Journal of Algebra 505 (2018) 1–32

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Wedderburn principal theorem for Jordan superalgebras I

F.A. Gómez-González [1]

Instituto de Matematicas, Universidad de Antioquia, Colombia

ARTICLE INFO

Article history:
Received 19 December 2014
Available online 8 March 2018
Communicated by Alberto Elduque

MSC:
17A15
17A70
17C50

Keywords:
Jordan superalgebras
Wedderburn Theorem
Decomposition theorem
Semisimple Jordan superalgebras

ABSTRACT

We consider finite dimensional Jordan superalgebras $\mathfrak{J}$ over an algebraically closed field of characteristic 0, with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$ and $\mathfrak{J}/\mathcal{N}$ is a simple Jordan superalgebra of one of the following types: Kac $\mathcal{K}_{10}$, Kaplansky $\mathcal{K}_3$, superform or $\mathcal{D}_t$.
We prove that an analogue of the *Wedderburn Principal Theorem (WPT)* holds if certain restrictions on the types of irreducible subbimodules of $\mathcal{N}$ are imposed, where $\mathcal{N}$ is considered as a $\mathfrak{J}/\mathcal{N}$-bimodule. Using counterexamples, it is shown that the imposed restrictions are essential.

1. Introduction

In 1892, Th. Molien [1,2] proved that for any finite-dimensional associative algebra $\mathcal{A}$ with nilpotent radical $\mathcal{N}$ over the complex field there exists a subalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$. This result was generalized by J. H. Maclagan-Wedderburnn [3] for all finite dimensional associative algebras over an arbitrary field. This result is known as the Wedderburn's Principal Theorem (WPT). Analogues of the

E-mail address: faber.gomez@udea.edu.co.
[1] Partially supported by CAPES/CNPq IEL Nacional, Brazil, and Universidad de Antioquia, Colombia.

https://doi.org/10.1016/j.jalgebra.2018.03.001
0021-8693/

WPT were proved for finite-dimensional alternative algebras by R. D. Schafer [4], and for finite-dimensional Jordan algebras by A. A. Albert [5], A. J. Penico [6], V. G. Askinuze [7], E. J. Taft [8]. Thus it is natural to try to extend this result to superalgebras.

In the case of finite dimensional alternative superalgebras $\mathcal{A}$ over a field of characteristic zero, N. A. Pisarenko [9] proved an analogue to the WPT. He proved that the theorem holds if some restrictions are imposed over summands in the semisimple superalgebra $\mathcal{A}/\mathcal{N}$. It was also shown with counter-examples that the restrictions are essential.

In the current paper, we consider finite dimensional Jordan superalgebras $\mathcal{A}$ over a field of characteristic zero with radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N}$ is a simple Jordan superalgebra of one of the following types: Kac $\mathcal{K}_{10}$, Kaplansky $\mathcal{K}_3$, superform or $\mathcal{D}_t$.

It's proved that a Wedderburn decomposition is possible with certain essential restrictions

This paper is organized as follows. In Section 2, the basic examples of Jordan superalgebras are given. Sections 3–6 contain the proof of the Main Theorem. In Section 3, the necessary reductions are done. Sections 4–6 are devoted to the proofs of the theorems for corresponding simple quotients. Finally, in Section 7, the main theorem is deduced.

Note that the cases $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$, $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ and JP_n $n \geq 3$ are considered in [20], [21] and [22] respectively. The other cases, when $\mathfrak{J}/\mathcal{N}$ is isomorphic to

$\mathrm{JP}_n(\mathbb{F})$, $\mathcal{Q}_n(\mathbb{F})^{(+)}$, $\mathcal{K}_3 \oplus \mathcal{K}_3 \oplus \cdots \oplus \mathcal{K}_3 \oplus \mathbb{F} \cdot 1$, and Kantor superalgebra, are to be considered in future papers.

We also stress that the Main Theorem implies that the second cohomology group $H^2(\mathfrak{J}, \mathcal{N})$ is not trivial for some simple Jordan superalgebra $\mathfrak{J}$ and some irreducible $\mathfrak{J}$-bimodule $\mathcal{N}$. This gives one more subject of interest to be considered in future papers.

2. Jordan superalgebras, definition and some examples

Throughout the paper, all algebras are considered over an algebraically closed field $\mathbb{F}$ of characteristic zero.

Recall that an algebra $\mathcal{A}$ is said to be a *superalgebra* if it is a direct sum $\mathcal{A} = \mathcal{A}_0 \dotplus \mathcal{A}_1$ of vector spaces satisfying the relation $\mathcal{A}_i\mathcal{A}_j \subseteq \mathcal{A}_{i+j(\mathrm{mod}\ 2)}$, i.e. $\mathcal{A}$ is a $\mathbb{Z}_2$-graded algebra. For an element $a \in \mathcal{A}_i$, $i = 0, 1$, the number $|a| = i$ denotes a *parity* of a.

Let $\Gamma = \mathbf{alg}\,\langle 1, e_i,\ i \in \mathbb{Z}^+ | e_ie_j + e_je_i = 0 \rangle$ be the Grassmann algebra. Then $\Gamma = \Gamma_0 \dotplus \Gamma_1$, where Γ_0 and Γ_1 are the spans of all monomials of even and odd lengths, respectively. It is not difficult to see that Γ has a superalgebra structure.

For a superalgebra $\mathcal{A} = \mathcal{A}_0 \dotplus \mathcal{A}_1$, we define the *Grassmann envelope of* $\mathcal{A}$ as follows: $\Gamma(\mathcal{A}) = \Gamma_0 \otimes \mathcal{A}_0 \dotplus \Gamma_1 \otimes \mathcal{A}_1$. We assume that $\mathfrak{M}$ is a homogeneous variety of algebras. The superalgebra $\mathcal{A}$ is said to be an $\mathfrak{M}$-*superalgebra* if the Grassmann envelope $\Gamma(\mathcal{A})$ lies in $\mathfrak{M}$. Following this definition, one can consider associative, alternative, Lie, Jordan, etc. superalgebras.

We recall that an algebra $\mathfrak{J}$ is said to be Jordan algebra if its multiplication satisfies the identity $ab = ba$ of commutativity and the Jordan identity $(a^2b)a = a^2(ba)$. In this paper, we consider algebras over a field characteristic zero. Thus, the Jordan identity is equivalent to its complete linearization

$$((ac)b)d + ((ad)b)c + ((cd)b)a = (ac)(bd) + (ad)(bc) + (cd)(ba).$$

An associative superalgebra is just a $\mathbb{Z}_2$-graded associative algebra, but it is not the case in general terms. It is easy to see that a Jordan superalgebra it is not always a Jordan algebra. One can verify that a superalgebra $\mathfrak{J} = \mathfrak{J}_0 \dotplus \mathfrak{J}_1$ is a Jordan superalgebra iff it satisfies the superidentities

$$a_i a_j = (-1)^{ij} a_j a_i, \tag{1}$$

$$\begin{aligned} ((a_i a_j)a_k)a_l + (-1)^{l(k+j)+kj}((a_i a_l)a_k)a_j + (-1)^{i(j+k+l)+kl}((a_j a_l)a_k)a_i = \\ = (a_i a_j)(a_k a_l) + (-1)^{l(k+j)}(a_i a_l)(a_j a_k) + (-1)^{jk}(a_i a_k)(a_j a_l) \end{aligned} \tag{2}$$

for homogeneous elements $a_t \in \mathfrak{J}_t$, $t \in \{i, j, k, l\}$.

We stress that, in view of the restriction on the characteristic of ground field, superidentity (1) yields that the Jordan superalgebra $\mathfrak{J} = \mathfrak{J}_0 \dotplus \mathfrak{J}_1$ is a ($\mathbb{Z}_2$-graded) Jordan algebra iff $(\mathfrak{J}_1)^2 = 0$.

Throughout the paper, we denote by $\dotplus$ a direct sum of vector space, by $+$ denote a sum of vector space and by $\oplus$ we denote a direct sum of superalgebras.

Some examples of Jordan Superalgebras.

Let $\mathcal{A}$ be an associative superalgebra with multiplication ab. We define on the vector space $\mathcal{A}$ a new multiplication $a \circ b = \frac{1}{2}(ab + (-1)^{|a||b|}ba)$ for a, $b \in \mathcal{A}_0 \cup \mathcal{A}_1$. It is not hard to verify that $\mathcal{A}$ gains a structure of Jordan superalgebra with respect to the defined multiplication. We denote this superalgebra by $\mathcal{A}^{(+)}$.

