Simply Typed Lambda Calculus with Opposite Types

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Abstract

In this article we present an extension of simply typed lambda calculus by introducing the idea of *opposite types* developed by Agudelo-Agudelo and Sicard-Ramírez (2021). The rules for these new types are based on the rules of a fragment of the logic system presented by the same authors. Two of the main properties of type systems are proven: The strong normalization theorem and the Curry-Howard correspondence.

Keywords: type theory, opposite types, the Curry-Howard correspondence, strong normalization.

1 Introduction

At the beginning of the twentieth century, Bertrand Russell shook the foundations of mathematics by finding a paradox which could enter in the formal systems of Frege, Cantor and Peano (Kamareddine, Laan, & Nederpelt, 2004). The discover of Russell's paradox implied that these formal systems were inconsistent, and therefore, they lost the meaning of what was true or what was false. To avoid this drawbacks, first Russell (1903, 1908) and later with Whitehead (1910, 1912) proposed the first type theory, the Ramified Type Theory (RTT), which was strongly influenced by Frege's concept of types (Kamareddine et al., 2004).

RTT was characterized by having a double hierarchy: one of *orders* and the other of *types*. However, this characterization had limitations such as the requirement of the so-called reducibility axiom. Hence, the idea of ramified types and the reducibility axiom were not fully accepted and received several critics from: Zermelo (1908), Wiener (1914), Skolem (1922) and von Neumann (1925) among others. For this reason, Ramsey (1926) (and independently Ackerman and Hilbert, 1928) proposed a simplification of RTT, the Simple Theory

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of Types (STT), which consisted of the removal of the hierarchy of *orders* and the maintenance of the hierarchy of *types*. Subsequently, Church (1940) developed one of the most influential type theories using his λ-calculus and STT: the *simply typed* λ*-calculus* ($\lambda_{\rightarrow}$). This theory was the starting point for developing different type systems such as the Pure Type Systems (PTS) (Kamareddine et al., 2004).

In 1934, Curry observed that a logical implication might be seen as a class of functions, whereby, every proposition $A \supset B$ could be interpreted as a function which took a proof of A as argument and returned a proof of B. In this way, for every proof of a proposition $A \supset B$ corresponds to a function of type $A \rightarrow B$. This principle is known as *Propositions As Types* (PAT). However, the correspondence is not merely between types and propositions, Curry and Feys (1958) extended it to terms and proofs and left the indications to extend the relation to a deeper level, i.e, between evaluation of terms and simplification of proofs. This deepest level of the correspondence was showed explicitly by Howard (1980) between the *simply typed* λ*-calculus* and intuitionistic natural deduction (Wadler, 2015). Furthermore, he extended the correspondence to quantifiers and dependent types. Therefore, the PAT principle is also known as the Curry-Howard correspondence.

Thanks to the Curry-Howard correspondence, the Intuitionistic First Order Logic (IL) can be formalized using type theory. Nevertheless, the mathematical constructivism perspective has various objections to IL, in particular against the definition of negation as $\neg A \overset{def}{=} A \supset \perp$, where $\perp$ represents the *bottom particle* (or an unprovable statement), consequently $\neg A$ can be interpreted as the impossibility of deriving a construction for A. From the criticism of IL, the following stand out: the Griss's criticism (Heyting, 1971, p. 124):

> Though agreeing completely with Brouwer's basic ideas on the nature of mathematics, he contends that every mathematical notion has its origin in a mathematical construction, which can actually be carried out; if the construction is impossible, then the notion cannot be clear.

which led him to propose a negationless intuitionistic mathematics (Griss, 1946); and the Johansson's criticism (Johansson, 1937) which points out that the *ex falso* rule (a rule establishing that every formula could be derived from $\perp$) can not be generally accepted from the constructivist viewpoint, therefore, it should be eliminated in IL (this system is known as *minimal logic*).

These and others objections against IL has motivated the development of formal systems with more robust constructivist denials. For instance, Nelson (1949) introduce a formal system, N_1, for number theory in which falsity has symmetric constructivist characteristics as the truth in IL. On the other hand, López-Escobar (1972) come up with a first-order logic through a sequent calculus called *refutability calculus* (RFC), where the rules for the negation of a compound formula A are defined based on the analysis of whether a construct c serves as a refutation of A. With this system López-Escobar (1972) formalizes number theory, using RFC as the underlying logic, obtaining a system known as PN.

Based on the above considerations and in the *intuitionistic type theory* (*constructive type theory* or *Martin Löf type theory*), Agudelo-Agudelo and Sicard-Ramírez (2021) obtained a type theory in which the negation, with constructive properties (corresponding to those developed by Nelson and López-Escobar), is formalized through a type constructor inside the theory which the authors called *opposite types*. The inclusion of this constructor allowed them to derive a *paraconsistent*[1] *type theory* (PTT). On the other side, Kamide (2010) introduces a type system for Nelson's paraconsistent logic N4 and gives an sketch of the proof for strong normalization, mentioning that the proof is almost the same as the one presented in Joachimski and Matthes (2003).

Following the ideas of Agudelo-Agudelo and Sicard-Ramírez (2021) and Kamide (2010), the objectives of this article are: to extend $\lambda_{\to}$ allowing *opposite types* (this extension will be denoted by $\bar{\lambda}_{\to}$), and to study two important properties of it, the strong normalization theorem and the Curry-Howard correspondence. For the former, since Kamide (2010) just provided a sketch of the proof and $\bar{\lambda}_{\to}$ is a fragment his type system, we are going to provide all the details of the demonstration because we consider that the proof is not as trivial as Kamide (2010) pointed out.

To achieve our goals, this article is structured as follows. In Section 2 we present a fragment of the logic system introduced by Agudelo-Agudelo and Sicard-Ramírez (2021). The rules for negation are going to be the basis for the definition of *opposite types*. So, in Section 3, we introduce $\bar{\lambda}_{\to}$. The Section 4 is dedicated to the strong normalization theorem and Section 5 to the Curry-Howard correspondence. At the end of this article (Section 6), some conclusions are presented. We hope these results will be helpful for future works.

2 A paraconsistent logic

Prawitz (1965) formalized Nelson's constructible falsity, in Nelson's constructive logic N, by introducing a natural deduction system. On one hand, Agudelo-Agudelo and Sicard-Ramírez (2021) considered a fragment of that system without $\perp$ and without the rule that connects intuitionistic negation and constructible falsity, which they called PL (*paraconsistent logic*). On the other hand, Kamide (2010) worked with the natural deduction system adapted for N4, a sublogic of N. Based on this ideas and due to the fact that we are study-

[1]The term *paraconsistent* refers to a type theory that allows the formalization of logically inconsistent and non-trivial theories. Moreover, this corresponds to the notion of paraconsistency in logic.

ing an extension of $\lambda_{\rightarrow}$ with opposite types, we shall consider a fragment of the Prawitz's natural deduction system, specifically, a fragment of the system considered by Agudelo-Agudelo and Sicard-Ramírez (2021), just with the logical constants for negation ($\neg$) and implication ($\supset$). This system shall be denoted by $\mathrm{PL}_{\neg,\supset}$.

Definition 2.1 (Set of formulas)**.** *Let* $\mathbf{PV} = \{a, b, c, \ldots\}$ *be an infinite set of propositional variables. The* set of formulas Φ *is defined by:*

- *If* $a \in \mathbf{PV}$*, then* $a \in \Phi$*.*
- *If* $\sigma, \tau \in \Phi$*, then* $\sigma \supset \tau \in \Phi$
- *If* $\sigma \in \Phi$ *then* $\neg\sigma \in \Phi$*.*

which can be summarized by the following grammar:

$$\sigma, \tau ::= a \mid \sigma \supset \tau \mid \neg\sigma$$

where $\sigma, \tau \in \Phi$ *and* $a \in \mathbf{PV}$*.*

Definition 2.2 (Judgement, formal proof)**.**

- *A* judgement *is a pair, written* $\Gamma \vdash_{\mathrm{PL}_{\neg,\supset}} \varphi$ *(and read "*Γ *proves* ϕ*"), consisting in a finite set of formulas* Γ *and a formula* φ*.*
- *A* formal proof *or* derivation *of* $\Gamma \vdash_{\mathrm{PL}_{\neg,\supset}} \varphi$ *is tree of judgements with root* $\Gamma \vdash_{\mathrm{PL}_{\neg,\supset}} \varphi$ *and where each judgment is obtained from the application of one of the derivation rules presented in Table 1.*

Remark: We are going to simplify the notation for judgments, writing:

- $\varphi_1, \varphi_2 \vdash_{\mathrm{PL}_{\neg,\supset}} \psi$ instead of $\{\varphi_1, \varphi_2\} \vdash_{\mathrm{PL}_{\neg,\supset}} \psi$.
- Γ, Δ instead of $\Gamma \cup \Delta$.
- Γ, φ instead of $\Gamma \cup \{\varphi\}$.
- $\vdash$ instead of $\vdash_{\mathrm{PL}_{\neg,\supset}}$.

Definition 2.3 (Derivation rules)**.**

$\Delta \vdash \sigma$ If $\sigma \in \Delta$ (Ax)	$\dfrac{\Delta, \sigma \vdash \tau}{\Delta \vdash \sigma \supset \tau}$ $(\supset I)$
$\dfrac{\Delta \vdash \sigma \supset \tau \quad \Delta \vdash \sigma}{\Delta \vdash \tau}$ $(\supset E)$	$\dfrac{\Delta \vdash \sigma \quad \Delta \vdash \neg\tau}{\Delta \vdash \neg(\sigma \supset \tau)}$ $(\neg \supset I)$
$\dfrac{\Delta \vdash \neg(\sigma \supset \tau)}{\Delta \vdash \sigma}$ $(\neg \supset E_1)$	$\dfrac{\Delta \vdash \neg(\sigma \supset \tau)}{\Delta \vdash \neg\tau}$ $(\neg \supset E_2)$
$\dfrac{\Delta \vdash \sigma}{\Delta \vdash \neg\neg\sigma}$ $(\neg\neg I)$	$\dfrac{\Delta \vdash \neg\neg\sigma}{\Delta \vdash \sigma}$ $(\neg\neg E)$

Table 1: Derivation rules for $\mathrm{PL}_{\neg,\supset}$

3 Simply typed lambda calculus with opposite types

Based on $\mathrm{PL}_{\neg,\supset}$, an extension of $\lambda_{\rightarrow}$ is introduced via the Curry-Howard correspondence, adding the so-called *opposite types*. This system will be denoted by $\bar{\lambda}_{\rightarrow}$. The system presented is constructed using five new term constructors: pairs $(\ , \ )$, projections π_1 and π_2, and identities Id and Id^{-1}, in order to interpret the rules in $\mathrm{PL}_{\neg,\supset}$, following the ideas of Kamide (2010). It is also necessary to point out that we adapt the definitions presented in Geuvers and Nederpelt (2014) for $\bar{\lambda}_{\rightarrow}$.

