ldots i_r}, \text{ where } n_{i_1 i_2 \ldots i_r} \geq n_{j_1 j_2 \ldots j_r}, \quad (1)$$

whenever $i_1 \leq j_1$, $i_2 \leq j_2$, ..., $i_r \leq j_r$ (all $n_{i_1 i_2 \ldots i_r}$ nonnegative integers) [1]. In particular, the plane partitions of n are two-dimensional arrays of nonnegative integers in the first quadrant subject to a nonincreasing condition along rows and columns.

We note that, up to date, there are very few results regarding partitions of dimension greater than 2. As an advance in the research of this topic, we use poset representation theory to describe some 3-dimensional partitions.

3. Categorification of the Sequence A002662

In this section, we give a brief description of the results obtained by Cañadas et al. in [5], where posets M_j^k, $1 \leq j \leq k$, $k \geq 1$ fixed and its corresponding derived posets $(M_j^k)' = \widehat{Y_j^k}$ (see Figure 1 below) were used to give a categorification of the sequence A002662 in the OEIS [10].

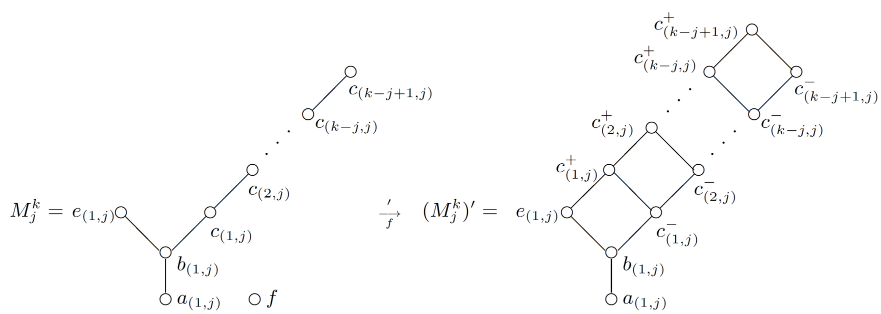

Figure 1

In this case, $(M_j^k)' = \{a_{(1,j)} < b_{(1,j)} < e_{(1,j)}\} + C^+_{(k-j+1,j)} + C^-_{(k-j+1,j)}$, $1 \leq j \leq k$, and $C^+_{(k-j+1,j)} = c^+_{(1,j)} < c^+_{(2,j)} < \cdots < c^+_{(k-j+1,j)}$, $C^-_{(k-j+1,j)}$ is defined as $C^+_{(k-j+1,j)}$ for each j.

In [5], k-linear representations of posets $\widehat{Y_j^k} = (M_j^k)'$ are defined with the following notation:

$$x_\Delta = \{y \in \mathcal{P} \mid y \leq x\}, \quad x_\blacktriangle = x_\Delta \setminus \{x\},$$

$$x^\nabla = \{y \in \mathcal{P} \mid x \leq y\}, \quad x^\blacktriangledown = x^\nabla \setminus \{x\},$$

$$\operatorname{rad} U_x = \sum_{y \in x_\blacktriangle} U_y,$$

$$\operatorname{rad}^- U_x = U_x / \sum_{y \in C^-_{(k-j+1,\, j)}} U_y,$$

$$\operatorname{rad}^+ U_x = U_x / \sum_{y \in C^+_{(k-j+1,\, j)}} U_y,$$

$$\partial_x^- = \dim \sum_{y \in x_\Delta \cap C^-_{(k-j+1,\, j)}} \operatorname{rad}^- U_y,$$

$$\partial_x^+ = \dim \sum_{y \in x_\Delta \cap C^+_{(k-j+1,\, j)}} \operatorname{rad}^+ U_y,$$

$$\Delta_j^k = |\operatorname{Ind} M_j^k| - |\operatorname{Ind}(M_j^k)'| - 1. \tag{2}$$

A finite nonnegative *lattice path* in the plane (with unit steps to the right and down) is a sequence $L = (v_1, v_2, ..., v_k)$, where $v_i \in \mathbb{N}^2$ and $v_{i+1} - v_i = (1, 0)$ or $(0, -1)$ [5].