C.T.C. Wall [14] proved that every associative simple finite-dimensional superalgebra over an algebraically closed field $\mathbb{F}$ is isomorphic to one of the following associative superalgebras:

(i) $\mathcal{A} = \mathcal{M}_{n|m}(\mathbb{F}), \quad \mathcal{A}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\}, \quad \mathcal{A}_1 = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\},$

(ii) $\mathcal{A} = \mathcal{Q}_n(\mathbb{F}) = \mathcal{Q}(n), \quad \mathcal{A}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\}, \quad \mathcal{A}_1 = \left\{ \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix} \right\}.$

where a, $h \in \mathcal{M}_n(\mathbb{F})$, $d \in \mathcal{M}_m(\mathbb{F})$, $b \in \mathcal{M}_{n\times m}(\mathbb{F})$, $c \in \mathcal{M}_{m\times n}(\mathbb{F})$.

(I) Applying the multiplication "$\circ$" to the associative superalgebras $\mathcal{Q}_n(\mathbb{F})$ and $\mathcal{M}_{n|m}(\mathbb{F})$, we get the Jordan superalgebras $\mathcal{Q}_n(\mathbb{F})^{(+)}$ and $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$ respectively.

(II) Let $\mathcal{A}$ be an associative superalgebra. A graded linear mapping $* : \mathcal{A} \longrightarrow \mathcal{A}$ is called *superinvolution* if $(a^*)^* = a$ and $(ab)^* = (-1)^{|a||b|}b^*a^*$. By $\mathcal{H}(\mathcal{A}, *)$ denote the set of symmetric elements of $\mathcal{A}$ relative to $*$. Then $\mathcal{H}(\mathcal{A}, *)$ is a Jordan superalgebra such that $\mathcal{H}(\mathcal{A}, *) \subseteq \mathcal{A}^{(+)}$.

Let I_n, I_m be the identity matrices of order n and m respectively, t be the transposition and

$$U = -U^t = -U^{-1} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}.$$

Consider linear mappings

$$\mathrm{Osp} : \mathcal{M}_{n|2m}(\mathbb{F}) \longrightarrow \mathcal{M}_{n|2m}(\mathbb{F}) \text{ and } \sigma : \mathcal{Q}_n(\mathbb{F}) \longrightarrow \mathcal{Q}_n(\mathbb{F})$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\mathrm{Osp}} = \begin{pmatrix} I_n & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} a^t & -c^t \\ b^t & d^t \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & U^{-1} \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sigma} = \begin{pmatrix} d^t & -b^t \\ c^t & a^t \end{pmatrix}. \tag{3}$$

It is easy to check that Osp and σ are superinvolutions and its Jordan superalgebras are $\mathcal{H}(\mathcal{M}_{n|2m}(\mathbb{F}), \mathrm{Osp})$ and $\mathcal{H}(\mathcal{Q}_n(\mathbb{F}), \sigma)$. We denote these superalgebras by $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$ and $\mathrm{JP}_n(\mathbb{F})$ respectively.

One also may consider the following Jordan superalgebras.

(III) The 4-dimensional 1-parametric family $\mathcal{D}_t = (\mathbb{F} \cdot e_1 + \mathbb{F} \cdot e_2) \dotplus (\mathbb{F} \cdot x + \mathbb{F} \cdot y)$, with nonzero products given by $e_i^2 = e_i$, $e_i x = x e_i = \frac{1}{2} x$, $e_i y = y e_i = \frac{1}{2} y$, $xy = -yx = e_1 + t e_2$. The superalgebra $\mathcal{D}_t$ is simple for $t \neq 0$.

(IV) The non unital 3-dimensional Kaplansky superalgebra $\mathcal{K}_3 = \mathbb{F} \cdot e \dotplus (\mathbb{F} \cdot x + \mathbb{F} \cdot y)$, with nonzero products $ex = xe = \frac{1}{2} x$, $ey = ye = \frac{1}{2} y$, $xy = -yx = e$. The superalgebra $\mathcal{K}_3$ is simple. Note also that the unital hull $\mathcal{K}_3 \oplus 1$ is isomorphic to $\mathcal{D}_0$.

(V) Let $V = V_0 \oplus V_1$ be a vector superspace. We say that a bilinear mapping $f : V \times V \longrightarrow \mathbb{F}$ is a superform if f is symmetric over V_0, skew-symmetric over V_1, and satisfies $f(V_0, V_1) = 0$. Consider a superalgebra $\mathfrak{J} = (\mathbb{F} \cdot 1 \oplus V_0) \dotplus V_1$ with the unit 1 and the multiplication $v \cdot w = f(v, w) \cdot 1$, $(v, w \in V)$. If f is a non-degenerate superform and $\dim V_0 > 1$, then $\mathfrak{J}$ is a simple Jordan superalgebra.

(VI) The introduced by Kac 10-dimensional superalgebra $\mathcal{K}_{10}$ is a simple Jordan superalgebra. A detailed description of $\mathcal{K}_{10}$ is given in Section 4.

(VII) I. L. Kantor [11] defined a simple Jordan superalgebra structure in the finite-dimensional Grassmann algebra generated by $e_1, \ldots, e_n$.

It is known [10,11], that every simple finite-dimensional Jordan superalgebra over $\mathbb{F}$ is isomorphic to one of the superalgebras $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$, $\mathcal{Q}_n(\mathbb{F})^{(+)}$, $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$, $\mathrm{JP}_n(\mathbb{F})$, $\mathcal{D}_t$, $\mathcal{K}_3$, $\mathcal{K}_{10}$, a superalgebra of superform or a Kantor superalgebra.

Let $\mathfrak{J}$ be a finite dimensional Jordan superalgebra, then a $\mathfrak{J}$ bimodule $\mathcal{M} = \mathcal{M}_0 \dot{+} \mathcal{M}_1$ is called a Jordan bimodule if the corresponding split null extension $\mathcal{E} = \mathfrak{J} \oplus \mathcal{M}$ is a Jordan superalgebra [17]. Recalling that a split null extension is a direct sum $\mathfrak{J} \oplus \mathcal{M}$ of vector spaces with a multiplication that extends the multiplication in $\mathfrak{J}$ through the action of $\mathfrak{J}$ on $\mathcal{M}$, while the product of two arbitrary elements in $\mathcal{M}$ is zero.

Let $\mathcal{M}$ be a $\mathfrak{J}$-bimodule. The opposite bimodule $\mathcal{M}^{\mathrm{op}} = \mathcal{M}_0^{\mathrm{op}} \dot{+} \mathcal{M}_1^{\mathrm{op}}$ is defined by the conditions $\mathcal{M}_0^{\mathrm{op}} = \mathcal{M}_1$, $\mathcal{M}_1^{\mathrm{op}} = \mathcal{M}_0$, and by the following action of $\mathfrak{J}$ over $\mathcal{M}^{\mathrm{op}}$: $a \cdot m^{\mathrm{op}} = (-1)^{|a|}(am)^{\mathrm{op}}$, $m^{\mathrm{op}} \cdot a = (ma)^{\mathrm{op}}$ for all $a \in \mathfrak{J}_0 \cup \mathfrak{J}_1$, $m \in \mathcal{M}_0^{\mathrm{op}} \cup \mathcal{M}_1^{\mathrm{op}}$. Whenever $\mathcal{M}$ is a Jordan $\mathfrak{J}$-superbimodule, $\mathcal{M}^{\mathrm{op}}$ is a Jordan one as well.