3.1 Terms and types

Definition 3.1 (The set $\mathbb{T}$ of types)**.** *Let $\mathbb{B} = \{a, b, c, \ldots\}$ be the set of basic types. The* set of types $\mathbb{T}$ *is defined by:*

- *If $a \in \mathbb{B}$, then $a \in \mathbb{T}$.*
- *If $\sigma, \tau \in \mathbb{T}$, then $\sigma \rightarrow \tau \in \mathbb{T}$.*
- *If $\sigma \in \mathbb{T}$, then $\overline{\sigma} \in \mathbb{T}$.*

which can be summarized by the following grammar:

$$\sigma, \tau ::= a \mid \sigma \rightarrow \tau \mid \overline{\sigma}$$

where $\sigma, \tau \in \mathbb{T}$ and $a \in \mathbb{B}$.

Definition 3.2 (The set $\mathbf{\Lambda}_{\mathbb{T}}$ of pre-typed λ-terms)**.** *Let* $\mathbf{V} = \{x, y, z, \dots\}$ *be the set of infinite term variables. The* set of pre-typed λ-terms $\mathbf{\Lambda}_{\mathbb{T}}$ *is defined by:*

- *If* $x \in \mathbf{V}$*, then* $x \in \mathbf{\Lambda}_{\mathbb{T}}$*.*
- *If* $M, N \in \mathbf{\Lambda}_{\mathbb{T}}$*, then* $MN \in \mathbf{\Lambda}_{\mathbb{T}}$*.*
- *If* $x \in \mathbf{V}$ *has type* $\sigma \in \mathbb{T}$ *and* $M \in \mathbf{\Lambda}_{\mathbb{T}}$*, then* $\lambda x : \sigma.M \in \mathbf{\Lambda}_{\mathbb{T}}$*.*
- *If* $M \in \mathbf{\Lambda}_{\mathbb{T}}$ *and* $N \in \mathbf{\Lambda}_{\mathbb{T}}$*, then* $\langle M, N\rangle \in \mathbf{\Lambda}_{\mathbb{T}}$*.*
- *If* $M \in \mathbf{\Lambda}_{\mathbb{T}}$*, then* $M\ \pi_1$ *and* $M\ \pi_2 \in \mathbf{\Lambda}_{\mathbb{T}}$*.*
- *If* $M \in \mathbf{\Lambda}_{\mathbb{T}}$*, then* $\mathrm{Id}\ M \in \mathbf{\Lambda}_{\mathbb{T}}$ *and* $M\ \mathrm{Id}^{-1} \in \mathbf{\Lambda}_{\mathbb{T}}$

which can be summarized by the following grammar:

$$M ::= x \mid MN \mid \lambda x : \sigma.M \mid \langle M, N\rangle \mid M\ \pi_1 \mid M\ \pi_2 \mid \mathrm{Id}\ M \mid M\ \mathrm{Id}^{-1}$$

where $M, N \in \mathbf{\Lambda}_{\mathbb{T}}$*.*

Definition 3.3 (Statement, declaration, context and judgment)**.**

- *A* statement *is of the form* $M : \sigma$*, where* $M \in \mathbf{\Lambda}_{\mathbb{T}}$ *and* $\sigma \in \mathbf{T}$*. In such statement,* M *is called the subject and* σ *the type.*
- *A* declaration *is a statement with a variable as a subject.*
- *A* context *is a list of declarations with different subjects.*
- *A* judgement *has the form* $\Gamma \vdash_{\bar{\lambda}_{\to}} M : \sigma$*, with* Γ *a context and* $M : \sigma$ *a statement.*

3.2 Derivation rules

Definition 3.4 (Derivation rules)**.**

$\Gamma \vdash_{\bar{\lambda}_\to} x : \sigma$ if $x : \sigma \in \Gamma$ (Ax^*)	$\dfrac{\Gamma, x : \sigma \vdash_{\bar{\lambda}_\to} M : \tau}{\Gamma \vdash_{\bar{\lambda}_\to} \lambda x : \sigma . M : \sigma \to \tau}\ (\to I)$
$\dfrac{\Gamma \vdash_{\bar{\lambda}_\to} M : \sigma \to \tau \quad \Gamma \vdash_{\bar{\lambda}_\to} N : \sigma}{\Gamma \vdash_{\bar{\lambda}_\to} MN : \tau}\ (\to E)$	$\dfrac{\Gamma \vdash_{\bar{\lambda}_\to} M : \sigma \quad \Gamma \vdash_{\bar{\lambda}_\to} N : \bar{\tau}}{\Gamma \vdash_{\bar{\lambda}_\to} \langle M, N \rangle : \overline{\sigma \to \tau}}\ (\overline{\to} I)$
$\dfrac{\Gamma \vdash_{\bar{\lambda}_\to} M : \overline{\sigma \to \tau}}{\Gamma \vdash_{\bar{\lambda}_\to} M\ \pi_1 : \sigma}\ (\overline{\to} E_1)$	$\dfrac{\Gamma \vdash_{\bar{\lambda}_\to} M : \overline{\sigma \to \tau}}{\Gamma \vdash_{\bar{\lambda}_\to} M\ \pi_2 : \bar{\tau}}\ (\overline{\to} E_2)$
$\dfrac{\Gamma \vdash_{\bar{\lambda}_\to} M : \sigma}{\Gamma \vdash_{\bar{\lambda}_\to} \mathrm{Id}\ M : \bar{\bar{\sigma}}}\ ({}^{=}I)$	$\dfrac{\Gamma \vdash_{\bar{\lambda}_\to} M : \bar{\bar{\sigma}}}{\Gamma \vdash_{\bar{\lambda}_\to} M\ \mathrm{Id}^{-1} : \sigma}\ ({}^{=}E)$

Table 2: Derivation rules for $\bar{\lambda}_\to$

3.3 α-conversion and substitution

Definition 3.5 (α-conversion or α-equivalence, $=_\alpha$)**.**

- ***(Renaming)****: If $y \notin FV(M)$[2] and y is not a binding variable in M, then $\lambda x : \sigma.M =_\alpha \lambda y : \sigma.M^{x \to y}$, where $M^{x \to y}$ denotes the result of replacing every free occurrence of x in M by y.*
- ***(Compatibility)****: If $M =_\alpha N$, then $ML =_\alpha NL, LM =_\alpha LN$, $\langle M, L\rangle =_\alpha \langle N, L\rangle$, $\langle L, M\rangle =_\alpha \langle L, N\rangle$, $M\ \pi_1 =_\alpha N\ \pi_1$, $M\ \pi_2 =_\alpha N\ \pi_2$, $\mathrm{Id}\ M =_\alpha \mathrm{Id}\ N$, $M\ \mathrm{Id}^{-1} =_\alpha N\ \mathrm{Id}^{-1}$, and for any arbitrary $z : \sigma$, $\lambda z : \sigma.M =_\alpha \lambda z : \sigma.N$.*
- ***(Reflexivity)****: $M =_\alpha M$.*
- ***(Symmetry)****: If $M =_\alpha N$, then $N =_\alpha M$.*
- ***(Transitivity)****: If $L =_\alpha M$ and $M =_\alpha N$, then $L =_\alpha N$.*

Definition 3.6 (Substitution)**.**

- $x[x := N] \equiv N$
- $y[x := N] \equiv y$
- $(PQ)[x := N] \equiv (P[x := N])(Q[x := N])$
- $\langle P, Q\rangle[x := N] \equiv \langle P[x := N], Q[x := N]\rangle$

[2] $FV(N)$ denote the set of free-variables in N.

- $(P\ \pi_1)[x := N] \equiv P[x := N]\ \pi_1$
- $(P\ \pi_2)[x := N] \equiv P[x := N]\ \pi_2$
- $(\mathrm{Id}\ P)[x := N] \equiv \mathrm{Id}\ P[x := N]$
- $(P[x := N]\ \mathrm{Id}^{-1})[x := N] \equiv P[x := N]\ \mathrm{Id}^{-1}$
- $(\lambda y : \sigma.P)[x := N] \equiv \lambda z : \sigma.(P^{z\to y}[x := N])$, *if* $\lambda z : \sigma.(P^{z\to y}$ *is an* α*-variant of* $\lambda y : \sigma.P$ *such that* $z \notin FV(N)$

3.4 β-reduction

Definition 3.7 (β-reduction $\to_\beta$). *Let* $M, N, L \in \mathbf{\Lambda_T}$. *The* β-reduction *is defined:*

- $(\lambda x : \sigma.M)N \to_\beta M[x := N]$.
- $\langle M, N\rangle\ \pi_1 \to_\beta M$.
- $\langle M, N\rangle\ \pi_2 \to_\beta N$.
- $(\mathrm{Id}\ M)\ \mathrm{Id}^{-1} \to_\beta M$.
- ***(Substitutivity)****: If* $M \to_\beta N$*, then* $M[x := L] \to_\beta N[x := L]$.
- ***(Compatibility)****: If* $M \to_\beta N$*, then* $LM \to_\beta LN$, $ML \to_\beta NL$, $\lambda x : \sigma.M \to_\beta \lambda x : \sigma.N$, $L[x := M] \to_\beta L[x := N]$, $M\ \pi_1 \to_\beta N\ \pi_1$, $M\ \pi_2 \to_\beta N\ \pi_2$, $\langle M, L\rangle \to_\beta \langle N, L\rangle$, $\langle L, M\rangle \to_\beta \langle L, N\rangle$, $\mathrm{Id}\ M \to_\beta \mathrm{Id}\ N$ *and* $M\ \mathrm{Id}^{-1} \to_\beta N\ \mathrm{Id}^{-1}$

4 Strong normalization theorem

As we mentioned in the introduction, in spite of $\bar{\lambda}_\to$ being a fragment of the type system for Nelson's paraconsistent logic N4 introduced by Kamide (2010), where just a sketch of the proof for strong normalization theorem is given due to the fact that the proof is almost the same as the one presented in Joachimski and Matthes (2003) for $\lambda_\to$ and other systems, we are going to provide all the details of the demonstration of the strong normalization theorem for $\bar{\lambda}_\to$, because we consider that the proof for this system is not as trivial as Kamide (2010) pointed out.

The idea of the proof relies on the definition of a set SW which, as it will be proved, is a subset of the set of all strongly normalizing terms and captures all the legal terms. However, we need some preliminaries before defining the set SW. The first step is to redefine $\mathbf{\Lambda}_\mathbb{T}$ in order to exhibit the leftmost outermost reducible expression. Then, we are going to provide an alternative definition for strongly normalizing terms which is equivalent to the usual one. After that, we will prove that the set of all strongly normalizing terms corresponds to the *well-founded part* (Subsection 4.3) of the β-reduction. Finally, we will define the set SW and proceed with the proof.

4.1 Alternative inductive characterization of $\mathbf{\Lambda}_{\mathbb{T}}$

Before proceeding with the redefinition of $\mathbf{\Lambda}_{\mathbb{T}}$, we need to define what we are going to call *eliminations.*

Definition 4.1. *We use $\mathcal{E}_i$ $(i = 0, 1, ...)$ for* eliminations*, which are either terms (M), projection-eliminations (π_1, π_2), or inverse-eliminations (*Id^{-1}*). This definition can be summarized by the following grammar:*

$$\mathcal{E}_i ::= M \mid \pi_1 \mid \pi_2 \mid \mathrm{Id}^{-1}$$

the term M is named elimination term.