Lattice paths in posets $\widehat{Y_j^k}$ which link the point $a_{(1,\, j)}$ with the final point of the subset $D^+_{k-j+1} = C^+_{k-j+1} \cup \{e_{(1,\, j)}\}$ have been considered. We let $P^k_{Y_j}$ denote such a set, i.e.,

$$P^k_{Y_j} = \{P = a_{(1,\, j)} \| x \mid x \in \{c^+_{k-j+1}, e_{(1,\, j)}\}\},$$

$$|P^k_{Y_j}| = L^k_{Y_j}. \tag{3}$$

For all *j*, we assume the notation $L^{k}_{Y_{k+1}}$ $(\widehat{Y^{k}_{k+1}} = a_{(1,\, j)} < b_{(1,\, j)} < e_{(1,\, j)})$ for the number of lattice paths linking points $a_{(1,\, j)}$ and $e_{(1,\, j)}$. Note that $L^{k}_{Y_{k+1}} = 1.$

For a given poset $\widehat{Y^{k}_{j}},\ 1 \leq j \leq k+1,$ we let $\Gamma^{m}_{a_{(1,\, j)}}$ denote a collection of sets of the form:

$$\Lambda^{m}_{a_{(1,\, j)}} = \{(p^{m}_{h_1},\ p^{m}_{h_2},\ \ldots,\ p^{m}_{h_j}) \mid 2 \leq h_1 < h_2 < \cdots < h_j$$

$$= |\, C^{+}_{k-j+1} \,| + 3,\ m = k+3\},$$

$|\, \lambda^{m}_{a_{(1,\, j)}} \,|$ and $w^{m}(\lambda^{m}_{a_{(1,\, j)}}) = \sum_{h_s} p^{m}_{h_s}$ denote the size and the weight of $\lambda^{m}_{a_{(1,\, j)}} \in \Lambda^{m}_{a_j}$, respectively. That is, $\lambda^{m}_{a_{(1,\, j)}}$ is a partition of the positive integer $w^{m}(\lambda^{m}_{a_{(1,\, j)}})$. $|\, C^{+}_{0} \,| = 0.$

We let $\underline{\mathit{dim}}$ U denote the *dimension vector* of a representation U of a poset $\mathcal{P}$ which is a vector whose coordinates are nonnegative integers such that $\underline{\dim} U = (d_0;\, d_x \mid x \in \mathcal{P})$, where $d_0 = \dim U_0$ and $d_x = \dim U_n / \sum_{y<x} U_x$, for each $x \in \mathcal{P}$.

For $k \geq 1$ and $1 \leq j \leq k+1$ fixed, we define the following dimension vector of a poset $M^{k}_{j} \backslash \{f\}$ $(M^{k}_{k+1} \backslash \{f\} = a_{(1,\, k+1)} < b_{(1,\, k+1)} < e_{(1,\, k+1)}$:

$$\dim U_{a_{(1,\, j)}} = w^{m}(\lambda^{m}_{a_{(1,\, j)}}),\ \text{for some } \lambda^{m}_{a_{(1,\, j)}} \in \Gamma^{m}_{a_{(1,\, j)}},$$

$$\dim U_{b_{(1,\, j)}} = |\, \lambda^{m}_{a_{(1,\, j)}} \,|\, p^{m}_{h_j},$$

$$\dim U_{c_{(\bar{i},\, j)}} = |\, \lambda^{m}_{a_{(1,\, j)}} \,|\, p^{m}_{i+2},$$

$$\dim U_{c^+_{(i,\,j)}} = (m - |\lambda^m_{a_{(1,\,j)}}|)\, p^m_{i+2},\ 1 \le i \le k,$$

$$\dim U_{e_{(1,\,j)}} = mp^m_{h_j},\ 1 \le i \le k. \tag{4}$$

Each lattice path $P \in P^k_{Y_j}$ defines a k-linear representation of $\widehat{Y^k_j}$ with a vector dimension $\underline{\dim}\widehat{Y^k_j}$ given by the following formulae:

$$d^m_x = \begin{cases} w^m(\lambda^m_{a_{(1,\,j)}}), & \text{if } x = a_{(1,\,j)},\ \lambda^m_{a_{(1,\,j)}} \in \Gamma^m_{a_{(1,\,j)}}, \\ |\lambda^m_{a_{(1,\,j)}}|\, p^m_{h_k} - w^m(\lambda^m_{a_{(1,\,j)}}), & \text{if } x = b_{(1,\,j)} \\ \partial^-_x, & \text{if } x \in P \cap C^-_{(k-j+1,\,j)}, \\ \partial^+_x, & \text{if } x \in (z^+)^{\blacktriangledown}, \\ (m - |\lambda^m_{a_{(1,\,j)}}|)\, p^m_{i+2}, & \text{if } x = z^+ = c^+_{(i,\,j)} = \min(P \cap D^+_{(k-j+1,\,j)}), \\ (m - |\lambda^m_{a_{(1,\,j)}}|)\, p^m_{h_j}, & \text{if } z = e_{(1,\,j)}, \\ 0, & \text{if } x \in z^+_{\blacktriangle} \text{ and } P \cap C^+_{(k-j+1,\,j)} \ne \varnothing, \\ 0, & \text{if } x \in (z^-)^{\blacktriangledown},\ z^- = \max(P \cap C^-_{(k-j+1,\,j)}), \end{cases} \tag{5}$$