Let $\mathcal{A} = \mathfrak{J}$ as a vector superspace and let am, ma with $m \in \mathfrak{J}$, $a \in \mathcal{A}$ be the products as defined in the superalgebra $\mathfrak{J}$. It is easy to see that $\mathcal{A}$ has a natural structure of $\mathfrak{J}$-bimodule. We call $\mathcal{A}$ a *regular bimodule.*

The irreducible bimodules over the Jordan superalgebras of superform, $\mathfrak{J}\mathrm{osp}_{n|2m}(\mathbb{F})$, $\mathrm{JP}_n(\mathbb{F})$, $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$, were classified by C. Martinez, E. Zelmanov [17]. C. Martinez, I. Shestakov, E. Zelmanov [18], classified the irreducible bimodules for Jordan superalgebras $\mathcal{Q}_n(\mathbb{F})^{(+)}$. Irreducible bimodules for Jordan superalgebra $\mathcal{D}_t$ and $\mathcal{K}_3$ were classified by C. Martinez, E. Zelmanov [16] and independently by Trushina [15]. C. Martinez [19], classified the irreducible bimodules over the Jordan superalgebra $\mathcal{M}_{1|1}(\mathbb{F})^{(+)}$ and A. S. Shtern [13] classified the irreducible bimodules over Jordan superalgebras of type $\mathcal{K}_{10}$, and Kantor superalgebra $\gamma(e_1, \ldots, e_n)$, $n \geq 4$

Peirce decomposition. Recall, that if $\mathfrak{J}$ is a Jordan (super)algebra with unity 1, and $\{e_1, \ldots, e_n\}$ is a set of pairwise orthogonal idempotents such that $1 = \sum_{i=1}^n e_i$, then $\mathfrak{J}$ admits Peirce decomposition [17], it is

$$\mathfrak{J} = \left(\bigoplus_{i=1}^{n} \mathfrak{J}_{ii}\right) \bigoplus \left(\bigoplus_{i<j} \mathfrak{J}_{ij}\right),$$

where

$$\mathfrak{J}_{ii} = \{\, x \in \mathfrak{J} : \quad e_i x = x, \}$$

and

$$\mathfrak{J}_{ij} = \{\, x \in \mathfrak{J} : \quad e_i x = \frac{1}{2}x, \quad e_j x = \frac{1}{2}x \ \text{ if } i \neq j\},$$

are the Peirce components of $\mathfrak{J}$ relative to the idempotents e_i, and e_j. Moreover the following relations hold when $i \neq k, l; j \neq k, l$

$$\mathfrak{J}_{ij}^2 \subseteq \mathfrak{J}_{ii} + \mathfrak{J}_{jj}, \quad \mathfrak{J}_{ij} \cdot \mathfrak{J}_{jk} \subseteq \mathfrak{J}_{ik}, \quad \mathfrak{J}_{ij} \cdot \mathfrak{J}_{kl} = 0.$$

3. Preliminary reductions for WPT

As in the case of Jordan algebras, we can make some restrictions before the main proof. To start we prove the following proposition.

Proposition 1. *Let $\mathfrak{J}$ be a Jordan superalgebra without 1 and with radical $\mathcal{N}$. If the WPT is valid for $\mathfrak{J}^{\#}$, then it is also valid for $\mathfrak{J}$.*

Proof. Let $\mathfrak{J}$ be a Jordan superalgebra without 1 and with radical $\mathcal{N}$. Consider $\mathfrak{J}^{\#} = \mathfrak{J} \oplus \mathbb{F} \cdot 1$. It is clear that $\mathcal{N}(\mathfrak{J}) = \mathcal{N}(\mathfrak{J}^{\#}) = \mathcal{N}$ and $\mathfrak{J}^{\#}/\mathcal{N} = (\mathfrak{J}/\mathcal{N})^{\#} = \mathfrak{J}/\mathcal{N} \oplus \mathbb{F} \cdot \bar{1}$. By assumption, there exists $\mathcal{S}_1 \subseteq \mathfrak{J}^{\#}$ such that $\mathcal{S}_1 \cong \mathfrak{J}^{\#}/\mathcal{N} \cong (\mathfrak{J}/\mathcal{N})^{\#}$, $\mathcal{S}_1 \cap \mathcal{N} = (0)$, $\mathfrak{J}^{\#} = \mathcal{S}_1 \oplus \mathcal{N}$. Denote $\mathcal{S} = \mathcal{S}_1 \cap \mathfrak{J}$, then $\mathcal{S} \cap \mathcal{N} = (0)$. Let us show that $\mathcal{S} \oplus \mathcal{N} = \mathfrak{J}$. Take $a \in \mathfrak{J}$, then $a = s_1 + n$, $s_1 \in \mathcal{S}_1$, $n \in \mathcal{N}$. But $s_1 = a - n \in \mathfrak{J}$. Hence, $s_1 \in \mathfrak{J} \cap \mathcal{S}_1 = \mathcal{S}$ and $a \in \mathcal{S} \oplus \mathcal{N}$. Finally, $\mathcal{S} \cong \mathcal{S}/(\mathcal{S} \cap \mathcal{N}) \cong (\mathcal{S} \oplus \mathcal{N})/\mathcal{N} \cong \mathfrak{J}/\mathcal{N}$. □

Let $\mathfrak{J}$ be a unital Jordan superalgebra of $\dim \mathfrak{J} = n$. Assume that for any unital Jordan superalgebra of dimension less that n the WPT is true. A base for induction is $\dim_{\mathbb{F}} \mathfrak{J} = 1$, $\mathfrak{J} = \mathbb{F} \cdot 1$.

Proposition 2. *Let $\mathfrak{J}/\mathcal{N} = \mathfrak{J}_1 \oplus \cdots \oplus \mathfrak{J}_k$, where $\mathfrak{J}_i$ are unital simple Jordan superalgebras with $\mathcal{N}(\mathfrak{J}_i) = 0$. If WPT is valid for $\mathfrak{J}_k$, $k > 1$, then the WPT is true for $\mathfrak{J}$.*

Proof. Denote by e_i the identity elements in $\mathfrak{J}_i$. Then (by Jordan algebras results) there are orthogonal idempotents $f_i \in \mathfrak{J}$ such that $e_i = f_i + \mathcal{N}$, $i = 1, 2, \ldots, k$. Consider $\mathfrak{J}_1(f_i) = \{f_i, \mathfrak{J}, f_i\}$, then $\mathfrak{J}_1(f_i)/(\mathfrak{J}_1(f_i) \cap \mathcal{N}) \cong \mathfrak{J}_i$. By virtue of $\mathcal{N}(\mathfrak{J}_i) = 0$, we have the inclusion $\mathcal{N}(\mathfrak{J}_1(f_i)) \subseteq \mathfrak{J}_1(f_i) \cap \mathcal{N}$. Since the inverse inclusion is obvious, we have the equality $\mathcal{N}(\mathfrak{J}_1(f_i)) = \mathfrak{J}_1(f_i) \cap \mathcal{N}$. If $k > 1$, then $\dim \mathfrak{J}_1(f_i) < \dim \mathfrak{J}$ and by the inductive hypothesis, there exists $\mathcal{S}_i \subseteq \mathfrak{J}_1(f_i)$, $\mathcal{S}_i \cong \mathfrak{J}_i/(\mathcal{N} \cap \mathfrak{J}_i)$. Note that $\mathcal{S}_i \cdot \mathcal{S}_j = 0$. Further, $\mathcal{S} = \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_k$ is a direct sum and $\mathcal{S} \cong \mathfrak{J}_1 \oplus \cdots \oplus \mathfrak{J}_k$. □

Now by the Zelmanov Theorem [12], in the case of characteristic zero, it is sufficient to prove the WPT for unital finite dimensional Jordan superalgebras $\mathfrak{J}$ satisfying one of the following conditions:

1. $\mathfrak{J}/\mathcal{N}$ is simple unital;
2. $\mathfrak{J}/\mathcal{N} = (\mathcal{K}_3 \oplus \mathcal{K}_3 \oplus \cdots \oplus \mathcal{K}_3) \oplus \mathbb{F} \cdot 1$, where $\mathcal{K}_3$ is the Kaplansky superalgebra.

Theorem 3. *Let $\mathfrak{J}$ be a finite dimensional semisimple Jordan superalgebra, i.e $\mathcal{N}(\mathfrak{J}) = 0$, where $\mathcal{N}$ is the solvable radical. Let $\mathfrak{M}(\mathfrak{J})$ be a class of finite dimensional Jordan $\mathfrak{J}$-bimodules $\mathcal{N}$ such that $\mathfrak{M}(\mathfrak{J})$ is closed with respect to subbimodules and homomor-*

phic images. Denote by $\mathcal{K}(\mathfrak{M}, \mathfrak{J})$ the class of finite dimensional Jordan superalgebras $\mathcal{A}$ that satisfy the following conditions:

1. $\mathcal{A}/\mathcal{N}(\mathcal{A}) \cong \mathfrak{J}$,
2. $\mathcal{N}(\mathcal{A})^2 = 0$,
3. $\mathcal{N}(\mathcal{A})$ *considered as a* $\mathfrak{J}$*-bimodule, in* $\mathfrak{M}(\mathfrak{J})$.

Then if WPT is true for all superalgebras $\mathcal{B} \in \mathcal{K}(\mathfrak{M}, \mathfrak{J})$ with the restriction that the radical $\mathcal{N}(\mathcal{B})$ is an irreducible $\mathfrak{J}$-bimodule, then it is true for all superalgebras $\mathcal{A}$ from $\mathcal{K}(\mathfrak{M}, \mathfrak{J})$.