Remark: The vector notation are going to apply for lists of terms and multiple eliminations (following Joachimski and Matthes, 2003).

Definition 4.2 (Alternative inductive characterization of $\mathbf{\Lambda}_{\mathbb{T}}$)**.** *Now, we can alternatively characterize the set $\mathbf{\Lambda}_{\mathbb{T}}$ inductively by the following grammar:*

$$\begin{aligned} M, N \quad ::= \quad & x \mid x\vec{M}\vec{\mathcal{E}_0} \mid \lambda x : \sigma.M \mid \langle M, N\rangle \mid \mathrm{Id}\ M \mid M\ \pi_1 \mid (\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \mid \\ & \langle M, N\rangle\ \pi_1\vec{\mathcal{E}_0} \mid \langle M, N\rangle\ \pi_2\vec{\mathcal{E}_0} \mid (\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\vec{\mathcal{E}_0} \end{aligned}$$

where $M, N \in \mathbf{\Lambda}_{\mathbb{T}}$.

This characterization differs from the first one in the definition of the application, i.e, instead of considering the application in the usual form, MN, we consider it distinguishing all the possible cases for M. Moreover, it exhibits the leftmost outermost reducible expression (*redex*). On the other side, it is necessary to point out that in the literature there are other notations for π_1, π_2 and Id^{-1}, case in point, Sørensen and Urzyczyn (2006). However, due to the fact that we are following the method of Joachimski and Matthes (2003) for proving strong normalization, this notation eases the proof of the theorem.

4.2 Normalization

The definition of the strongly normalizing terms requires defining the concept of a term in normal form. Instead of the usual definition of a term in normal form which is the one that allows no further reduction, we are going to consider the inductive one in Joachimski and Matthes (2003), which is based on the alternative inductive characterization of $\mathbf{\Lambda}_{\mathbb{T}}$ (Definition 4.2). After doing that, and following the ideas of the same authors, we will give an alternative definition of the strongly normalizing terms and we will show that this definition is equivalent to the usual one.

Definition 4.3 (The set NF of normal forms)**.** *The* set NF of typed terms in normal form *is inductively defined by the following grammar:*

$$M, N ::= x \mid x\vec{M}\vec{\mathcal{E}_0} \mid \lambda x : \sigma.M \mid \langle M, N\rangle \mid \text{Id } M$$

where $\vec{M}\vec{\mathcal{E}_0}, N \in \text{NF}$.

Remark: Notice how the alternative inductive characterization of $\mathbf{\Lambda}_{\mathbb{T}}$ uniquely determines the canonical *redex* of non-normal terms.

Definition 4.4 (Strongly normalizing terms)**.** $M \in \mathbf{\Lambda}_{\mathbb{T}}$ *is* strongly normalizing ($M \Downarrow$) *if* $\forall M^{'}(M \to_{\beta} M^{'} \implies M^{'} \Downarrow)$.

Remark: We will denote a *non-strongly normalizing term* by $M \not\Downarrow$.

Proposition 4.1. *The latest definition is equivalent to the usual definition of strong normalization:* There is no infinite reduction sequence starting with M.

Proof. $\Rightarrow$ Let M be strongly normalizing according to Definition 4.4, i.e, $M \Downarrow$. We need to prove that there is no infinite reduction sequence starting with M. For doing that, we are going to proceed by induction on the possible reduction of M:

1. *Basis*: In this case, we have to show that all the terms in normal form satisfy the property, i.e, there is no infinite reduction sequence starting with them. But, this is trivially true since they do not admit any reductions, so they do not have any reduction sequence starting with themselves.
2. *Step*: We must prove that there is no infinite reduction sequence starting with M assuming that for all $M^{'}$ such that $M \to_{\beta} M^{'}$, $M^{'}$ does not admit any infinite reduction sequence starting with itself. Nevertheless, it follows immediately since the length of all the reduction sequences starting with M is equal to the length of a reduction sequence starting with $M^{'}$ (which are all finite by induction hypothesis) plus one, for any $M^{'}$ such that $M \to_{\beta} M^{'}$.

$\Leftarrow$ We are going to prove this statement by *counter-reciprocal.* Assume that M is not strongly normalizing according to the Definition 4.4, i.e, $M \not\Downarrow$. We need to prove that there is at least one infinite reduction sequence starting with M. Due to $M \not\Downarrow$, there is $M^{'}$ such that $M \to_{\beta} M^{'}$ and $M^{'} \not\Downarrow$. Again, since $M^{'} \not\Downarrow$, there is $M^{''}$ such that $M^{'} \to M^{''}$ and $M^{''} \not\Downarrow$. This leads to the following reduction sequence:

$$M \to_{\beta} M^{'} \to_{\beta} M^{''}$$

Continuing with the same process, we can construct an infinite reduction sequence starting with M.

■

4.3 The well-founded part of the β-reduction

In this subsection, we are going to introduce the concept of the *well-founded part* of a binary relation $\succ$ on a fixed set H, in order to prove that the set of all strongly normalizing terms corresponds to the *well-founded part* of the β-reduction.

Definition 4.5 ($\succ$-Progressive)**.** *$X \subseteq H$ is* $\succ$-progressive *($Prog_{\succ}(X)$) if and only if $\forall x \in H(\forall y(x \succ y \Longrightarrow y \in X) \Longrightarrow x \in X)$.*

Definition 4.6 (Well Founded part (WF))**.**

$$WF_{\succ} := \cap\{X \subseteq H | Prog_{\succ}(X)\}$$

Definition 4.7 (Properties of $WF_{\succ}$)**.**

1. *The definition of the well-founded part $WF_{\succ}$ yields the schema $Prog_{\succ}(X) \Longrightarrow WF_{\succ} \subseteq X$ of* accessible part induction.
2. *$WF_{\succ}$ is $\succ$-progressive itself.*

Remark: In the case of the β-reduction:

- $X \subseteq \mathbf{\Lambda}_{\mathbb{T}}$ is $\rightarrow_{\beta}$*-progressive* if and only if $\forall M \in \mathbf{\Lambda}_{\mathbb{T}}(\forall M^{'} \in \mathbf{\Lambda}_{\mathbb{T}}(M \rightarrow_{\beta} M^{'} \Longrightarrow M^{'} \in X) \Longrightarrow M \in X)$.
- $WF_{\rightarrow_{\beta}} := \cap\{X \subseteq A | Prog_{\rightarrow_{\beta}}(X)\}$.
- $Prog_{\rightarrow_{\beta}}(X) \Longrightarrow WF_{\rightarrow_{\beta}} \subseteq X$.
- $WF_{\rightarrow_{\beta}}$ is $\rightarrow_{\beta}$*-progressive* itself.

Proposition 4.2.

$$WF_{\rightarrow_{\beta}} = \{M \in \mathbf{\Lambda}_{\mathbb{T}} | M \Downarrow\}$$

Proof.

- ($\subseteq$): We just have to prove that $Prog_{\rightarrow_{\beta}}(\{M \in \mathbf{\Lambda}_{\mathbb{T}} | M \Downarrow\})$. However $\{M \in \mathbf{\Lambda}_{\mathbb{T}} | M \Downarrow\}$ is progressive by Definition 4.4. Therefore, due to the first property of $WF_{\rightarrow_{\beta}}$ (Definition 4.7) we have the first inclusion.
- ($\supseteq$): Let $M \in \{M \in \mathbf{\Lambda}_{\mathbb{T}} | M \Downarrow\}$. We need to prove that $M \in WF_{\rightarrow_{\beta}}$, i.e, $M \in \cap\{X \subseteq \mathbf{\Lambda}_{\mathbb{T}} | Prog_{\succ}(X)\}$. Let $X \subseteq \mathbf{\Lambda}_{\mathbb{T}}$ such that $Prog_{\rightarrow_{\beta}}(X)$. By *reductio ad absurdum* assume $M \notin X$. Then, there is $M^{'}$ such that $M \rightarrow_{\beta} M^{'}$ and $M^{'} \notin X$. Since $M^{'} \notin X$, there is $M^{''}$ such that $M^{'} \rightarrow_{\beta} M^{''}$ and $M^{''} \notin X$. This reasoning leads to the following reduction sequence:

  $$M \rightarrow_{\beta} M^{'} \rightarrow_{\beta} M^{''}$$

  Continuing in this way we can construct an infinitive reduction sequence starting with M. However, the existence of the infinite reduction sequence contradicts the fact that $M \Downarrow$. Thus, $M \in X$ and therefore $M \in WF_{\rightarrow_{\beta}}$ since X was chosen arbitrarily.

$\blacksquare$

4.4 The set SW

Definition 4.8 (The set SW)**.** *The set* SW *is inductively defined by the following rules:*

$x \in \mathrm{SW}\ (var_0)$	$\dfrac{\overrightarrow{M}\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}{x\overrightarrow{M}\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}\ (var)$
$\dfrac{M \in \mathrm{SW}}{\lambda x : \sigma.M \in \mathrm{SW}}\ (abs)$	$\dfrac{M\overrightarrow{\mathcal{E}_0} \in \mathrm{SW} \quad N\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}{\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}\ (pair\_red_1)$
$\dfrac{M[x := N]\overrightarrow{\mathcal{E}_0} \in \mathrm{SW} \quad N \in \mathrm{SW}}{(\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0}}\ (appl)$	$\dfrac{M\overrightarrow{\mathcal{E}_0} \in \mathrm{SW} \quad N\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}{\langle M, N\rangle\ \pi_2\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}\ (pair\_red_2)$
$\dfrac{M, N \in \mathrm{SW}}{\langle M, N\rangle \in \mathrm{SW}}\ (pair)$	$\dfrac{M \in \mathrm{SW}}{\mathrm{Id}\ M \in \mathrm{SW}}\ (neg)$
$\dfrac{M\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}{(\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}}\ (neg\_red)$	

Table 3: Derivation rules for SW

Remark: *Notice that each of the rules defining* SW *corresponds to one of the grammar rules defined in the alternative inductive definition of* $\mathbf{\Lambda}_{\mathbb{T}}$ *(Definition 4.2). Additionally, it is clear that* NF $\subseteq$ SW*.*

Lemma 4.3. *If* $L \equiv (\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}$ *is derived from the appl rule and the premises of it are strongly normalizing, then* $(\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0} \Downarrow$*.*

Proof. Assume that $L \equiv (\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0} \in \mathrm{SW}$ is the derived from the *appl* rule and the premises of it are strongly normalizing, i.e, $M[x := N]\overrightarrow{\mathcal{E}_0}, N \Downarrow$. Let us show that $(\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0} \Downarrow$. For doing that, we are going to proceed by induction on $N \Downarrow$. As Joachimski and Matthes (2003, footnote 18, p. 68) mention, that induction on $N \Downarrow$ amounts to showing $\to_\beta$-progressivity of the set:

$$A := \{N \mid \forall M, \overrightarrow{\mathcal{E}_0}(M[x := N]\overrightarrow{\mathcal{E}_0} \Downarrow) \Longrightarrow (\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0} \Downarrow\}$$

i.e, $\forall N \in \mathbf{\Lambda}_{\mathbb{T}}(\forall N^{'}(N \to_\beta N^{'} \Longrightarrow N^{'} \in A) \Longrightarrow N \in A)$. So, assume that any one-step-reduct of N is in A. To prove that $N \in A$, we do induction on