where $D^+_{k-j+1} = C^+_{k-j+1} \cup \{e_{(1,\,j)}\}$.

Support of each dimension vector of a poset $\widehat{Y^k_j}$ defines a composition (of type $\mathcal{L}$) of $(k+3)\,p^{k+3}_{k+3}$.

Henceforth, we let $\mathcal{L}^k_j$ denote the number of compositions of type $\mathcal{L}$ of $(k+3)\,p^{k+3}_{k+3}$ induced by the set $\bigcup_{1 \le j \le k+1} P^{k-j+1}_j$.

The following results regarding $\mathcal{L}^k_j$, $k \ge 2$ were obtained in [5].

Theorem 1. $\mathcal{L}_j^k = \sum_{h=1}^{k+1}(2^h - 1)(k + 2 - h)$ *for each* $k \geq 2$ *and* $m = k + 3$ *fixed*.

Corollary 2. *For each* $k \geq 2$, *the following identity holds*:

$$\mathcal{L}_j^k = \sum_{h=1}^{k+1}(2^h - 1)\left\lfloor \frac{\Delta_h^k}{2} \right\rfloor.$$

Remark 3. Note that $\mathcal{L}_j^k = 2(2^{k+2} - 1) - \frac{(k+2)(k+5)}{2}$. Therefore, Theorem 1 and Corollary 2 give a formula partition for elements $\mathcal{L}_j^k \in \{1, 5, 16, 42, 99, 219, 466, 968, \ldots\}$ which is the sequence A002662 in OEIS [10]. In this case,

$$\begin{aligned}
16 &= (1)(3) + (3)(2) + (7)(1),\\
42 &= (1)(4) + (3)(3) + (7)(2) + (15)(1),\\
99 &= (1)(5) + (3)(4) + (7)(3) + (15)(2) + (31)(1),\\
219 &= (1)(6) + (3)(5) + (7)(4) + (15)(3) + (31)(2) + (63)(1),\\
968 &= (1)(8) + (3)(7) + (15)(5) + (31)(4) + (63)(3) + (127)(2) + (255)(1),\\
\vdots &= \vdots \quad \vdots \quad \vdots
\end{aligned} \tag{6}$$

4. Main Results

In this section, for each $1 \leq j \leq k + 1$, we consider completed derived posets $\widehat{Z_j^k} = \widehat{X_j^k} + \widehat{Y_j^k}$ $(\widehat{X_1^k} = \varnothing)$ with a Hasse diagram as shown in the following Figure 2.

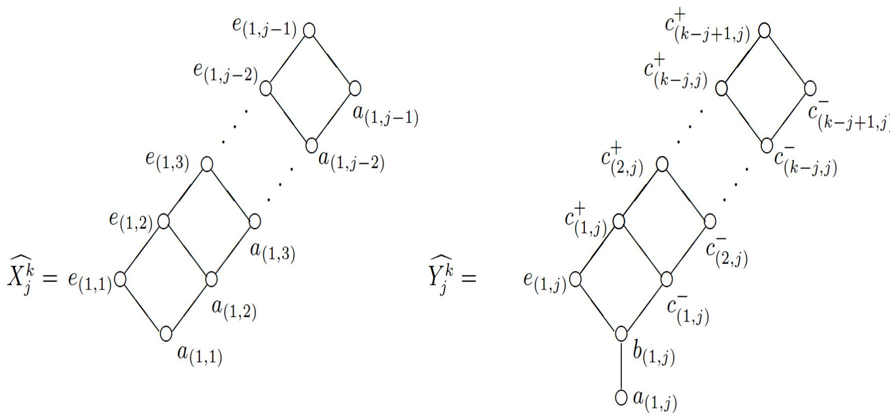

Figure 2

Let us define posets $\widehat{T_j^k}$ in such a way that

$$(\widehat{Z_j^k})^k = \widehat{T_j^k} = \sum_{j=1}^{k+1} \widehat{Z_j^k},$$

i.e., $\widehat{T_j^k}$ is a sum of k-copies of poset $\widehat{Z_j^k}$. Each point $x_{(r,s)}$ in the jth copy $\widehat{Z_j^k}$ is denoted $x_{(r,j)}$.