Proof. We use the induction on $\dim \mathcal{A}$. The base of induction is provide by the case $\dim \mathcal{A} = \dim \mathfrak{J}$, so $\mathcal{A} = \mathfrak{J}$, $\mathcal{N}(\mathcal{A}) = 0$. Assume that the theorem is true for all Jordan superalgebras $\mathcal{B} \in \mathcal{K}(\mathfrak{M}, \mathfrak{J})$ with $\dim \mathcal{B} < \dim \mathcal{A}$. Let us set by $\mathcal{N} = \mathcal{N}(\mathcal{A})$. If $\mathcal{N}$ is an irreducible $\mathfrak{J}$-bimodule, then the theorem is true by the conjecture. Suppose that $\mathcal{N}$ is not irreducible, then let us take a minimal $\mathfrak{J}$-bimodule $\mathcal{M}$ contained in $\mathcal{N}$. Since that $\mathcal{A}$ is unital $\mathfrak{J}\mathcal{M} = \mathcal{A}\mathcal{M} = \mathcal{M}$, therefore $\mathcal{M}$ is irreducible. Observe that $\mathcal{N}/\mathcal{M} \neq 0$, otherwise $\mathcal{N} = \mathcal{M}$ would be irreducible. We see that $\frac{\mathcal{A}/\mathcal{M}}{\mathcal{N}/\mathcal{M}} \cong \mathcal{A}/\mathcal{N} \cong \mathfrak{J}$.

Since $\mathcal{A}/\mathcal{N}$ is semisimple, we have that $\mathcal{N}(\mathcal{A}/\mathcal{M}) \subseteq \mathcal{N}/\mathcal{M}$. But $(\mathcal{N}/\mathcal{M})^2 = 0$. Thus $\mathcal{N}(\mathcal{A}/\mathcal{M}) = \mathcal{N}/\mathcal{M}$. Observe that $\mathcal{A}/\mathcal{M} \in \mathcal{K}(\mathfrak{M}, \mathfrak{J})$ and $\dim \mathcal{A}/\mathcal{M} \leq \dim \mathcal{A}$. Therefore there exists a subsuperalgebra $\overline{\mathcal{S}} \subseteq \mathcal{A}/\mathcal{M}$ such that $\overline{\mathcal{S}} \cong \frac{\mathcal{A}/\mathcal{M}}{\mathcal{N}/\mathcal{M}} \cong \mathcal{A}/\mathcal{N}$ and $\mathcal{A}/\mathcal{M} = \overline{\mathcal{S}} \oplus \mathcal{N}/\mathcal{M}$. By the main theorems on homomorphisms, there is a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{M} \subseteq \mathcal{S}$ and $\mathcal{S}/\mathcal{M} \cong \overline{\mathcal{S}} \cong \mathcal{A}/\mathcal{N} \cong \mathfrak{J}$. We observe that $\mathcal{S} \in \mathcal{K}(\mathfrak{M}, \mathfrak{J})$ and $\mathcal{N}(\mathcal{S}) = \mathcal{M}$ is an irreducible $\mathfrak{J}$-bimodule. By the assumption, WPT is true for $\mathcal{S}$, hence there is a subsuperalgebra $\mathcal{S}_1 \subseteq \mathcal{S} \subseteq \mathcal{A}$, such that $\mathcal{S}_1 \cong \mathcal{S}/\mathcal{M} \cong \mathcal{A}/\mathcal{N}$. Since $\mathcal{S}_1$ is semisimple, $\mathcal{N} \cap \mathcal{S} \subseteq \mathcal{N}(\mathcal{S}_1) = 0$. Furthermore, $\dim \mathcal{S}_1 = \dim \mathcal{A} - \dim \mathcal{N}$. Hence, $\dim(\mathcal{N} + \mathcal{S}_1) = \dim \mathcal{A}$ and $\mathcal{A} = \mathcal{N} \oplus \mathcal{S}_1$. □

Let $V_1, \ldots, V_k$ be irreducible $\mathfrak{J}$-bimodules, and $\mathfrak{J}$ be a simple Jordan superalgebra. Let $\overline{\mathfrak{M}}(\mathfrak{J}; V_1, \ldots, V_k) = \{V \,/\, V$ is a $\mathfrak{J}$-bimodule, doesn't containing amoung its irreducible subbimodule any copy isomorphic to one of the bimodules $V_1, \ldots, V_k\}$ It is clear that $\mathfrak{M}$ is closed with respect to taking of subbimodules and homomorphic images. Thus it satisfies the conditions of Theorem 3.

In each section, we assume that $\mathcal{A}$ is a finite dimensional Jordan superalgebra over $\mathbb{F}$, with radical $\mathcal{N}$ and such that $\mathcal{N}^2 = 0$, $\mathcal{A}/\mathcal{N} \cong \mathfrak{J}$, where $\mathfrak{J}$ is a simple Jordan superalgebra and $\mathcal{N}$ is an irreducible $\mathfrak{J}$-bimodule. Moreover, if $b_1, b_2, \ldots, b_n$ is an additive base of $\mathfrak{J}_0$, then we assume that $\widetilde{b}_1, \widetilde{b}_2, \ldots, \widetilde{b}_n$ is an additive base of $\mathcal{S}_0 \subseteq \mathcal{A}_0$, $\mathcal{A}_0 = \mathcal{S}_0 \oplus \mathcal{N}_0$ and, $\widetilde{b}_i \cdot \widetilde{b}_j = \widetilde{b_i b_j}$. If $\mathcal{A}_1/\mathcal{N}_1 \cong \mathfrak{J}_1$ and $v_1, \ldots, v_k$ is an additive base of $\mathfrak{J}_1$, we can assume that $\bar{v}_1, \ldots, \bar{v}_k$ is an additive base of $\mathcal{A}_1/\mathcal{N}_1$, and we shall find $\widetilde{v}_1, \ldots, \widetilde{v}_k$ additive base of $\mathcal{S}_1 \subseteq \mathcal{A}_1$ such that $\widetilde{v}_i \cdot \widetilde{v}_j = \widetilde{v_i v_j}$ and $\widetilde{v}_i \cdot \widetilde{b}_j = \widetilde{v_i b_j}$, and $\mathcal{A}_1 = \mathcal{S}_1 \oplus \mathcal{N}_1$. Therefore, we obtain that there exist $\mathcal{S} = \mathcal{S}_0 \dotplus \mathcal{S}_1 \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathfrak{J}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$. In each case we can assume that $\widetilde{a} \cdot n = an$, where $\widetilde{a} \in \mathcal{A}_0 \dot\cup \mathcal{A}_1$, $a \in \mathfrak{J}_0 \dot\cup \mathfrak{J}_1$, $n \in \mathcal{N}_0 \dot\cup \mathcal{N}_1$

4. Kac superalgebra

In this section, we consider the 10-dimensional Kac superalgebra $\mathcal{K}_{10} = \mathfrak{J}_0 \dot{+} \mathfrak{J}_1$, where

$$\mathfrak{J}_0 = (\mathbb{F} \cdot e + \sum_{i=1}^{4} \mathbb{F} \cdot v_i) \oplus \mathbb{F} \cdot f, \quad \mathfrak{J}_1 = \mathbb{F} \cdot x_1 + \mathbb{F} \cdot x_2 + \mathbb{F} \cdot y_1 + \mathbb{F} \cdot y_2,$$

and all nonzero products of the basis elements are the following

$$e^2 = e, \quad e \cdot v_i = v_i, \quad f^2 = f, \quad v_1 \cdot v_2 = v_3 \cdot v_4 = 2e. \tag{4}$$

$$\begin{array}{llll} f \cdot x_j = \frac{1}{2}x_j, & f \cdot y_j = \frac{1}{2}y_j, & e \cdot x_j = \frac{1}{2}x_j, & e \cdot y_j = \frac{1}{2}y_j, \\ y_1 \cdot v_1 = x_2, & y_2 \cdot v_1 = -x_1, & x_1 \cdot v_2 = -y_2, & x_2 \cdot v_2 = y_1, \\ x_2 \cdot v_3 = x_1, & y_1 \cdot v_3 = y_2, & x_1 \cdot v_4 = x_2, & y_2 \cdot v_4 = y_1. \end{array} \tag{5}$$

$$\begin{array}{l} x_1 \cdot x_2 = v_1, \quad x_1 \cdot y_2 = v_3, x_2 \cdot y_1 = v_4, \quad y_1 \cdot y_2 = v_2, \\ x_i \cdot y_i = e - 3f. \end{array} \tag{6}$$