$M[x := N]\vec{\mathcal{E}_0} \Downarrow$, i.e, we are going to show $\rightarrow_\beta$-progressivity of the set:

$$B := \{M[x := N]\vec{\mathcal{E}_0} \mid M[x := N]\vec{\mathcal{E}_0} \Downarrow \Longrightarrow (\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \Downarrow\}$$

for all $M, \vec{\mathcal{E}_0}$. Assume $T \equiv M[x := N]\vec{\mathcal{E}_0}$ and that any one-step-reduct of T is in B. We must show $T \in B$. Clearly, $T \Downarrow$ because every one-step-reduct of T is strongly normalizing since they belong to B. Then, we just need to show that $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \Downarrow$ proving that each of its one-step-reducts is strongly normalizing (Definition 4.4):

- $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \rightarrow_\beta (\lambda x : \sigma.M^{'})N\vec{\mathcal{E}_0}$: In this case, note that $T \equiv M[x := N]\vec{\mathcal{E}_0} \rightarrow_\beta M^{'}[x := N]\vec{\mathcal{E}_0}$ by *substitutivity*, hence, by the second induction hypothesis ($\rightarrow_\beta$-progressivity of B) the reduct $(\lambda x : \sigma.M^{'})N\vec{\mathcal{E}_0}$ is strongly normalizing.

- $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \rightarrow_\beta (\lambda x : \sigma.M)N^{'}\vec{\mathcal{E}_0}$: In this case, $T \equiv M[x := N]\vec{\mathcal{E}_0} \rightarrow_\beta M[x := N^{'}]\vec{\mathcal{E}_0}$ by *compatibility* of the β-reduction, hence, by the second induction hypothesis $M[x := N^{'}]\vec{\mathcal{E}_0} \Downarrow$. Thus, by the first induction hypothesis ($\rightarrow_\beta$-progressivity of A) yields $(\lambda x : \sigma.M)N^{'}\vec{\mathcal{E}_0} \Downarrow$.

- $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \rightarrow_\beta (\lambda x : \sigma.M)N\vec{\mathcal{E}_0^{'}}$: In this case, by *compatibility* of β-reduction, $T \equiv M[x := N]\vec{\mathcal{E}_0} \rightarrow_\beta M[x := N]\vec{\mathcal{E}_0^{'}}$, hence, by the second induction hypothesis ($\rightarrow_\beta$-progressivity of B) $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0^{'}} \Downarrow$.

- $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \rightarrow_\beta M[x := N]\vec{\mathcal{E}_0}$: By assumption $M[x := N]\vec{\mathcal{E}_0}$ is strongly normalizing.

Since all the reducts are strongly normalizing, $T \in B$. Therefore, $N \in A$, and finally, we can conclude that $L \equiv (\lambda x : \sigma.M)N\vec{\mathcal{E}_0} \Downarrow$. ■

Lemma 4.4.

$$\mathrm{SW} \subseteq WF_{\rightarrow_\beta}$$

Proof. Let $L \in \mathrm{SW}$. Let us proceed by induction on L.

1. *Basis*: The basis of the induction is the var_0 rule and it is trivial since any variable is a term in normal form and consequently is strongly normalizing.

2. *Step*: We need to prove $L \Downarrow$ assuming that all the premises showing $L \in \mathrm{SW}$ are strongly normalizing. We have the following cases:

   (var) ($L \equiv x\vec{M}\vec{\mathcal{E}_0}$): Since each reduction on $x\vec{M}\vec{\mathcal{E}_0}$ must take place in $\vec{M}\vec{\mathcal{E}_0}$ and $\vec{M}\vec{\mathcal{E}_0} \Downarrow$ because of the induction hypothesis ($\vec{M}\vec{\mathcal{E}_0}$ is the premise of the var rule), then $x\vec{M}\vec{\mathcal{E}_0} \Downarrow$.

(abs) ($L \equiv \lambda x : \sigma.M$): Each reduction on $\lambda x : \sigma.M$ occurs in M and $M \Downarrow$ due to the induction hypothesis (M is the premise of the *abs* rule), we have $\lambda x : \sigma.M \Downarrow$.

($appl$) ($L \equiv (\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0}$): In this case, $(\lambda x : \sigma.M)N\overrightarrow{\mathcal{E}_0} \Downarrow$ by the Lemma 4.3.

($pair$) ($L \equiv \langle M, N\rangle$): Since each reduction has to take place in M or in N and both are strongly normalizing by the induction hypothesis, $\langle M, N\rangle \Downarrow$.

($pair\_red_1$) ($L \equiv \langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0}$): In this case, by the induction hypothesis $M\overrightarrow{\mathcal{E}_0} \Downarrow$ and $N\overrightarrow{\mathcal{E}_0} \Downarrow$ since they are the premises of the *pair_red*$_1$ rule. Then, following the same idea as in the *appl* case (Lemma 4.3), we proceed by induction on $M\overrightarrow{\mathcal{E}_0} \Downarrow$, i.e, we are going to prove the $\rightarrow_\beta$-progressivity of the set:

$$C = \{M\overrightarrow{\mathcal{E}_0} \mid M\overrightarrow{\mathcal{E}_0} \Downarrow \Longrightarrow \langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \Downarrow\}$$

for all $M, \overrightarrow{\mathcal{E}_0}$. So, assume $T \equiv M\overrightarrow{\mathcal{E}_0}$ and that every-one-step-reduct of it is in C. We must show that $T \in C$. Clearly, $T \Downarrow$ due to every one-step-reduct of it is strongly normalizing for belonging to C. Therefore, we just need to prove that $\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \Downarrow$ showing that each of its one-step-reducts is strongly normalizing (Definition 4.4):

- $\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \rightarrow_\beta \langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}'_0}$: In this case, $T \equiv M\overrightarrow{\mathcal{E}_0} \rightarrow_\beta M\overrightarrow{\mathcal{E}'_0}$ by *compatibility* of the β-reduction, hence, by induction hypothesis ($\rightarrow_\beta$-progressivity of C) $\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}'_0} \Downarrow$.
- $\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \rightarrow_\beta \langle M', N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0}$: In this case, $T \equiv M\overrightarrow{\mathcal{E}_0} \rightarrow_\beta M'\overrightarrow{\mathcal{E}_0}$ by *compatibility* of the β-reduction, hence, by induction hypothesis ($\rightarrow_\beta$-progressivity of C) $\langle M', N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \Downarrow$.
- $\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \rightarrow_\beta \langle M, N'\rangle\ \pi_1\overrightarrow{\mathcal{E}_0}$: We have $M\overrightarrow{\mathcal{E}_0}$ and by assumption $M\overrightarrow{\mathcal{E}_0} \Downarrow$.
- $\langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \rightarrow_\beta M\overrightarrow{\mathcal{E}_0}$: By assumption $M\overrightarrow{\mathcal{E}_0} \Downarrow$.

Since all the one-step-reducts are strongly normalizing, $T \in C$, and thus, $L \equiv \langle M, N\rangle\ \pi_1\overrightarrow{\mathcal{E}_0} \Downarrow$.

($pair\_red_2$) ($L \equiv \langle M, N\rangle\ \pi_2\overrightarrow{\mathcal{E}_0}$): Analogous to the previous case.

(neg) ($L \equiv \mathrm{Id}\ M$): Clearly $\mathrm{Id}\ M \Downarrow$ since every possible reduction happens in M which is strongly normalizing.

(neg_red) ($L \equiv (\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\overrightarrow{\mathcal{E}_0}$): In this case, by the induction hypothesis $M\overrightarrow{\mathcal{E}_0} \Downarrow$ since it is the premise of the *neg_red* rule. Again, we are going to follow the same idea as in the *appl* and *pair_red*$_1$ rules, i.e, we are going to proceed by induction on $M\overrightarrow{\mathcal{E}_0} \Downarrow$ showing the $\rightarrow_\beta$-progressivity of the set:

$$D = \{M\overrightarrow{\mathcal{E}_0} \mid M\overrightarrow{\mathcal{E}_0} \Downarrow \Longrightarrow (\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \Downarrow\}$$

for all $M, \overrightarrow{\mathcal{E}_0}$. So, assume $T \equiv M\overrightarrow{\mathcal{E}_0}$ and that every one-step-reduct of it is in D. Clearly $T \Downarrow$, because every one-step-reduct of it is strongly normalizing for belonging to D. Therefore, we just need to prove that $(\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \Downarrow$ by showing that every one-step-reduct is strongly normalizing (Definition 4.4):

* $(\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \rightarrow_\beta (\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0'}$: In this case, $T \equiv M\overrightarrow{\mathcal{E}_0} \rightarrow_\beta M\overrightarrow{\mathcal{E}_0'}$ by *compatibility* of the β-reduction, hence, by induction hypothesis ($\rightarrow_\beta$-progressivity of D) $M\overrightarrow{\mathcal{E}_0'} \Downarrow$ and therefore $(\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0'} \Downarrow$.
* $(\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \rightarrow_\beta (\text{Id } M')\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0}$: In this case, $T \equiv M\overrightarrow{\mathcal{E}_0} \rightarrow_\beta M'\overrightarrow{\mathcal{E}_0}$ by *compatibility* of the β-reduction, hence, by induction hypothesis ($\rightarrow_\beta$-progressivity of D) $M'\overrightarrow{\mathcal{E}_0} \Downarrow$ and therefore $(\text{Id } M')\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \Downarrow$.
* $(\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \rightarrow_\beta M\overrightarrow{\mathcal{E}_0}$: By assumption $M\overrightarrow{\mathcal{E}_0} \Downarrow$.

Since all the one-step-reducts are strongly normalizing, $T \in D$, and therefore, $L \equiv (\text{Id } M)\ \text{Id}^{-1}\overrightarrow{\mathcal{E}_0} \Downarrow$.

Since for all cases $L \Downarrow$, we have proven the step part of the induction.

Due to induction principle, SW $\subseteq WF_{\rightarrow_\beta}$. ∎

4.5 Strong normalization proof

Before proceeding with the proof of the strong normalization theorem, we are going to prove some lemmas.

Lemma 4.5. *If $L, N \in$ SW are legal, $L \equiv y\overrightarrow{M}$ and $\overrightarrow{M}[y := N] \in$ SW, then $L[y := N] \in$ SW.*

Proof. We are going to proceed by induction on the length of $\overrightarrow{M}$.