A vector dimension of a representation of $\widehat{T_j^k}$ is defined by vector dimensions of copies $\widehat{Z_j^k}$, $1 \leq j \leq k+1$. Moreover, for j fixed, a vector dimension of $\widehat{Z_j^k}$ is induced by the corresponding representation of $\widehat{Y_j^k}$. That is, if $d^j_{x_{(r,j)}} = \dim U_{x_{(r,j)}} / \operatorname{rad} U_{x_{(r,j)}}$ ($d^j_{x_{(r,j)}}$ is the dimension of the quotient $U_{x_{(r,j)}} / \operatorname{rad} U_{x_{(r,j)}}$ in the jth copy $\widehat{Z_j^k}$), then:

$$d^{j}_{x_{(r,j)}}$$

$$=\begin{cases} w^{j+4}(\lambda^{j+4}_{a_{(1,j)}}), & \text{if } x_{(r,j)} = a_{(1,j)},\ \lambda^{j+4}_{a_{(1,j)}} \in \Gamma^{j+4}_{a_{(1,j)}}, \\ |\lambda^{j+4}_{a_{(1,j)}}|\, p^{j+4}_{h_k} - w^{j+4}(\lambda^{j+4}_{a_{(1,j)}}), & \text{if } x_{(r,j)} = b_{(1,j)} \\ \partial^{-}_{x_{(r,j)}}, & \text{if } x_{(r,j)} \in P_j \cap C^{-}_{(k-j+1,j)}, \\ \partial^{+}_{x_{(r,j)}}, & \text{if } x_{(r,j)} \in (z^{+}_j)^{\blacktriangledown}, \\ (m - |\lambda^{j+4}_{a_{(1,j)}}|)\, p^{j+4}_{h+2}, & \text{if } x_{(r,j)} = z^{+}_j = c^{+}_{(h,j)} = \min(P_j \cap D^{+}_{(k-j+1,j)}), \\ (m - |\lambda^{j+4}_{a_{(1,j)}}|)\, p^{j+4}_{h_j}, & \text{if } z_j = e_{(1,j)}, \\ |\lambda^{j+4}_{a_{(1,j)}}|\, p^{j+4}_{s+2}, & \text{if } x_{(r,j)} = a_{(1,s)},\ 1 \le s \le j-1, \\ (m - |\lambda^{j+4}_{a_{(1,j)}}|)\, p^{j+4}_{s+2}, & \text{if } x_{(r,j)} = e_{(1,s)},\ 1 \le s \le j-1, \\ 0, & \text{if } x_{(r,j)} \in (z_j)^{+}_{\blacktriangle} \text{ and } P \cap C^{+}_{(k-j+1,j)} \neq \emptyset, \\ 0, & \text{if } x_{(r,j)} \in (z^{-}_j)^{\blacktriangledown},\ z^{-}_j = \max(P_j \cap C^{-}_{(k-j+1,j)}). \end{cases} \tag{7}$$

Each dimension of a subspace U^{i}_{x}, x in a representation of the ith copy of $\widehat{Y^{k}_{j}}$ is defined as in formulas (4).

$P_j \in \widehat{Y^{k}_{j}}$ is a fixed lattice path for all $1 \le j \le k+1$. Therefore, as in the case for posets $\widehat{Y^{k}_{j}}$, the support of vector dimension of representations of posets $\widehat{Z^{k}_{j}}$ induces 3-dimensional partitions (of type $\mathcal{H}$) of numbers $t_m = (m+2)[(\rho_{m+1}-1)(p^{3}_{m+1}-1) + (\rho_{m+2}-4)m]\left(\rho_n = \dfrac{n(n+1)(n+2)}{6}\right)$.

Remark 4. We note that the sequence $\left\{\frac{t_m}{m+2}\right\} = \{12, 77, 264, 684, 1500, 2937, ...\}$ does not appear in the OEIS.