The zero characteristic of the ground field implies that $\mathcal{K}_{10}$ is a simple Jordan superalgebra. Using Theorem 3 and the classifications of irreducible bimodules over $\mathcal{K}_{10}$ we need to consider two cases, the regular bimodule and its opposite. Assume that $a \leftrightarrow e$, $b \leftrightarrow f$, $u_i \leftrightarrow v_i$, $m_j \leftrightarrow x_j$, and $n_j \leftrightarrow y_j$ for $i = 1, 2, 3, 4$, $j = 1, 2$, thus

$$\begin{array}{l} (\mathcal{R}\text{eg}\, K_{10})_0 = (\mathbb{F} \cdot a + \mathbb{F} \cdot u_1 + \mathbb{F} \cdot u_2 + \mathbb{F} \cdot u_3 + \mathbb{F} \cdot u_4) \oplus \mathbb{F} \cdot b, \\ (\mathcal{R}\text{eg}\, K_{10})_1 = \mathbb{F} \cdot m_1 + \mathbb{F} \cdot m_2 + \mathbb{F} \cdot n_1 + \mathbb{F} \cdot n_2 \end{array}$$

Let $\mathcal{A}_0 = (\mathcal{S}_0 \oplus \mathcal{N}_0)$ and $\mathcal{A}_1/\mathcal{N}_1 \cong (\mathcal{K}_{10})_1$ where $\mathcal{S}_0 = \mathbb{F} \cdot \widetilde{e} + \sum_{i=1}^{4} \mathbb{F} \cdot \widetilde{v}_i \oplus \mathbb{F} \cdot \widetilde{f}$, and $(\mathcal{K}_{10})_0 \cong \mathcal{S}_0$, and $\mathcal{A}_1/\mathcal{N}_1 = \mathbb{F} \cdot \bar{x}_1 + \mathbb{F} \cdot \bar{x}_2 + \mathbb{F} \cdot \bar{y}_1 + \mathbb{F} \cdot \bar{y}_2$.

4.1. $\mathcal{N}$ is isomorphic to regular bimodule

Lemma 4.

$$\begin{array}{llll} \widetilde{f} \cdot \widetilde{x}_j = \frac{1}{2}\widetilde{x}_j, & \widetilde{f} \cdot \widetilde{y}_j = \frac{1}{2}\widetilde{y}_j, & \widetilde{e} \cdot \widetilde{x}_j = \frac{1}{2}\widetilde{x}_j, & \widetilde{e} \cdot \widetilde{y}_j = \frac{1}{2}\widetilde{y}_j, \\ \widetilde{y}_1 \cdot \widetilde{v}_1 = \widetilde{x}_2, & \widetilde{y}_2 \cdot \widetilde{v}_1 = -\widetilde{x}_1, & \widetilde{x}_1 \cdot \widetilde{v}_2 = -\widetilde{y}_2, & \widetilde{x}_2 \cdot \widetilde{v}_2 = \widetilde{y}_1, \\ \widetilde{x}_2 \cdot \widetilde{v}_3 = \widetilde{x}_1, & \widetilde{y}_1 \cdot \widetilde{v}_3 = \widetilde{y}_2, & \widetilde{x}_1 \cdot \widetilde{v}_4 = \widetilde{x}_2, & \widetilde{y}_2 \cdot \widetilde{v}_4 = \widetilde{y}_1, \end{array} \tag{7}$$

Proof. First prove $\widetilde{e}\widetilde{x}_1 = \frac{1}{2}\widetilde{x}_1$. To start, we can assume that there exist scalars $\lambda_{s_1}^{x_1 e} s_1$, such that $\widetilde{e}\widetilde{x}_1 = \frac{1}{2}\widetilde{x}_1 + \Lambda_e^{x_1} = \frac{1}{2}\widetilde{x}_1 + \lambda_{m_1}^{x_1 e} m_1 + \lambda_{m_2}^{x_1 e} m_2 + \lambda_{n_1}^{x_1 e} n_1 + \lambda_{n_2}^{x_1 e} n_2$.

It is easy to see that $\Lambda_e^{x_i} \cdot \widetilde{e} = \frac{1}{2}\Lambda_e^{x_i}$. Substituting $a_i = \widetilde{x}_1$ and $a_j = a_k = a_l = \widetilde{e}$ in (2), we get

$$2((\widetilde{x}_1 \cdot \widetilde{e}) \cdot \widetilde{e}) \cdot \widetilde{e} + \widetilde{x}_1 \cdot \widetilde{e} = 3(\widetilde{x}_1 \cdot \widetilde{e}) \cdot \widetilde{e}.$$

Combining the above equality with $\widetilde{x}_1 \cdot \widetilde{e} = \frac{1}{2}\widetilde{x}_1 + \Lambda_e^{x_1}$, we have $\frac{5}{2}\Lambda_e^{x_1} = 3\Lambda_e^{x_1}$, therefore, $\lambda_{m_1}^{x_1 e} = \lambda_{m_2}^{x_1 e} = \lambda_{n_1}^{x_1 e} = \lambda_{n_2}^{x_1 e} = 0$. Thus, $\Lambda_e^{x_1} = 0$ and $\widetilde{x}_1 \cdot \widetilde{e} = \frac{1}{2}\widetilde{x}_1$. Similarly one can prove the equalities $\widetilde{x}_2 \cdot \widetilde{e} = \frac{1}{2}\widetilde{x}_2$, $\widetilde{x}_i \cdot \widetilde{f} = \frac{1}{2}\widetilde{x}_i$, $\widetilde{y}_i \cdot \widetilde{e} = \frac{1}{2}\widetilde{y}_i$ and $\widetilde{y}_i \cdot \widetilde{f} = \frac{1}{2}\widetilde{y}_i$.

Now we shall prove that other equalities in (7) hold. Let Λ_x^{ij} be the radical part in the product $\widetilde{x}_i \cdot \widetilde{v}_j$ where $\Lambda_x^{ij} = \lambda_{m_1}^{ijx} m_1 + \lambda_{m_2}^{ijx} m_2 + \lambda_{n_1}^{ijx} n_1 + \lambda_{n_2}^{ijx} n_2$ for some scalars $\lambda_{m_1}^{ijx}$, $\lambda_{m_2}^{ijx}$, $\lambda_{n_1}^{ijx}$ and $\lambda_{n_2}^{ijx}$. (Similarly, Λ_y^{ij}.)

First note that $(\Lambda_s^{ij} \cdot \widetilde{v}_j) \cdot \widetilde{v}_j = 0$ for $s = x$ or $s = y$.

We set $a_i = \widetilde{y}_1$ and $a_j = a_k = a_l = \widetilde{v}_1$ in (2). Since $\widetilde{v}_i^2 = 0$, we have

$$0 = ((\widetilde{y}_1 \cdot \widetilde{v}_1) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = ((\widetilde{x}_2 + \Lambda_y^{11}) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = (\widetilde{x}_2 \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = \Lambda_x^{21} \cdot \widetilde{v}_1.$$

Thus, $\lambda_{n_1}^{21x} m_2 - \lambda_{n_2}^{21x} m_1 = 0$. The linear independence of m_1 and m_2 implies $\lambda_{n_1}^{21x} = \lambda_{n_2}^{21x} = 0$ and therefore $\Lambda_x^{21} = \lambda_{m_1}^{21x} m_1 + \lambda_{m_2}^{21x} m_2$. Similarly one can prove that $\Lambda_x^{11} = \lambda_{m_1}^{11x} m_1 + \lambda_{m_2}^{11x} m_2$, $\Lambda_y^{12} = \lambda_{n_1}^{12y} n_1 + \lambda_{n_2}^{12y} n_2$, $\Lambda_y^{22} = \lambda_{n_1}^{22y} n_1 + \lambda_{n_2}^{22y} n_2$, $\Lambda_x^{13} = \lambda_{m_2}^{13x} m_2 + \lambda_{n_1}^{13x} n_1$, $\Lambda_y^{23} = \lambda_{m_2}^{23y} m_2 + \lambda_{n_1}^{23y} n_1$, $\Lambda_x^{24} = \lambda_{m_1}^{24x} m_1 + \lambda_{n_2}^{24x} n_2$ and $\Lambda_y^{14} = \lambda_{m_1}^{14y} m_1 + \lambda_{n_2}^{14y} n_2$.