- *Basis*: $\overrightarrow{M}$ is void and therefore $y[y := N] \equiv N \in$ SW.
- *Step*: Assume that the statement holds for lengths up to n. Let us prove that it holds when the length of $\overrightarrow{M}$ is $n+1$. Therefore, we can decompose $\overrightarrow{M}$ in $\overrightarrow{O}$ (of length n) and P, i.e, $\overrightarrow{M} \equiv \overrightarrow{O}P$. Because of $\overrightarrow{M}[y := N] \in$ SW by hypothesis, therefore, $\overrightarrow{O}[y := N], P[y := N] \in$ SW. Then, on one

hand, since $P[y := N] \in$ SW, $(zP)[y := N] \in$ SW for a new variable z by the *var* rule (note that $(zP)[y := N] \equiv zP[y := N]$ since z is a new variable). On the other hand, due to $\vec{O}[y := N] \in$ SW and its length is equal to n, by the induction hypothesis, $(y\vec{O})[y := N] \in$ SW. Then, because of $(y\vec{O})[y := N] \in$ SW, $(zP[y := N]) \in$ SW and the length of $zP[y := N]$ is less than n, applying the induction hypothesis on $P[y := N]$ yields $(zP[y := N])[z := (y\vec{O})[x := N]] \in$ SW. Finally, note that

$$
\begin{aligned}
(zP[y := N])[z := (y\vec{O})[y := N]] &\equiv \\
(y\vec{O})[y := N]P[y := N] &\equiv \quad \text{(substitution of } z\text{)} \\
(y\vec{O}P)[y := N] &\equiv \quad \text{(Definition 3.6)} \\
(y\vec{M})[y := N] &\equiv
\end{aligned}
$$

Therefore, we can conclude that $(y\vec{M})[y := N] \in$ SW. Hence, by the induction principle the statement has been proven.

■

Lemma 4.6. *If $L, N \in$ SW are legal, $L \equiv y\vec{M}\vec{\mathcal{E}_0}$ and $(\vec{M}\vec{\mathcal{E}_0})[y := N] \in$ SW, then $L[y := N] \in$ SW.*

Proof. We are going to proceed by induction on the length of $\vec{\mathcal{E}_0}$:

- *Basis:* $\vec{\mathcal{E}_0}$ is void and therefore $L \equiv y\vec{M}$. By the Lemma 4.5, $L[y := N] \in$ SW.
- *Step:* Assume that the statement holds for lengths up to m. Let us prove that it holds when the length of $\vec{\mathcal{E}_0}$ is $m+1$. Therefore, we can decompose $\vec{\mathcal{E}_0}$ in $\vec{\mathcal{E}_1}$ (of length m) and $\mathcal{E}_2$, i.e, $\vec{\mathcal{E}_0} \equiv \vec{\mathcal{E}_1}\mathcal{E}_2$. The following cases can be distinguished:
  - $\mathcal{E}_2$ is one of: π_1, π_2 or Id^{-1}. In this cases, $(y\vec{M}\vec{\mathcal{E}_1}\mathcal{E}_2)[y := N] \equiv (y\vec{M}\vec{\mathcal{E}_1})[y := N]\mathcal{E}_2$ by Definition 3.6. Therefore, $(y\vec{M}\vec{\mathcal{E}_1})[y := N]\mathcal{E}_2 \in$ SW since the inductive hypothesis yields $(y\vec{M}\vec{\mathcal{E}_1})[y := N] \in$ SW because the length of $\vec{\mathcal{E}_1}$ is m.
  - $\mathcal{E}_2 = S$. In this case, because $(\vec{M}\vec{\mathcal{E}_0})[y := N] \in$ SW by hypothesis, therefore, $(\vec{M}\vec{\mathcal{E}_1})[y := N], \mathcal{E}_2[y := N] \in$ SW, i.e, $(\vec{M}\vec{\mathcal{E}_1})[y := N], S[y := N] \in$ SW. Then, on one hand, since $S[y := N] \in$ SW, $(zS)[y := N] \in$ SW for a new variable z by the *var* rule (note that $(zS)[y := N] \equiv zS[y := N]$ due to the fact that z is a new variable). On the other hand, due to $\vec{M}\vec{\mathcal{E}_1}[y := N]$ and its length is equal to

n, by the induction hypothesis, $(y\vec{M}\vec{\mathcal{E}_1})[y := N] \in \mathrm{SW}$. Then, because of $(y\vec{M}\vec{\mathcal{E}_1})[y := N] \in \mathrm{SW}$, $zS[y := N] \in \mathrm{SW}$ and the length of $zS[y := N]$ is less than n, applying the induction hypothesis on $S[y := N]$ yields $(zS[y := N])[z := (y\vec{M}\vec{\mathcal{E}_1})[y := N]] \in \mathrm{SW}$. Finally, note that

$$\begin{aligned}
(zS[y := N])[z := (y\vec{M}\vec{\mathcal{E}_1})[y := N]] &\equiv \\
(y\vec{M}\vec{\mathcal{E}_1})[y := N]S[y := N] &\equiv \text{(substitution of } z) \\
(y\vec{M}\vec{\mathcal{E}_1}S)[y := N] &\equiv \text{(Definition 3.6)} \\
(y\vec{M}\vec{\mathcal{E}_1}\mathcal{E}_2)[y := N] &\equiv \text{(substitution of } S) \\
(y\vec{M}\vec{\mathcal{E}_0})[y := N]
\end{aligned}$$

Thus, we can conclude that $(y\vec{M}\vec{\mathcal{E}_0})[y := N] \in \mathrm{SW}$.

Because in all the cases the statement holds, the proof of the lemma has been completed.

Hence, by the induction principle the statement has been proven. ■

Lemma 4.7. *If $L \in \mathrm{SW}$ and $N \in \mathrm{SW}$ are legal, then $LN \in \mathrm{SW}$ and $L[x := N] \in \mathrm{SW}$.*

Proof. We will do the proof by induction on the structure of $L \in \mathrm{SW}$.

1. *Basis*: The basis step of the induction part refers to the var_0 rule in the Definition 4.8 and it is trivially true since $yN \in \mathrm{SW}$ (for all term variables y), by the *var* rule.

   The proof of $y[x := N] \in \mathrm{SW}$ also follows immediately because if $y \neq x$, then $y[x := N] \equiv y$; on the other hand, if $y \equiv x$, $y[x := N] \equiv N$ and $N \in \mathrm{SW}$.

2. *Step*: Assume that the proposition holds for all the premises in the derivation of $L \in \mathrm{SW}$. We need to prove $LN, L[x := N] \in \mathrm{SW}$. The following cases are possible:

   (*var*) $L \equiv y\vec{M}\vec{\mathcal{E}_0}$: In this case, due to the inductive hypothesis $\vec{M}\vec{\mathcal{E}_0}, N \in \mathrm{SW}$, and by the *var* rule $LN \equiv y\vec{M}N \in \mathrm{SW}$.

   To prove $(y\vec{M}\vec{\mathcal{E}_0})[x := N] \in \mathrm{SW}$ we will distinguish the following cases:

   * ($y \not\equiv x$): By the inductive hypothesis $(\vec{M}\vec{\mathcal{E}_0})[x := N] \in \mathrm{SW}$. Applying the *var* rule, we can conclude that $y(\vec{M}\vec{\mathcal{E}_0})[x := N] \in \mathrm{SW}$ and it is equivalent to $(y\vec{M}\vec{\mathcal{E}_0})[x := N]$.

* $(y \equiv x)$: By the Lemma 4.6 $(y\overrightarrow{M}\overrightarrow{\mathcal{E}_0})[x := N] \in \mathrm{SW}$.

(abs) $L \equiv \lambda y : \sigma.M$: In this case $M[y := N] \in \mathrm{SW}$ by the inductive hypothesis for substitution since it is the premise of the *abs* rule. Therefore, applying *appl* rule $(\lambda y : \sigma.M)N \in \mathrm{SW}$.

The proof of $(\lambda y : \sigma.M)[x := N]$ comes from the fact that all the substitutions occurs in M. By the inductive hypothesis $M[x := N] \in \mathrm{SW}$. Thus, by *abs* rule $\lambda y : \sigma.M[x := N] \equiv (\lambda y : \sigma.M)[x := N] \in \mathrm{SW}$.

($appl$) $L \equiv (\lambda y := \sigma.M)O\overrightarrow{\mathcal{E}_0}$: Since $M[y := O]\overrightarrow{\mathcal{E}_0}N, O \in \mathrm{SW}$ by induction hypothesis, then by the *appl* rule $(\lambda y := \sigma.M)O\overrightarrow{\mathcal{E}_0}N \in \mathrm{SW}$.

To prove $((\lambda y := \sigma.M)O\overrightarrow{\mathcal{E}_0})[x := N] \in \mathrm{SW}$, consider the induction hypothesis, i.e, $(M[y := O[x := N]]\overrightarrow{\mathcal{E}_0})[x := N], O[x := N] \in \mathrm{SW}$. Then, by the application of the *appl* rule, conclude $(\lambda y : \sigma.M[x := N])O[x := N]\overrightarrow{\mathcal{E}_0}[x := N] \equiv ((\lambda y : \sigma.M)O\overrightarrow{\mathcal{E}_0})[x := N] \in \mathrm{SW}$.

($pair$) $(L \equiv \langle M, O\rangle)$: Since $\langle M, O\rangle$ has a type of the form $\overline{\sigma \to \tau}$ and $N : \sigma$ and applications requires a function type only the substitution case is possible. Then, due to $M[x := N], O[x := N] \in \mathrm{SW}$ by induction hypothesis and all the substitutions in $\langle M, O\rangle$ take place in M and O, the term $\langle M, O\rangle[x := N] \in \mathrm{SW}$.

($pair\_red_1$) $L \equiv \langle M, O\rangle\ \pi_1\overrightarrow{\mathcal{E}_0}$: By induction hypothesis $M\overrightarrow{\mathcal{E}_0}N, O\overrightarrow{\mathcal{E}_0}N \in \mathrm{SW}$, hence $\langle M, O\rangle\ \pi_1\overrightarrow{\mathcal{E}_0}N$ applying the *pair_red*$_1$ rule.

In the substitution, again, by the induction hypothesis $(M\overrightarrow{\mathcal{E}_0})[x := N], (O\overrightarrow{\mathcal{E}_0})[x := N] \in \mathrm{SW}$. Then, by the *pair_red*$_1$ rule $(\langle M, O\rangle\ \pi_1\overrightarrow{\mathcal{E}_0})[x := N] \equiv \langle M[x := N], O[x := N]\rangle\ \pi_1\overrightarrow{\mathcal{E}_0}[x := N] \in \mathrm{SW}$.

($pair\_red_2$) $L \equiv \langle M, O\rangle\ \pi_2\overrightarrow{\mathcal{E}_0}$: Analogous to the previous one.

(neg) $L \equiv \mathrm{Id}\ M$: Only the substitution case is possible. Since $M[x := N] \in \mathrm{SW}$, then $\mathrm{Id}\ M[x := N] \equiv \mathrm{Id}\ M[x := N] \in \mathrm{SW}$.