Parts of partitions of type $\mathcal{H}$ are defined as follows:

$$M^1 = (M_{ih}^1) = \begin{cases} d_{a_{(1,h)}}^i, & 1 \leq h \leq j-1, \\ d_{a_{(1,j)}}^i, & j \leq h \leq k+1, \end{cases} \tag{8}$$

$$M^2 = (M_{ih}^2) = \begin{cases} 0, & 1 \leq h \leq j-1, \\ d_{b_{(1,j)}}^i, & j \leq h \leq k+1, \end{cases} \tag{9}$$

$$M^3 = (M_{ih}^3) = \begin{cases} d_{e_{(1,h)}}^i, & 1 \leq h \leq j-1, \\ d_{z_j^+}^i, & j \leq h \leq k+1, \end{cases} \tag{10}$$

$$M^4 = (M_{ih}^4) = \begin{cases} 0, & 1 \leq h \leq j-1, \\ \partial_{x_{(r,h)}}^-, & j \leq h \leq k+1, \end{cases} \tag{11}$$

$$M^5 = (M_{ih}^5) = \begin{cases} 0, & 1 \leq h \leq j-1, \\ \partial_{x_{(r,h)}}^+, & j \leq h \leq k+1. \end{cases} \tag{12}$$

We let $\mathfrak{M}_j^k$ denote the number of partitions of the number $(k+2)[(\rho_{k+1}-1)(p_{k+1}^3-1)+(\rho_{k+2}-4)k]$ into parts of type $\mathcal{H}$.

The relationship between partitions of type $\mathcal{L}$ and partitions of type $\mathcal{H}$ allows us to establish the following result:

Theorem 5. *For* $k \geq 2$, $\mathcal{L}_j^k = \mathfrak{M}_j^k$.

Proof. Each partition of type $\mathcal{H}$ is induced by a fixed lattice path $P_i \in \widehat{Y_j^k}$ in a fixed copy of $\widehat{Z_j^k}$. □

The following is a consequence of Remark 4 and the definition of partitions of type $\mathcal{H}$.

Corollary 6. *The next identity holds for every* $m \geq 1$:

$$[(\rho_{m+1}-1)(p_{m+1}^{3}-1)+(\rho_{m+2}-4)m]=\sum_{i=1}^{m}\sum_{j=1}^{m}p_{j+2}^{i+4}.$$

As an example, the following are the 16 partitions of type $\mathcal{H}$ of 1320:

$$\lambda_1=\left\{M_{ij}^{1}=\begin{bmatrix}7&7&7\\9&9&9\\11&11&11\end{bmatrix},\ M_{ij}^{2}=\begin{bmatrix}p_2^5+p_3^5&p_2^5+p_3^5&p_2^5+p_3^5\\p_2^6+p_3^6&p_2^6+p_3^6&p_2^6+p_3^6\\p_2^7+p_3^7&p_2^7+p_3^7&p_2^7+p_3^7\end{bmatrix},\right.$$

$$\left.M_{ij}^{3}=\begin{bmatrix}3p_3^5&3p_3^5&3p_3^5\\3p_3^6&3p_3^6&3p_3^6\\3p_3^7&3p_3^7&3p_3^7\end{bmatrix},M_{ij}^{4}=0,1\leq i,j\leq 3,M_{ij}^{5}=\begin{bmatrix}0&50&50+65\\0&65&65+85\\0&80&80+105\end{bmatrix}\right\},$$

$$\lambda_2=\left\{M_{ij}^{1}=\begin{bmatrix}7&7&7\\9&9&9\\11&11&11\end{bmatrix},\ M_{ij}^{2}=\begin{bmatrix}p_2^5+p_3^5&p_2^5+p_3^5&p_2^5+p_3^5\\p_2^6+p_3^6&p_2^6+p_3^6&p_2^6+p_3^6\\p_2^7+p_3^7&p_2^7+p_3^7&p_2^7+p_3^7\end{bmatrix},\right.$$

$$\left.M_{ij}^{3}=\begin{bmatrix}3p_3^5&3p_4^5&3p_4^5\\3p_3^6&3p_4^6&3p_4^6\\3p_3^7&3p_4^7&3p_4^7\end{bmatrix},M_{ij}^{4}=\begin{bmatrix}0&20&20\\0&26&26\\0&32&32\end{bmatrix},\ M_{ij}^{5}=\begin{bmatrix}0&0&65\\0&0&85\\0&0&105\end{bmatrix}\right\},$$