Substituting $a_i = \widetilde{y}_1$, $a_j = \widetilde{v}_1$ and $a_k = a_l = \widetilde{v}_2$ in (2), we have,

$$((\widetilde{y}_1 \cdot \widetilde{v}_1) \cdot \widetilde{v}_2) \cdot \widetilde{v}_2 + ((\widetilde{y}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_2) \cdot \widetilde{v}_1 + ((\widetilde{v}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_2) \cdot \widetilde{y}_1 = 2(\widetilde{v}_1 \cdot \widetilde{v}_2) \cdot (\widetilde{y}_1 \cdot \widetilde{v}_2) \tag{8}$$

Observe that $\Lambda_y^{12} = \lambda_{n_1}^{12y} n_1 + \lambda_{n_2}^{12y} n_2$, therefore $\Lambda_y^{12} \cdot \widetilde{v}_2 = 0$. Recall that $\widetilde{v}_1 \cdot \widetilde{v}_2 = 2\widetilde{e}$, $\widetilde{e} \cdot \widetilde{v}_i = \widetilde{v}_i$ and $\widetilde{e} \cdot (\widetilde{y}_1 \cdot \widetilde{v}_2) = \frac{1}{2}\widetilde{y}_1 \cdot \widetilde{v}_2$. Thus, combining the above observation with (8), we obtain the equality

$$0 = ((\widetilde{y}_1 \cdot \widetilde{v}_1) \cdot \widetilde{v}_2) \cdot \widetilde{v}_2 = \Lambda_y^{12} + \Lambda_x^{22} \cdot \widetilde{v}_2 = \Lambda_y^{12} + \Lambda_x^{22} \cdot \widetilde{v}_2,$$

therefore, $(\lambda_{n_1}^{12y} + \lambda_{m_2}^{22x})\, n_1 + (\lambda_{n_2}^{12y} - \lambda_{m_1}^{22x})\, n_2 = 0$. Using the fact that n_1 and n_2 are linearly independent, we obtain $\lambda_{n_1}^{12y} = -\lambda_{m_2}^{22x}$ and $\lambda_{n_2}^{12y} = \lambda_{m_1}^{22x}$.

Taking $a_i = \widetilde{y}_1$, $a_j = \widetilde{v}_1$ and $a_k = a_l = \widetilde{v}_3$ in (2) we have,

$$0 = ((\widetilde{y}_1 \cdot \widetilde{v}_1) \cdot \widetilde{v}_3) \cdot \widetilde{v}_3 + ((\widetilde{y}_1 \cdot \widetilde{v}_3) \cdot \widetilde{v}_3) \cdot \widetilde{v}_1.$$

Thus we obtain $\lambda_{n_2}^{13x} = -\lambda_{n_1}^{23x}$.

Using $0 = ((\widetilde{y}_2 \cdot \widetilde{v}_1) \cdot \widetilde{v}_2) \cdot \widetilde{v}_2 + ((\widetilde{y}_2 \cdot \widetilde{v}_2) \cdot \widetilde{v}_2) \cdot \widetilde{v}_1$, we obtain $\lambda_{m_1}^{12x} = \lambda_{n_2}^{22y}$ and $\lambda_{m_2}^{12x} = -\lambda_{n_1}^{22y}$. Since, $0 = ((\widetilde{y}_2 \cdot \widetilde{v}_1) \cdot \widetilde{v}_4) \cdot \widetilde{v}_4 + ((\widetilde{y}_2 \cdot \widetilde{v}_4) \cdot \widetilde{v}_4) \cdot \widetilde{v}_1$, then $\lambda_{n_1}^{24x} = -\lambda_{n_2}^{14x}$.

Similarly, we obtain $\lambda_{m_1}^{21x} = \lambda_{n_2}^{11y}$, $\lambda_{m_2}^{21x} = -\lambda_{n_1}^{11y}$, $\lambda_{m_1}^{23y} = -\lambda_{m_2}^{13y}$, $\lambda_{n_2}^{23y} = -\lambda_{n_1}^{13y}$, $\lambda_{m_2}^{23x} = -\lambda_{m_1}^{13x}$, $\lambda_{n_1}^{12y} = -\lambda_{m_2}^{12x}$, $\lambda_{n_2}^{12y} = \lambda_{m_1}^{12x}$, $\lambda_{m_2}^{24x} = -\lambda_{m_1}^{14x}$, $\lambda_{m_2}^{11x} = \lambda_{n_1}^{21y}$, $\lambda_{m_1}^{11x} = -\lambda_{n_2}^{21y}$, $\lambda_{m_2}^{14y} = -\lambda_{m_1}^{24y}$, $\lambda_{n_1}^{14y} = -\lambda_{n_2}^{24y}$, $\lambda_{n_2}^{22y} = -\lambda_{m_1}^{12x}$, $\lambda_{m_1}^{13x} = -\lambda_{m_2}^{23x}$, $\lambda_{n_2}^{11y} = \lambda_{n_1}^{11y} = 0$. Thus, we have $\Lambda_x^{21} = 0$, $\Lambda^{12y} = \Lambda^{22y} = \lambda_{n_1}^{12y} n_1$ and $\Lambda_y^{11} = \lambda_{m_1}^{11y} m_1 + \lambda_{m_2}^{11y} m_2$.

Let $a_i = \widetilde{y}_1$, $a_j = a_l = \widetilde{v}_1$ and $a_k = \widetilde{v}_2$ in (2). Then we have $\widetilde{y}_1 \cdot \widetilde{v}_1 = ((\widetilde{y}_1 \cdot \widetilde{v}_1) \cdot \widetilde{v}_2) \cdot \widetilde{v}_1$, therefore, $\lambda_{n_1}^{22x} = -\lambda_{m_2}^{11y}$ and $\lambda_{n_2}^{22x} = \lambda_{m_1}^{11y}$. Similarly, we can obtain $\lambda_{m_1}^{11x} = \lambda_{n_2}^{23x}$, $\lambda_{m_2}^{11x} = -\lambda_{n_1}^{23x}$, $\lambda_{n_1}^{24x} = \lambda_{n_2}^{14x} = 0$, $\lambda_{n_1}^{12x} = \lambda_{n_2}^{12x} = 0$, $\lambda_{m_1}^{13x} = -\lambda_{n_1}^{21y}$, $\lambda_{n_2}^{13x} = 0$, $\lambda_{n_1}^{14x} = \lambda_{n_2}^{21y}$, $\lambda_{m_1}^{21y} = \lambda_{m_2}^{21y} = 0$, $\lambda_{n_1}^{23y} = \lambda_{m_2}^{12x}$, $\lambda_{m_1}^{24y} = 0$, $\lambda_{n_1}^{12y} = -\lambda_{m_2}^{24y}$, $\lambda_{n_1}^{22y} = -\lambda_{m_2}^{13x}$, $\lambda_{m_1}^{13y} = -\lambda_{m_2}^{22x}$, $\lambda_{n_2}^{23y} = 0$, $\lambda_{m_2}^{14x} = -\lambda_{m_1}^{23x}$, $\lambda_{n_1}^{24y} = -\lambda_{n_2}^{13y}$, $\lambda_{n_1}^{14y} = -\lambda_{n_2}^{22x}$, $\lambda_{m_1}^{11y} = \lambda_{n_2}^{24y}$.

Thus, we have $\Lambda_x^{12} = \Lambda_x^{13} = \Lambda_x^{24} = \Lambda_y^{12} = \Lambda_y^{22} = \Lambda_y^{23} = 0$.

Setting $a_i = \widetilde{y}_1$, $a_j = \widetilde{v}_1$, $a_k = \widetilde{v}_3$ and $a_l = \widetilde{v}_2$ in (2), we obtain

$$((\widetilde{y}_1 \cdot \widetilde{v}_1) \cdot \widetilde{v}_3) \cdot \widetilde{v}_2 + ((\widetilde{y}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_3) \cdot \widetilde{v}_1 + \widetilde{y}_1 \cdot \widetilde{v}_3 = 0.$$

Therefore we have $\lambda_{m_2}^{11y} = \lambda_{n_2}^{13y}$. Similarly one can prove the equalities $\lambda_{m_1}^{11x} = \lambda_{m_1}^{23x} = \lambda_{n_1}^{22x} = 0$. Thus $\Lambda_x^{11} = \Lambda_x^{14} = \Lambda_x^{22} = \Lambda_x^{23} = \Lambda_y^{11} = \Lambda_y^{13} = \Lambda_y^{14} = \Lambda_y^{21} = \Lambda_y^{24} = 0$. □

Lemma 5. *There exist $\alpha \in \mathbb{F}$ such that*

$$\begin{aligned} &\textbf{(i)}\ \widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 + \alpha u_1, && \textbf{(ii)}\ \widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2 + \alpha u_2 \\ &\textbf{(iii)}\ \widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3 + \alpha u_3, && \textbf{(iv)}\ \widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4 + \alpha u_4 \\ &\textbf{(v)}\ \widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{e} - 3\widetilde{f} + \alpha a - 3\alpha b && \textbf{(vi)}\ \widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} + \alpha a - 3\alpha b \end{aligned} \tag{9}$$

Proof. We can assume that there exist $\Lambda_x^{12}, \Lambda_y^{12}, \Lambda_{xy}^{12}, \Lambda_{xy}^{21}, \Lambda_{xy}^{11}$ and $\Lambda_{xy}^{22} \in \mathcal{N}_0$ such that $\widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 + \Lambda_x^{12}$, $\widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2 + \Lambda_x^{12}$, $\widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3 + \Lambda_{xy}^{12}$, $\widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4 + \Lambda_{xy}^{21}$, $\widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{e} - 3\widetilde{f} + \Lambda_{xy}^{11}$ and $\widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} + \Lambda_{xy}^{22}$.