(neg_red) $L \equiv (\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\overrightarrow{\mathcal{E}_0}$: By induction hypothesis $M\overrightarrow{\mathcal{E}_0}N \in \mathrm{SW}$. Therefore, by the *neg_red* rule $(\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\overrightarrow{\mathcal{E}_0}N \in \mathrm{SW}$.

Since the induction hypothesis yields $(M\overrightarrow{\mathcal{E}_0})[x := N] \equiv M[x := N]\overrightarrow{\mathcal{E}_0}[x := N] \in \mathrm{SW}$, then $((\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\overrightarrow{\mathcal{E}_0})[x := N] \equiv (\mathrm{Id}\ M[x := N])\ \mathrm{Id}^{-1}\overrightarrow{\mathcal{E}_0}[x := N] \in \mathrm{SW}$.

Thus, because the statement holds in all the cases, the step part of the induction has been proven.

By the induction principle the proof is finished. ■

Lemma 4.8. *If $L \in \mathrm{SW}$ is legal, then $L\ \pi_1 \in \mathrm{SW}$ and $L\ \pi_2 \in \mathrm{SW}$.*

Proof. We will proceed by induction on the structure of $L \in \mathrm{SW}$.

1. *Basis*: Since $x \in \mathrm{SW}$ $(L \equiv x)$ due to the var_0 rule of the Definition 4.8, the application of the var yields $L\ \pi_1 \in \mathrm{SW}$ and $L\ \pi_2 \in \mathrm{SW}$.

2. *Step*: Assume that the proposition holds for all the premises in the derivation of $L \in \mathrm{SW}$. We need to prove $L\ \pi_1 \in \mathrm{SW}$ and $L\ \pi_2 \in \mathrm{SW}$. Then, we have the following cases:

   (var) $L \equiv x\vec{M}\vec{\mathcal{E}_0}$: In this case, by the inductive hypothesis, we have that $\vec{M}\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$. Hence, applying the var rule yields $x\vec{M}\vec{\mathcal{E}_0}\ \pi_1$. The case for π_2 is analogous.

   (abs) $L \equiv \lambda x : \sigma.M$: This case does not make sense due to the fact that π_1 and π_2 require that $\lambda x : \sigma.M$ has type $\overline{\alpha \to \gamma}$ and this is impossible since $\lambda x : \sigma.M$ has a function type.

   ($appl$) $L \equiv (\lambda x : \sigma.M)N\vec{\mathcal{E}_0}$: In this case, by the induction hypothesis $M[x := N]\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$. Hence, by the $appl$ rule $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$. The case for π_2 is analogous.

   ($pair$) $L \equiv \langle M, N\rangle$: In this case, is derived directly using the $pair\_red_1$ rule for π_1. In the case of π_2, we just need to apply the $pair\_red_2$ rule.

   ($pair\_red_1$) $L \equiv \langle M, N\rangle\ \pi_1\vec{\mathcal{E}_0}$: in this case, the induction hypothesis yields $M\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$ and $N\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$. Hence, using the $pair\_red_1$ rule, we can conclude $\langle M, N\rangle\ \pi_1\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$. The case for π_2 is obtained applying the $pair\_red_2$ rule instead of $pair\_red_1$ rule.

   ($pair\_red_2$) $L \equiv \langle M, N\rangle\ \pi_2\vec{\mathcal{E}_0}$: This case is analogous to the previous one.

   (neg) $L \equiv \mathrm{Id}\ M$: This case is not possible due to the fact that π_1, π_2 requires a type of the form $\overline{\alpha \to \gamma}$ and L must have a type of the form $\overline{\overline{\sigma}}$.

   (neg_red) $L \equiv (\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\vec{\mathcal{E}_0}$: In this case, the induction hypothesis allows us to conclude $M\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$. Therefore, $(\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\vec{\mathcal{E}_0}\ \pi_1 \in \mathrm{SW}$ by the application of the neg_red rule. For the case of π_2, the proof is analogous.

   Since the statement holds for all the cases, the step part of the induction has been proven.

By the induction principle the proof is finished. ■

Lemma 4.9. *If $L \in \mathrm{SW}$ is legal, then $L\ \mathrm{Id}^{-1} \in \mathrm{SW}$*

Proof. We will proceed by induction on the structure of $L \in \mathrm{SW}$.

1. *Basis*: Since $x \in \mathrm{SW}$ ($L \equiv x$) due to the var_0 rule of the Definition 4.8, the application of the var yields $L\ \mathrm{Id}^{-1} \in \mathrm{SW}$.

2. *Step*: Assume that the proposition holds for all the premises in the derivation of $L \in \mathrm{SW}$. We need to prove $L\ \mathrm{Id}^{-1} \in \mathrm{SW}$. Then, we have the following cases:

   (var) $L \equiv x\vec{M}\vec{\mathcal{E}_0}$: In this case, by the inductive hypothesis, we have that $\vec{M}\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$. Hence, applying the var rule yields $x\vec{M}\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1}$.

   (abs) $L \equiv \lambda x : \sigma.M$: This case does not make sense due to the fact that Id^{-1} requires that $\lambda x : \sigma.M$ has type $\overline{\overline{\alpha}}$ and this is impossible since $\lambda x : \sigma.M$ has a function type.

   ($appl$) $L \equiv (\lambda x : \sigma.M)N\vec{\mathcal{E}_0}$: In this case, by the induction hypothesis $M[x := N]\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$. Hence, by the *appl* rule $(\lambda x : \sigma.M)N\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$.

   ($pair$) $L \equiv \langle M, N\rangle$: This case does not make sense since pairs has types of the form $\overline{\alpha \to \gamma}$ and Id^{-1} requires a type of the form $\overline{\overline{\sigma}}$.

   ($pair\_red_1$) $L \equiv \langle M, N\rangle\ \pi_1\vec{\mathcal{E}_0}$: In this case, the induction hypothesis yields $M\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$ and $N\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$. Hence, using the $pair\_red_1$ rule, we can conclude $\langle M, N\rangle\ \pi_1\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$.

   ($pair\_red_2$) $L \equiv \langle M, N\rangle\ \pi_2\vec{\mathcal{E}_0}$: This case is analogous to the previous one.

   (neg) $L \equiv \mathrm{Id}\ M$: This case can be derived directly applying the neg_red.

   (neg_red) $L \equiv (\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\vec{\mathcal{E}_0}$: In this case, the induction hypothesis allows us to conclude $M\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$. Therefore, $(\mathrm{Id}\ M)\ \mathrm{Id}^{-1}\vec{\mathcal{E}_0}\ \mathrm{Id}^{-1} \in \mathrm{SW}$ by the application of the neg_red rule.

   Since the statement holds for all the cases, the step part of the induction has been proven.

By the induction principle the proof is finished. ■

Theorem 4.10 (Strong Normalization Theorem)**.** *All legal terms are strongly normalizing.*

Proof. Assume L is a legal term, then exists Γ and α such that $\Gamma \vdash_{\bar{\lambda}_{\to}} L : \alpha$. Firstly, we are going to prove that all the legal terms belong to SW proceeding by induction on the derivation of L, following the rules in Definition 3.4:

1. *Basis*: In this case $L \equiv x$ (and $\alpha = \sigma$) is derived from the assumption that $x : \sigma \in \Gamma$. $x : \sigma \in \mathrm{SW}$ is followed by the var_0 rule.

2. *Step*: Assume that the statement holds for all the premises in the derivation of L. We have the following cases:

($\to I$) In this case $L \equiv \lambda x : \sigma.M$ (and $\alpha = \sigma \to \tau$) is derived from $M : \tau$ in the context $\Gamma, x : \sigma$. Since $M : \tau \in \text{SW}$ by induction hypothesis, then, applying the *abs* rule, $\lambda x : \sigma.M : \sigma \to \tau \in \text{SW}$.

($\to E$) $L \equiv MN$ (and $\alpha = \tau$) is derived from $M : \sigma \to \tau$ and $N : \sigma$. By the induction hypothesis $M : \sigma \to \tau, N : \sigma \in \text{SW}$. Hence, by Lemma 4.7 $MN : \tau \in \text{SW}$.

($\overline{\to} I$) $L \equiv \langle M, N\rangle$ (and $\alpha = \overline{\sigma \to \tau}$) is derived from $M : \sigma$ and $N : \overline{\tau}$. By the induction hypothesis $M : \sigma, N : \overline{\tau} \in \text{SW}$. By the application of the *pair* rule, $\langle M, N\rangle : \overline{\sigma \to \tau} \in \text{SW}$.

($\overline{\to} E_1$) $L \equiv M\ \pi_1$ (and $\alpha = \sigma$) is derived from $M : \overline{\sigma \to \tau}$. Hence, by Lemma 4.8, $L \in \text{SW}$.

($\overline{\to} E_2$) $L \equiv M\ \pi_2$ (and $\alpha = \overline{\tau}$) is derived from $M : \overline{\sigma \to \tau}$. Therefore, by the Lemma 4.8, $L \in \text{SW}$.

($^{=}I$) $L \equiv \text{Id}\ M : \overline{\overline{\sigma}}$ in the context Γ is derived from $M : \sigma$. The induction hypothesis yields $M : \sigma \in \text{SW}$, therefore, by the *neg* rule $\text{Id}\ M : \overline{\overline{\sigma}} \in \text{SW}$.

($^{=}E$) $L \equiv M\ \text{Id}^{-1} : \sigma$ in the context Γ is derived from $M : \overline{\overline{\sigma}}$. By the Lemma 4.9, $L \in \text{SW}$.

By the induction principle, all the legal terms are in SW. Therefore, by the lemma 4.3.1 all the legal terms are strongly normalizing since $\text{SW} \subseteq WF_{\to_\beta}$. ∎

5 Curry-Howard correspondence

As we mentioned in the introduction, the Curry-Howard correspondence has three levels:

1. Propositions as types.
2. Proofs as terms.
3. Simplifications of proofs as reduction of terms.

In this section[3], we are just going to show the first two levels of the correspondence. For the third level, we will just provide the idea of the proof in this system with β-reduction.

[3]The definitions and theorems of this section are based on Sørensen and Urzyczyn (2006).

5.1 Propositions as types and proofs as terms

The proof for the first two levels of the Curry-Howard correspondence involves the introduction of translation functions that allows us to interpret the *types* in $\bar{\lambda}_{\rightarrow}$ as *propositions* in $\mathrm{PL}_{\neg,\supset}$ and *vice versa.* Once, we have defined these translation functions, we are going to proceed with the proof of the correspondence using the induction principle.

Definition 5.1 (Translation functions: tr and tr^{-1})**.** *The* translation function *(denoted by* tr*) is recursively defined by:*

$$\begin{array}{rcl} \mathrm{tr} : \mathbb{T} & \longrightarrow & \mathrm{PL}_{\neg,\supset} \\ a & \mapsto & a \\ \sigma \rightarrow \tau & \mapsto & \mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau) \\ \overline{\sigma} & \mapsto & \neg\mathrm{tr}(\sigma) \end{array}$$

where $a \in \mathbb{B}$ and $\sigma, \tau \in \mathbb{T}$. Clearly, this function is bijective an its inverse (denoted by tr^{-1}*) is recursively defined:*

$$\begin{array}{rcl} \mathrm{tr}^{-1} : \; \mathrm{PL}_{\neg,\supset} & \longrightarrow & \mathbb{T} \\ a & \mapsto & a \\ \sigma \supset \tau & \mapsto & \mathrm{tr}^{-1}(\sigma) \rightarrow \mathrm{tr}^{-1}(\tau) \\ \neg\sigma & \mapsto & \overline{\mathrm{tr}^{-1}(\sigma)} \end{array}$$

where $a \in \mathbf{PV}$ and $\sigma, \tau \in \Phi$.