$$\lambda_3=\left\{M_{ij}^{1}=\begin{bmatrix}7&7&7\\9&9&9\\11&11&11\end{bmatrix},\ M_{ij}^{2}=\begin{bmatrix}p_2^5+p_3^5&p_2^5+p_3^5&p_2^5+p_3^5\\p_2^6+p_3^6&p_2^6+p_3^6&p_2^6+p_3^6\\p_2^7+p_3^7&p_2^7+p_3^7&p_2^7+p_3^7\end{bmatrix},\right.$$

$$\left.M_{ij}^{3}=\begin{bmatrix}3p_3^5&3p_4^5&3p_5^5\\3p_3^6&3p_4^6&3p_5^6\\3p_3^7&3p_4^7&3p_5^7\end{bmatrix},\ M_{ij}^{4}=\begin{bmatrix}0&20&20+26\\0&26&26+34\\0&32&32+42\end{bmatrix},\ M_{ij}^{5}=0,\ 1\leq i,j\leq 3\right\},$$

$$\lambda_4=\left\{M_{ij}^{1}=\begin{bmatrix}2p_3^5&17&17\\2p_3^6&22&22\\2p_3^7&27&27\end{bmatrix},\ M_{ij}^{2}=\begin{bmatrix}0&p_2^5+p_4^5&p_2^5+p_4^5\\0&p_2^6+p_4^6&p_2^6+p_4^6\\0&p_2^7+p_4^7&p_2^7+p_4^7\end{bmatrix},\right.$$

$$M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_4^5 \\ 3p_3^6 & 3p_4^6 & 3p_4^6 \\ 3p_3^7 & 3p_4^7 & 3p_4^7 \end{bmatrix}, M_{ij}^4 = 0, 1 \le i,\ j \le 3,\ M_{ij}^5 = \left.\begin{bmatrix} 0 & 0 & 65 \\ 0 & 0 & 85 \\ 0 & 0 & 105 \end{bmatrix}\right\},$$

$$\lambda_5 = \left\{ M_{ij}^1 = \begin{bmatrix} 2p_3^5 & 17 & 17 \\ 2p_3^6 & 22 & 22 \\ 2p_3^7 & 27 & 27 \end{bmatrix}, M_{ij}^2 = \begin{bmatrix} 0 & p_2^5 + p_4^5 & p_2^5 + p_4^5 \\ 0 & p_2^6 + p_4^6 & p_2^6 + p_4^6 \\ 0 & p_2^7 + p_4^7 & p_2^7 + p_4^7 \end{bmatrix}, \right.$$

$$\left. M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_5^5 \\ 3p_3^6 & 3p_4^6 & 3p_5^6 \\ 3p_3^7 & 3p_4^7 & 3p_5^7 \end{bmatrix}, M_{ij}^4 = \begin{bmatrix} 0 & 0 & 26 \\ 0 & 0 & 34 \\ 0 & 0 & 42 \end{bmatrix}, M_{ij}^5 = 0,\ \ 1 \le i,\ j \le 3 \right\},$$

$$\lambda_6 = \left\{ M_{ij}^1 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 30 \\ 2p_3^6 & 2p_4^6 & 39 \\ 2p_3^7 & 2p_4^7 & 48 \end{bmatrix},\ M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_2^5 + p_5^5 \\ 0 & 0 & p_2^6 + p_5^6 \\ 0 & 0 & p_2^7 + p_5^7 \end{bmatrix}, \right.$$

$$\left. M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_5^5 \\ 3p_3^6 & 3p_4^6 & 3p_5^6 \\ 3p_3^7 & 3p_4^7 & 3p_5^7 \end{bmatrix},\ \ M_{ij}^4 = M_{ij}^5 = 0,\ \ \ 1 \le i,\ j \le 3 \right\},$$

$$\lambda_7 = \left\{ M_{ij}^1 = \begin{bmatrix} 2p_3^5 & 10 & 10 \\ 2p_3^6 & 13 & 13 \\ 2p_3^7 & 16 & 16 \end{bmatrix},\ M_{ij}^2 = \begin{bmatrix} 0 & p_3^5 + p_4^5 & p_3^5 + p_4^5 \\ 0 & p_3^6 + p_4^6 & p_3^6 + p_4^6 \\ 0 & p_3^7 + p_4^7 & p_3^7 + p_4^7 \end{bmatrix}, \right.$$