We assume that there exist η_a^{tij}, η_b^{tij}, $\eta_{u_1}^{tij}$, $\eta_{u_2}^{tij}$, $\eta_{u_3}^{tij}$ and $\eta_{u_4}^{tij} \in \mathbb{F}$ such that $\Lambda_t^{ij} = \eta_a^{tij} a + \eta_b^{tij} b + \eta_{u_1}^{tij} u_1 + \eta_{u_2}^{tij} u_2 + \eta_{u_3}^{tij} u_3 + \eta_{u_4}^{tij} u_4$ for $i, j \in \{1, 2\}$ and $t \in \{x, y, xy\}$.

Replacing $a_i = \widetilde{x}_1$, $a_j = \widetilde{x}_2$, $a_k = a_l = \widetilde{v}_1$ in equation (2) and using (7); we have $((\widetilde{x}_1 \cdot \widetilde{x}_2) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = 0$, thus

$$\begin{aligned} 0 &= ((\widetilde{v}_1 + \eta_a^{x12} a + \eta_b^{x12} b + \eta_{u_1}^{x12} u_1 + \eta_{u_2}^{x12} u_2 + \eta_{u_3}^{x12} u_3 + \eta_{u_4}^{x12} u_4) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 \\ &= (\eta_a^{x12} u_1 + 2\eta_{u_2}^{x12} a) \cdot \widetilde{v}_1 = 2\eta_{u_2}^{x12} u_1. \end{aligned}$$

Therefore, $\eta_{u_2}^{x12} = 0$. In the same way one can prove that $\eta_{u_4}^{x12} = \eta_{u_3}^{x12} = 0$, thus

$$\widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 + \eta_a^{x12} a + \eta_b^{x12} b + \eta_{u_1}^{x12} u_1. \tag{10}$$

Similarly, we obtain that $\widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2 + \eta_a^{y12} a + \eta_b^{y12} b + \eta_{u_2}^{y12} u_2$.

Since (7) and replacing $a_i = \widetilde{x}_1$, $a_j = \widetilde{x}_2$ and $a_k = a_t = \widetilde{v}_2$ in (2), we obtain

$$((\widetilde{x}_1 \cdot \widetilde{x}_2) \cdot \widetilde{v}_2) \cdot \widetilde{v}_2 = 2(\widetilde{x}_1 \cdot \widetilde{v}_2) \cdot (\widetilde{v}_2 \cdot \widetilde{x}_2) = 2\widetilde{y}_1 \cdot \widetilde{y}_2. \tag{11}$$

Replacing (10) and its equivalent for $\widetilde{y}_1 \cdot \widetilde{y}_2$ in (11), we obtain $2\widetilde{v}_2 + \eta_{u_1}^{x12} u_2 = 2(\widetilde{v}_2 + \eta_a^{y12} a + \eta_b^{y12} b + \eta_{u_2}^{y12} u_2)$, therefore $\eta_a^{y12} = \eta_b^{y12} = 0$ and $\eta_{u_1}^{x12} = \eta_{u_2}^{y12}$. If we take $a_i = \widetilde{y}_1$, $a_j = \widetilde{y}_2$ and $a_k = a_t = \widetilde{v}_1$ in (2) we obtain $\eta_a^{x12} = \eta_b^{x12} = 0$.

Let $a_i = \widetilde{x}_1$, $a_j = \widetilde{y}_2$ and $a_k = a_t = \widetilde{v}_1$ in (2), thus, we obtain $\eta_{u_2}^{xy12} = 0$. If we shall take $a_k = a_t = \widetilde{v}_2$ or $a_k = a_t = \widetilde{v}_3$ we obtain $\eta_{u_1}^{xy12} = \eta_{u_4}^{xy12} = 0$. Similarly to the case above, we obtain $\eta_{u_1}^{xy21} = \eta_{u_2}^{xy21} = \eta_{u_3}^{xy21} = 0$.

Setting $a_i = \widetilde{v}_1$, $a_j = \widetilde{y}_2$ and $a_k = a_t = \widetilde{v}_4$ (respectively, $a_i = \widetilde{x}_2$, $a_j = \widetilde{y}_1$ and $a_k = a_t = \widetilde{v}_3$) in (2), we obtain $\eta_a^{xy21} = \eta_b^{xy21} = 0$ (respectively $\eta_a^{xy12} = \eta_b^{xy12} = 0$) and $\eta_{u_4}^{xy21} = \eta_{u_3}^{xy12}$.

If we take $a_i = \widetilde{x}_1$, $a_j = \widetilde{x}_2$, $a_k = \widetilde{v}_2$ and $a_t = \widetilde{v}_4$ in (2), then using (7) we have $((\widetilde{x}_1 \cdot \widetilde{x}_2) \cdot \widetilde{v}_2) \cdot \widetilde{v}_4 = 2\widetilde{x}_2 \cdot \widetilde{y}_1$. Therefore, $\eta^{xy21}_{u_4} = \eta^{x12}_{u_1}$. Thus we get $\widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 + \alpha u_1$, $\widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2 + \alpha u_2$, $\widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3 + \alpha u_3$ and $\widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4 + \alpha u_4$ for some $\alpha \in \mathbb{F}$.

Let $a_i = \widetilde{x}_1$, $a_j = \widetilde{y}_1$, and $a_k = a_t = \widetilde{v}_1$ in (2). Using the products in $\mathcal{S}_0$ and (7), we obtain $((\widetilde{x}_1 \cdot \widetilde{y}_1) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = 0$. Thus $((\widetilde{x}_1 \cdot \widetilde{y}_1) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = 0$ and therefore $\eta^{xy11}_{u_2} = 0$. Analogously, one can verify that $\eta^{xy11}_{u_1} = \eta^{xy11}_{u_3} = \eta^{xy11}_{u_4} = 0$. Thus $\widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{e} - 3\widetilde{f} + \eta^{xy11}_a a + \eta^{xy11}_b$. Similarly one can show that $\widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} + \eta^{xy22}_a a + \eta^{xy22}_b$.

Taking $a_i = \widetilde{x}_1$, $a_j = \widetilde{y}_1$, $a_k = \widetilde{v}_1$ and $a_t = \widetilde{v}_2$, in (2), we obtain

$$((\widetilde{x}_1 \cdot \widetilde{y}_1) \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 + \widetilde{x}_1 \cdot \widetilde{y}_1 = 2\widetilde{e} \cdot (\widetilde{x}_1 \cdot \widetilde{y}_1) + \widetilde{x}_2 \cdot \widetilde{y}_2. \tag{12}$$

From the above equality, it is easy to see that $\eta^{xy11}_a = \eta^{xy22}_a$ and $\eta^{xy11}_b = \eta^{xy22}_b$. Thus we have that $\widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{x}_2 \cdot \widetilde{y}_2$.