Definition 5.2 (Range, rg)**.** *Let Γ be a context in $\bar{\lambda}_{\rightarrow}$. Then the* range *of Γ, denoted by* $\mathrm{rg}(\Gamma)$*, is* $\mathrm{rg}(\Gamma) = \{\sigma \mid x : \sigma \in \Gamma\}$*.*

Theorem 5.1 (Curry-Howard correspondence)**.**

1. *If $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \varphi$ in $\bar{\lambda}_{\rightarrow}$, then $\Delta \vdash \mathrm{tr}(\varphi)$ in $\mathrm{PL}_{\neg,\supset}$, with $\mathrm{tr}(\mathrm{rg}(\Gamma)) = \Delta$.*
2. *If $\Delta \vdash \varphi$ in $\mathrm{PL}_{\neg,\supset}$, then $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\varphi)$ in $\bar{\lambda}_{\rightarrow}$, for some term M and context Γ with $\mathrm{tr}(\mathrm{rg}(\Gamma)) = \Delta$.*

Proof. 1. We are going to proceed by induction on the derivation of the judgement $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \varphi$.

1. *Basis:* In this case $\mathcal{J}$ is derived from the (Ax^*), i.e, $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_{\rightarrow}} x : \varphi$ and is obtained from $x : \varphi \in \Gamma$. Then, given that $x : \varphi \in \Gamma$ and $\mathrm{tr}(\mathrm{rg}(\Gamma)) = \Delta$, we can conclude that $\mathrm{tr}(\varphi) \in \Delta$, and therefore, applying the (Ax) rule of $\mathrm{PL}_{\neg,\supset}$, $\Delta \vdash \mathrm{tr}(\varphi)$.
2. *Step*: Assume that $\mathcal{J}$ is the final deduction of a derivation, and suppose that the first part of the Theorem 5.1 is true for the premises which have been used to conclude $\mathcal{J}$. The following cases can be distinguished:

($\to I$) In this case $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_\to} \lambda x : \sigma.M : \sigma \to \tau$ is derived from $M : \tau$ in the context $\Gamma, x : \sigma$; hence, we need to prove that $\Delta \vdash \mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau)$. Due to the inductive hypothesis $\Delta, \mathrm{tr}(\sigma) \vdash \mathrm{tr}(\tau)$. Therefore, the rule ($\supset I$) of $\mathrm{PL}_{\neg,\supset}$ yields $\Delta \vdash \mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau)$.

($\to E$) In this case $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_\to} MN : \tau$ is derived from $M : \sigma \to \tau$ and $N : \sigma$; hence, we need to prove $\Delta \vdash \mathrm{tr}(\tau)$. By the inductive hypothesis $\Delta \vdash \mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau)$ and $\Delta \vdash \mathrm{tr}(\sigma)$. Then, using the ($\supset E$) rule of $\mathrm{PL}_{\neg,\supset}$ yields $\Delta \vdash \mathrm{tr}(\tau)$.

($\overline{\to} I$) In this case $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_\to} \langle M, N \rangle : \overline{\sigma \to \tau}$ is derived from $M : \sigma$ and $N : \overline{\tau}$; hence, we should prove $\Delta \vdash \neg(\mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau))$. The inductive hypothesis leads to $\Delta \vdash \mathrm{tr}(\sigma)$ and $\Delta \vdash \neg\mathrm{tr}(\tau)$. Therefore, applying the ($\neg \supset I$) rule of $\mathrm{PL}_{\neg,\supset}$ yields $\Delta \vdash \neg(\mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau))$.

($\overline{\to} E_1$) In this case $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_\to} M\ \pi_1 : \sigma$ is concluded from $M : \overline{\sigma \to \tau}$; hence, we need to prove $\Delta \vdash \mathrm{tr}(\sigma)$. By the inductive hypothesis $\Delta \vdash \neg(\mathrm{tr}(\sigma) \supset \mathrm{tr}(\tau))$. Therefore, from the rule ($\neg \supset E_1$) we can conclude $\Delta \vdash \mathrm{tr}(\sigma)$.

($\overline{\to} E_2$) This case is analogous to the previous one.

($^{=}I$) In this case $\mathcal{J} \equiv \Gamma \vdash_{\bar{\lambda}_\to} \mathrm{Id}\ M : \overline{\overline{\sigma}}$ is derived from $M : \sigma$; hence, it is necessary to prove $\Delta \vdash \neg\neg\mathrm{tr}(\sigma)$. By the induction hypothesis, $\Delta \vdash \mathrm{tr}(\sigma)$, therefore, using the ($\neg\neg I$) rule of $\mathrm{PL}_{\neg,\supset}$ yields $\Delta \vdash \neg\neg\mathrm{tr}(\sigma)$.

($^{=}E$) This case is analogous to the previous one.

Thus, by the induction principle the first part of the theorem has been proven.

2. The second part of the theorem will also be showed applying induction on the judgment $\mathcal{I} \equiv \Delta \vdash \varphi$ and considering $\Gamma = \{x_i : \mathrm{tr}^{-1}(\varphi_i) \mid \varphi_i \in \Delta \text{ and } i = 1, 2, \ldots n\}$, where n is the number of formulas in Δ.

   1. *Basis:* In this case $\mathcal{I}$ is obtained from the (Ax) rule of $\mathrm{PL}_{\neg,\supset}$, i.e, $\varphi \in \Delta$. Then, since $\mathrm{tr}(\mathrm{rg}(\Gamma)) = \Delta$, there is a term variable $x : \mathrm{tr}^{-1}(\varphi) \in \Gamma$. Hence, applying the ($Ax^*$) rule $\Gamma \vdash_{\bar{\lambda}_\to} x : \mathrm{tr}^{-1}(\varphi)$.
   2. *Step:* Assume that $\mathcal{I}$ is the last conclusion of a derivation in $\mathrm{PL}_{\neg,\supset}$ and that the second part of the Theorem 5.1 holds for all the premises used to conclude $\mathcal{I}$. We have the following cases:

($\supset I$) In this case $\mathcal{I} \equiv \Delta \vdash \sigma \supset \tau$ is derived from $\Delta, \sigma \vdash \tau$; therefore, it is necessary to prove that there is a term L such that $\Gamma \vdash_{\bar{\lambda}_\to} L : \mathrm{tr}^{-1}(\sigma) \to \mathrm{tr}^{-1}(\tau)$. By the inductive hypothesis there are a term M and a variable x such that $\Gamma, x : \mathrm{tr}^{-1}(\sigma) \vdash_{\bar{\lambda}_\to} M : \mathrm{tr}^{-1}(\tau)$. Therefore, applying the ($\to I$) rule yields $\Gamma \vdash_{\bar{\lambda}_\to} \lambda x : \mathrm{tr}^{-1}(\sigma).M : \mathrm{tr}^{-1}(\sigma) \to \mathrm{tr}^{-1}(\tau)$.

($\supset E$) In this case $\mathcal{I} \equiv \Delta \vdash_{\mathrm{PL}_{\neg,\supset}} \tau$ is concluded from $\Delta \vdash \sigma \supset \tau$ and $\Delta \vdash \sigma$; hence, we should prove that there is a term L such that $\Gamma \vdash_{\bar{\lambda}_\to} L : \mathrm{tr}^{-1}(\tau)$. Using the inductive hypothesis there are terms M and N such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\sigma) \rightarrow \mathrm{tr}^{-1}(\tau)$ and $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} N : \mathrm{tr}^{-1}(\sigma)$. Therefore, by the $(\rightarrow E)$ rule, we can conclude that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} MN : \mathrm{tr}^{-1}(\tau)$.

$(\neg \supset I)$ In this case $\mathcal{I} \equiv \Delta \vdash \neg(\sigma \supset \tau)$ is derived from $\Delta \vdash \sigma$ and $\Delta \vdash \neg\tau$; hence, we need to prove that there is a term L such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} L : \overline{\mathrm{tr}^{-1}(\sigma) \rightarrow \mathrm{tr}^{-1}(\tau)}$. By the induction hypothesis there are terms M and N such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\sigma)$ and $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} N : \overline{\mathrm{tr}^{-1}(\tau)}$. Therefore, applying the $(\overline{\rightarrow} I)$ yields $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \langle M, N\rangle : \overline{\mathrm{tr}^{-1}(\sigma) \rightarrow \mathrm{tr}^{-1}(\tau)}$.

$(\neg \supset E_1)$ In this case $\mathcal{I} \equiv \Delta \vdash \sigma$ is concluded from $\Delta \vdash \neg(\sigma \supset \tau)$; hence, it is necessary to show that there is a term L such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} L : \mathrm{tr}^{-1}(\sigma)$. By induction hypothesis there is a term M such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \overline{\mathrm{tr}^{-1}(\sigma) \rightarrow \mathrm{tr}^{-1}(\tau)}$. Therefore, using the $(\overline{\rightarrow} E_1)$ rule, we can conclude that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M\ \pi_1 : \mathrm{tr}^{-1}(\sigma)$.

$(\neg \supset E_2)$ This case is analogous to the previous one.

$(\neg\neg I)$ In this case $\mathcal{I} \equiv \Delta \vdash \neg\neg\sigma$ is derived from $\Delta \vdash \sigma$; hence, we need to show that there is a term L such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} L : \overline{\overline{\mathrm{tr}^{-1}(\sigma)}}$. By the induction hypothesis there is a term M such that $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\sigma)$. Therefore, using the $\left(^{=}I\right)$ yields $\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \mathrm{Id}\ M : \overline{\overline{\mathrm{tr}^{-1}(\sigma)}}$.

$(\neg\neg E)$ This is case is analogous to the previous one.

Thus, by the induction principle, the second part of the theorem has been proven.

Since we have already proven both parts of the theorem, the proof is completed. ■

5.2 Is this an exact correspondence?

Talking about a correspondence, implies that the statement of the theorem should be an equivalence, i.e, it should have the form of an exact bijective correspondence between proofs and terms (Sørensen & Urzyczyn, 2006). However, as the same authors point out, proofs can be annotated in various ways if we interpret terms as annotated proofs. Moreover, they (Sørensen & Urzyczyn, 2006, p. 82) said:

> [...] the difference between terms and proofs is that the former carry more information than the latter. The reason is that in logic, the primary issue is usually to determine provability of a formula [...]. It does not matter whether we use the same assumption twice or if we use two different assumptions about the same formula. [...]. On the contrary, in lambda-calculus we can have many variables of the same type φ, and this corresponds to making a difference between various assumptions about the same formula.