$$\left. M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_5^5 \\ 3p_3^6 & 3p_4^6 & 3p_5^6 \\ 3p_3^7 & 3p_4^7 & 3p_5^7 \end{bmatrix}, M_{ij}^4 = \begin{bmatrix} 0 & 0 & 26 \\ 0 & 0 & 34 \\ 0 & 0 & 42 \end{bmatrix},\ M_{ij}^5 = 0, 1 \le i,\ j \le 3 \right\},$$

$$\lambda_8 = \left\{ M_{ij}^1 = \begin{bmatrix} 2p_3^5 & 10 & 10 \\ 2p_3^6 & 13 & 13 \\ 2p_3^7 & 16 & 16 \end{bmatrix},\ M_{ij}^2 = \begin{bmatrix} 0 & p_3^5 + p_4^5 & p_3^5 + p_4^5 \\ 0 & p_3^6 + p_4^6 & p_3^6 + p_4^6 \\ 0 & p_3^7 + p_4^7 & p_3^7 + p_4^7 \end{bmatrix}, \right.$$

$$M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_4^5 \\ 3p_3^6 & 3p_4^6 & 3p_4^6 \\ 3p_3^7 & 3p_4^7 & 3p_4^7 \end{bmatrix}, \ M_{ij}^4 = 0, 1 \le i, \ j \le 3, \ M_{ij}^5 = \begin{bmatrix} 0 & 0 & 65 \\ 0 & 0 & 85 \\ 0 & 0 & 105 \end{bmatrix}\Bigg\},$$

$$\lambda_9 = \left\{ M_{ij}^1 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 23 \\ 2p_3^6 & 2p_4^6 & 30 \\ 2p_3^7 & 2p_4^7 & 37 \end{bmatrix}, \ M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_3^5 + p_5^5 \\ 0 & 0 & p_3^6 + p_5^6 \\ 0 & 0 & p_3^7 + p_5^7 \end{bmatrix}, \right.$$

$$\left. M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_5^5 \\ 3p_3^6 & 3p_4^6 & 3p_5^6 \\ 3p_3^7 & 3p_4^7 & 3p_5^7 \end{bmatrix}, \ M_{ij}^4 = M_{ij}^5 = 0, \quad 1 \le i, \ j \le 3 \right\},$$

$$\lambda_{10} = \left\{ M_{ij}^1 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 13 \\ 2p_3^6 & 2p_4^6 & 17 \\ 2p_3^7 & 2p_4^7 & 21 \end{bmatrix}, \ M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_4^5 + p_5^5 \\ 0 & 0 & p_4^6 + p_5^6 \\ 0 & 0 & p_4^7 + p_5^7 \end{bmatrix}, \right.$$

$$\left. M_{ij}^3 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 3p_5^5 \\ 3p_3^6 & 3p_4^6 & 3p_5^6 \\ 3p_3^7 & 3p_4^7 & 3p_5^7 \end{bmatrix}, \ M_{ij}^4 = M_{ij}^5 = 0, \quad 1 \le i, \ j \le 3 \right\},$$

$$\lambda_{11} = \left\{ M_{ij}^1 = \begin{bmatrix} 3p_3^5 & 27 & 27 \\ 3p_3^6 & 35 & 35 \\ 3p_3^7 & 43 & 43 \end{bmatrix}, \ M_{ij}^2 = \begin{bmatrix} 0 & p_2^5 + p_3^5 + p_4^5 & p_2^5 + p_3^5 + p_4^5 \\ 0 & p_2^6 + p_3^6 + p_4^6 & p_2^6 + p_3^6 + p_4^6 \\ 0 & p_2^7 + p_3^7 + p_4^7 & p_2^7 + p_3^7 + p_4^7 \end{bmatrix}, \right.$$

$$\left. M_{ij}^3 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 2p_4^5 \\ 2p_3^6 & 2p_4^6 & 2p_4^6 \\ 2p_3^7 & 2p_4^7 & 2p_4^7 \end{bmatrix}, \ M_{ij}^4 = 0, \ 1 \le i, \ j \le 3, \ M_{ij}^5 = \begin{bmatrix} 0 & 0 & 65 \\ 0 & 0 & 85 \\ 0 & 0 & 105 \end{bmatrix} \right\},$$

$$\lambda_{12} = \left\{ M_{ij}^1 = \begin{bmatrix} 3p_3^5 & 27 & 27 \\ 3p_3^6 & 35 & 35 \\ 3p_3^7 & 43 & 43 \end{bmatrix}, M_{ij}^2 = \begin{bmatrix} 0 & p_2^5 + p_3^5 + p_4^5 & p_2^5 + p_3^5 + p_4^5 \\ 0 & p_2^6 + p_3^6 + p_4^6 & p_2^6 + p_3^6 + p_4^6 \\ 0 & p_2^7 + p_3^7 + p_4^7 & p_2^7 + p_3^7 + p_4^7 \end{bmatrix}, \right.$$