In spite of this facts, for the natural deduction system the exact correspondence has been showed but just for closed terms and proofs with no free assumptions (Hindley, 1997). Nevertheless, if we want to extend the Curry-Howard correspondence to arbitrary terms, we require a proof system with labeled free assumptions, and even in this case, we would merely find out that "the resulting proofs, up to syntactic sugar, are... another representation of lambda terms" (Sørensen & Urzyczyn, 2006, p. 83).

5.3 Simplification of proofs and reduction of terms

The first two levels of the Curry-Howard correspondence induce that there should be a relation between reduction of terms and simplification of proofs. The simplifications of proofs, better known as *proof normalization*, is an important issue in proof theory and has been studied independently by logicians (Sørensen & Urzyczyn, 2006). It is interesting, then, to prove the correspondence between these two processes. Although, in this article we are not going to prove this correspondence, the idea behind the proof will be presented. For doing that, we will define the so-called *detours* in logic proofs and the relation of them with the *redexes* in λ-terms.

Definition 5.3 (Detours in $\mathrm{PL}_{\neg,\supset}$)**.** *A pair consisting of an introduction step followed by an elimination step (applied to the formula just introduced) is called* detour *in* $\mathrm{PL}_{\neg,\supset}$.

All the possible *detours* in $\mathrm{PL}_{\neg,\supset}$ are presented in table 4.

$\dfrac{\dfrac{\Delta, \sigma \vdash \tau}{\Delta \vdash \sigma \supset \tau}\,(\supset I) \qquad \Delta \vdash \sigma}{\Delta \vdash \tau}\,(\supset E)$
$\dfrac{\dfrac{\Delta \vdash \sigma \qquad \Delta \vdash \neg\tau}{\Delta \vdash \neg(\sigma \supset \neg\tau)}\,(\neg \supset I)}{\Delta \vdash \sigma}\,(\neg \supset E_1)$
$\dfrac{\dfrac{\Delta \vdash \sigma \qquad \Delta \vdash \neg\tau}{\Delta \vdash \neg(\sigma \supset \neg\tau)}\,(\neg \supset I)}{\Delta \vdash \neg\tau}\,(\neg \supset E_2)$
$\dfrac{\dfrac{\Delta \vdash \sigma}{\Delta \vdash \neg\neg\sigma}\,(\neg\neg I)}{\Delta \vdash \sigma}\,(\neg\neg E)$

Table 4: *Detours* in $\mathrm{PL}_{\neg,\supset}$

These *detours* have a correspondence with the *redexes* in $\bar{\lambda}_{\rightarrow}$ using the Theorem 5.1:

$\dfrac{\dfrac{\Gamma, x : \mathrm{tr}^{-1}(\sigma) \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\tau)}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \lambda x : \mathrm{tr}^{-1}(\sigma).M : \mathrm{tr}^{-1}(\sigma) \rightarrow \mathrm{tr}^{-1}(\tau)}\ (\rightarrow I) \qquad \Gamma \vdash_{\bar{\lambda}_{\rightarrow}} N : \mathrm{tr}^{-1}(\sigma)}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} (\lambda x : \mathrm{tr}^{-1}(\sigma).M)N : \mathrm{tr}^{-1}(\tau)}\ (\rightarrow E)$
$\dfrac{\dfrac{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\sigma) \qquad \Gamma \vdash_{\bar{\lambda}_{\rightarrow}} N : \overline{\mathrm{tr}^{-1}(\tau)}}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \langle M, N \rangle : \overline{(\mathrm{tr}^{-1}(\sigma) \rightarrow \overline{\mathrm{tr}^{-1}(\tau)})}}\ (\overline{\rightarrow} I)}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \langle M, N \rangle\ \pi_1 : \mathrm{tr}^{-1}(\sigma)}\ (\overline{\rightarrow} E_1)$
$\dfrac{\dfrac{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\sigma) \qquad \Gamma \vdash_{\bar{\lambda}_{\rightarrow}} N : \overline{\mathrm{tr}^{-1}(\tau)}}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \langle M, N \rangle : \overline{\mathrm{tr}^{-1}(\sigma) \rightarrow \overline{\mathrm{tr}^{-1}(\tau)}}}\ (\overline{\rightarrow} I)}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \langle M, N \rangle\ \pi_2 : \overline{\mathrm{tr}^{-1}(\tau)}}\ (\overline{\rightarrow} E_2)$
$\dfrac{\dfrac{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} M : \mathrm{tr}^{-1}(\sigma)}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} \mathrm{Id}\ M : \overline{\overline{\mathrm{tr}^{-1}(\sigma)}}}\ ({}^{=}I)}{\Gamma \vdash_{\bar{\lambda}_{\rightarrow}} (\mathrm{Id}\ M)\ \mathrm{Id}^{-1} : \mathrm{tr}^{-1}(\sigma)}\ ({}^{=}E)$

Table 5: *Redexes* in $\bar{\lambda}_{\rightarrow}$

Once we have showed the connection between *redexes* and its counterparts in $\mathrm{PL}_{\neg,\supset}$ (*detours*), we are able to proceed showing the counterpart for the β-reduction in $\mathrm{PL}_{\neg,\supset}$.

Definition 5.4 (Proof normalization and normal forms)**.** *The process of eliminating proof detours is called* proof normalization*, and a derivation with no detours is said to be in* normal form*.*

When we are talking about an exact Curry-Howard correspondence, especially for the third level of the correspondence, "It can be more informative to write natural deduction proofs in the "traditional" way" (Sørensen & Urzyczyn, 2006, p. 82). So, we present the *proof normalization* rules in the "traditional" way in the Table 6. In this style, instead of associate a set of assumptions to every node of the derivation, we will write all the assumptions at the top, and put in brackets the ones that have been discharged by the implication rule. We also mark proofs by either $(*)$ or $(+)$.

$\dfrac{\dfrac{\begin{matrix}(*)\\ \vdots\\ \sigma\end{matrix}}{\neg\neg\sigma}\,(\neg\neg I)}{\sigma}\,(\neg\neg E)$	$\Longrightarrow$	$\begin{matrix}(*)\\ \vdots\\ \sigma\end{matrix}$
$\dfrac{\dfrac{\begin{matrix}(*)\\ \vdots\\ \sigma\end{matrix} \quad \begin{matrix}(+)\\ \vdots\\ \neg\tau\end{matrix}}{\neg(\sigma \supset \neg\tau)}\,(\neg \supset I)}{\sigma}\,(\neg \supset E_1)$	$\Longrightarrow$	$\begin{matrix}(*)\\ \vdots\\ \sigma\end{matrix}$
$\dfrac{\dfrac{\begin{matrix}(*)\\ \vdots\\ \sigma\end{matrix} \quad \begin{matrix}(+)\\ \vdots\\ \neg\tau\end{matrix}}{\neg(\sigma \supset \neg\tau)}\,(\neg \supset I)}{\neg\tau}\,(\neg \supset E_2)$	$\Longrightarrow$	$\begin{matrix}(+)\\ \vdots\\ \neg\tau\end{matrix}$
$\dfrac{\dfrac{\begin{matrix}[\sigma]\\ \vdots\\ \tau\end{matrix}}{\sigma \supset \tau}\,(\supset I) \quad \begin{matrix}(*)\\ \vdots\\ \sigma\end{matrix}}{\tau}\,(\supset E)$	$\Longrightarrow$	$\begin{matrix}(*)\\ \vdots\\ \sigma\\ \vdots\\ \tau\end{matrix}$

Table 6: Rules for *proof normalization* in the "traditional" way

Notice that the rules for *proof normalization*, i.e, the rules for eliminating *detours* in $\mathrm{PL}_{\neg,\supset}$, correspond to the rules of β-reduction (Definition 3.7), i.e, the rules of reducing *redexes* in $\bar{\lambda}_{\to}$ (Table 7). This correspondence allow us to show the deepest level of the Curry-Howard correspondence: proofs simplifications as reduction of terms. Moreover, it is possible to prove that every proof can be converted to a normal form due to every typed term is normalizing. In fact, the order of eliminating *detours* is irrelevant since every typed term is also strongly normalizing (Theorem 4.10).

$\dfrac{\dfrac{\begin{matrix}[x:\sigma]\\ \vdots \\ M:\tau\end{matrix}}{\lambda x:\sigma.M:\sigma\to\tau}(\to I) \quad \begin{matrix}(*)\\ \vdots \\ N:\sigma\end{matrix}}{(\lambda x:\sigma.M)N:\tau}(\to E)$	$\Longrightarrow$	$\begin{matrix}(*)\\ \vdots \\ N:\sigma \\ \vdots \\ M[x:=N]:\tau\end{matrix}$
$\dfrac{\dfrac{\begin{matrix}(*)\\ \vdots \\ M:\sigma\end{matrix} \quad \begin{matrix}(+)\\ \vdots \\ N:\overline{\tau}\end{matrix}}{\langle M,N\rangle:\overline{\sigma\to\tau}}(\overline{\to}I)}{\langle M,N\rangle\ \pi_1:\sigma}(\overline{\to}E_1)$	$\Longrightarrow$	$\begin{matrix}(*)\\ \vdots \\ M:\sigma\end{matrix}$
$\dfrac{\dfrac{\begin{matrix}(*)\\ \vdots \\ M:\sigma\end{matrix} \quad \begin{matrix}(+)\\ \vdots \\ \overline{\tau}\end{matrix}}{\langle M,N\rangle:\overline{\sigma\to\tau}}(\overline{\to}I)}{\langle M,N\rangle\ \pi_2:\overline{\tau}}(\overline{\to}E_2)$	$\Longrightarrow$	$\begin{matrix}(+)\\ \vdots \\ N:\overline{\tau}\end{matrix}$
$\dfrac{\dfrac{\begin{matrix}(*)\\ \vdots \\ M:\sigma\end{matrix}}{\mathrm{Id}\ M:\overline{\overline{\sigma}}}(^{=}I)}{(\mathrm{Id}\ M)\ \mathrm{Id}^{-1}:\sigma}(^{=}E)$	$\Longrightarrow$	$\begin{matrix}(*)\\ \vdots \\ M:\sigma\end{matrix}$

Table 7: Rules for term normalization in $\bar{\lambda}_{\to}$

6 Conclusions

In this article we have presented $\bar{\lambda}_{\to}$, a extension of $\lambda_{\to}$ using *opposite types*. The rules for the *opposite types* are defined based on the rules for negation in $\mathrm{PL}_{\neg,\supset}$. Therefore, it is not a surprise the Curry-Howard correspondence in its three levels. Additionally, we have presented all the details for the strong normalization theorem (Theorem 4.10), which does not follow immediately from the proof made by Joachimski and Matthes (2003) as Kamide (2010) pointed out. It has require a few more lemmas and, in some cases, the extension of the proof is too different to the original one for $\lambda_{\to}$.

For future works, the extension of the idea of *opposite types* to more complex systems, such as PTS, where we have dependent types, can be addressed. Also, the possibility of applying the method of Joachimski and Matthes (2003) for strong normalization to other systems is an interesting issue.

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