$$M_{ij}^3 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 2p_5^5 \\ 2p_3^6 & 2p_4^6 & 2p_5^6 \\ 2p_3^7 & 2p_4^7 & 2p_5^7 \end{bmatrix}, \; M_{ij}^4 = \begin{bmatrix} 0 & 0 & 39 \\ 0 & 0 & 51 \\ 0 & 0 & 63 \end{bmatrix}, \; M_{ij}^5 = 0, \; 1 \le i, j \le 3 \Bigg\},$$

$$\lambda_{13} = \Bigg\{ M_{ij}^1 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 53 \\ 3p_3^6 & 3p_4^6 & 69 \\ 3p_3^7 & 3p_4^7 & 85 \end{bmatrix}, \; M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_2^5 + p_3^5 + p_5^5 \\ 0 & 0 & p_2^6 + p_3^6 + p_5^6 \\ 0 & 0 & p_2^7 + p_3^7 + p_5^7 \end{bmatrix},$$

$$M_{ij}^3 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 2p_5^5 \\ 2p_3^6 & 2p_4^6 & 2p_5^6 \\ 2p_3^7 & 2p_4^7 & 2p_5^7 \end{bmatrix}, \; M_{ij}^4 = M_{ij}^5 = 0, \; 1 \le i, \; j \le 3 \Bigg\},$$

$$\lambda_{14} = \Bigg\{ M_{ij}^1 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 43 \\ 3p_3^6 & 3p_4^6 & 56 \\ 3p_3^7 & 3p_4^7 & 69 \end{bmatrix}, \; M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_2^5 + p_4^5 + p_5^5 \\ 0 & 0 & p_2^6 + p_4^6 + p_5^6 \\ 0 & 0 & p_2^7 + p_4^7 + p_5^7 \end{bmatrix},$$

$$M_{ij}^3 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 2p_5^5 \\ 2p_3^6 & 2p_4^6 & 2p_5^6 \\ 2p_3^7 & 2p_4^7 & 2p_5^7 \end{bmatrix}, \; M_{ij}^4 = M_{ij}^5 = 0, \quad 1 \le i, \; j \le 3 \Bigg\},$$

$$\lambda_{15} = \Bigg\{ M_{ij}^1 = \begin{bmatrix} 3p_3^5 & 3p_4^5 & 36 \\ 3p_3^6 & 3p_4^6 & 47 \\ 3p_3^7 & 3p_4^7 & 58 \end{bmatrix}, \; M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_3^5 + p_4^5 + p_5^5 \\ 0 & 0 & p_3^6 + p_4^6 + p_5^6 \\ 0 & 0 & p_3^7 + p_4^7 + p_5^7 \end{bmatrix},$$

$$M_{ij}^3 = \begin{bmatrix} 2p_3^5 & 2p_4^5 & 2p_5^5 \\ 2p_3^6 & 2p_4^6 & 2p_5^6 \\ 2p_3^7 & 2p_4^7 & 2p_5^7 \end{bmatrix}, \; M_{ij}^4 = M_{ij}^5 = 0, \quad 1 \le i, \; j \le 3 \Bigg\},$$

$$\lambda_{16} = \Bigg\{ M_{ij}^1 = \begin{bmatrix} 4p_3^5 & 4p_4^5 & 101 \\ 4p_3^6 & 4p_4^6 & 131 \\ 4p_3^7 & 4p_4^7 & 161 \end{bmatrix}, \; M_{ij}^2 = \begin{bmatrix} 0 & 0 & p_2^5 + p_3^5 + p_4^5 + p_5^5 \\ 0 & 0 & p_2^6 + p_3^6 + p_4^6 + p_5^6 \\ 0 & 0 & p_2^7 + p_3^7 + p_4^7 + p_5^7 \end{bmatrix},$$

$$M_{ij}^3 = \begin{bmatrix} p_3^5 & p_4^5 & p_5^5 \\ p_3^6 & p_4^6 & p_5^6 \\ p_3^7 & p_4^7 & p_5^7 \end{bmatrix}, \quad M_{ij}^4 = M_{ij}^5 = 0, \quad 1 \leq i, \ j \leq 3 \Bigg\}.$$

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