Let $a_i = \widetilde{x}_1$, $a_j = \widetilde{x}_2$ and $a_k = a_t = \widetilde{y}_1$ in (2), hence

$$\begin{aligned}
0 =&((\widetilde{x}_1 \cdot \widetilde{x}_2) \cdot \widetilde{y}_1) \cdot \widetilde{y}_1 - ((\widetilde{x}_1 \cdot \widetilde{y}_1) \cdot \widetilde{y}_1) \cdot \widetilde{x}_2 + ((\widetilde{x}_2 \cdot \widetilde{y}_1) \cdot \widetilde{y}_1) \cdot \widetilde{x}_1 \\
=&((\widetilde{v}_1 + \alpha u_1) \cdot \widetilde{y}_1) \cdot \widetilde{y}_1 - ((\widetilde{e} - 3\widetilde{f} + \eta^{xy11}_a a + \eta^{xy11}_b b) \cdot \widetilde{y}_1) \cdot \widetilde{x}_2 + ((\widetilde{v}_4 + \alpha u_4) \cdot \widetilde{y}_1) \cdot \widetilde{y}_1 \\
=&(\widetilde{y}_1 \cdot \widetilde{v}_1 + \alpha \widetilde{y}_1 \cdot u_1) \cdot \widetilde{y}_1 - (-\widetilde{y}_1 + \frac{1}{2}(\eta^{xy11}_a + \eta^{xy11}_b) n_1) \cdot \widetilde{x}_2 \\
=&\widetilde{x}_2 \cdot \widetilde{y}_1 + \alpha m_2 \cdot \widetilde{y}_1 - \widetilde{x}_2 \cdot \widetilde{y}_1 + \frac{1}{2}(\eta^{xy11}_a + \eta^{xy11}_b)\widetilde{x}_2 \cdot n_1 = (\alpha + \frac{1}{2}(\eta^{xy11}_a + \eta^{pq11}_b))u_4,
\end{aligned}$$

thus, $2\alpha = -(\eta^{xy11}_a + \eta^{xy11}_b)$.

If we take $a_i = \widetilde{x}_1$, $a_j = \widetilde{y}_1$, $a_k = \widetilde{x}_2$ and $a_t = \widetilde{y}_2$ in (2), we obtain $\eta^{xy11}_a = \alpha$ and $\eta^{xy11}_b = -3\alpha$, therefore $\widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} + \alpha a - 3\alpha b$. $\square$

Lemma 6. *For $i = 1, 2$ there exists $\beta \in \mathbb{F}$ such that $\widetilde{x}_i = x_i + \beta\, m_i$, and $\widetilde{y}_i = y_i + \beta\, n_i$.*

Proof. Assume that there exist Λ^i_x and $\Lambda^i_y \in \mathcal{N}_1$ such that $\widetilde{x}_i = x_i + \Lambda^i_x$ and $\widetilde{y}_i = y_i + \Lambda^i_y$, where $\Lambda^i_t = \lambda^{t_i}_{m_1} m_1 + \lambda^{t_i}_{m_2} m_2 + \lambda^{t_i}_{n_1} n_1 + \lambda^{t_i}_{n_2} n_2$ and $\lambda^{t_i}_{m_1}$, $\lambda^{t_i}_{m_2}$, $\lambda^{t_i}_{n_1}$ and $\lambda^{t_i}_{n_2} \in \mathbb{F}$. It is easy to see that $\widetilde{x}_i = x_i + \lambda^{x_i}_{m_i} m_i$ and $\widetilde{y}_i = y_i + \lambda^{y_i}_{n_i} n_i$.

Using the Lemma 4, we have that $\widetilde{x}_1 \cdot \widetilde{v}_2 = -\widetilde{y}_2$, $\widetilde{x}_1 \cdot \widetilde{v}_4 = -\widetilde{x}_2$ and $\widetilde{y}_2 \cdot \widetilde{v}_4 = \widetilde{y}_1$. Thus one easily verifies that $\lambda^{x_1}_{m_1} = \lambda^{y_2}_{n_2}$, $\lambda^{x_1}_{m_1} = \lambda^{x_2}_{m_2}$ and $\lambda^{y_2}_{n_2} = \lambda^{y_1}_{n_1}$. Therefore, $\lambda^{y_1}_{n_1} = \lambda^{y_2}_{n_2} = \lambda^{x_1}_{m_1} = \lambda^{x_2}_{m_2}$. $\square$

4.2. $\mathcal{N}$ is isomorphic to opposite regular bimodule

Lemma 7.

$$\begin{array}{llll}
\widetilde{f} \cdot \widetilde{x}_j = \frac{1}{2}\widetilde{x}_j, & \widetilde{f} \cdot \widetilde{y}_j = \frac{1}{2}\widetilde{y}_j, & \widetilde{e} \cdot \widetilde{x}_j = \frac{1}{2}\widetilde{x}_j, & \widetilde{e} \cdot \widetilde{y}_j = \frac{1}{2}\widetilde{y}_j, \\
\widetilde{y}_1 \cdot \widetilde{v}_1 = \widetilde{x}_2, & \widetilde{y}_2 \cdot \widetilde{v}_1 = -\widetilde{x}_1, & \widetilde{x}_1 \cdot \widetilde{v}_2 = -\widetilde{y}_2, & \widetilde{x}_2 \cdot \widetilde{v}_2 = \widetilde{y}_1, \\
\widetilde{x}_2 \cdot \widetilde{v}_3 = \widetilde{x}_1, & \widetilde{y}_1 \cdot \widetilde{v}_3 = \widetilde{y}_2, & \widetilde{x}_1 \cdot \widetilde{v}_4 = \widetilde{x}_2, & \widetilde{y}_2 \cdot \widetilde{v}_4 = \widetilde{y}_1,
\end{array} \tag{13}$$

Proof. To start, we proof the first line in (13). Let us denote $W = \{x_1, x_2, y_1, y_2\}$ and $T = \{e, f\}$. We can assume that for $t \in T$ and $w \in W$ there exist scalars λ_s^{tw} such that

$$\widetilde{w} \cdot \widetilde{t} = \frac{1}{2}\widetilde{w} + \Lambda_t^w = \frac{1}{2}\widetilde{w} + \lambda_a^{tw} a + \lambda_{u_1}^{tw} u_1 + \lambda_{u_2}^{tw} u_2 + \lambda_{u_3}^{tw} u_3 + \lambda_{u_4}^{tw} u_4 + \lambda_b^{tw} b$$

where $s \in \{a, b, u_1, u_2, u_3, u_4\}$.

Substituting $a_i = \widetilde{w}$, $a_j = a_l = \widetilde{e}$ and $a_k = \widetilde{e}$ in (2), we get

$$2((\widetilde{w} \cdot \widetilde{e}) \cdot \widetilde{f}) \cdot \widetilde{e} = (\widetilde{w} \cdot \widetilde{f}) \cdot \widetilde{e}.$$

Replacing $\widetilde{w} \cdot \widetilde{e} = \frac{1}{2}\widetilde{w} + \Lambda_e^w$ and $\widetilde{w} \cdot \widetilde{f} = \frac{1}{2}\widetilde{w} + \Lambda_f^w$ in the above equality we obtain

$$0 = (\Lambda_e^w \cdot \widetilde{f}) \cdot \widetilde{e} = \lambda_{u_1}^{ew} u_1 + \lambda_{u_2}^{ew} u_2 + \lambda_{u_3}^{ew} u_3 + \lambda_{u_4}^{ew} u_4,$$

therefore, $\widetilde{w} \cdot \widetilde{t} = \frac{1}{2}\widetilde{w} + \lambda_a^{tw} a + \lambda_b^{tw} b$. Since $\widetilde{w} \cdot \widetilde{e} = \widetilde{w} \cdot \widetilde{f}$ we obtain $\Lambda_e^w = \Lambda_f^w$.

Considering $a_i = \widetilde{w}$, $a_j = \widetilde{f}$ and $a_k = a_l = \widetilde{e}$ in equation (2), we obtain $((\widetilde{w} \cdot \widetilde{f}) \cdot \widetilde{e}) \cdot \widetilde{e} + ((\widetilde{w} \cdot \widetilde{e}) \cdot \widetilde{e}) \cdot \widetilde{f} = (\widetilde{w} \cdot \widetilde{f}) \cdot \widetilde{e}$, replacing $\widetilde{w} \cdot \widetilde{f} = \widetilde{w} \cdot \widetilde{f} = \frac{1}{2}\widetilde{w} + \Lambda_e^w$ in the above equality, we obtain $\Lambda_e^w \cdot f + \frac{1}{2}\Lambda_e^w \cdot e = 0$, therefore $\lambda_a^{tw} = \lambda_b^{tw} = 0$ and we have $\widetilde{w} \cdot \widetilde{t} = \frac{1}{2}\widetilde{w}$.

We consider the set $V = \{v_1, v_2, v_3, v_4\}$. Without lost of generality, we can assume that for $v \in V$ and $w \in W$ there exist $\Lambda_v^w \in \mathcal{N}_1$ and $\widetilde{w}' \in W$ such that $\widetilde{w} \cdot \widetilde{v} = \delta\widetilde{w}' + \Lambda_v^w$ where $\delta \in \{0, 1\}$ and $\Lambda_v^w = \lambda_a^{vw} a + \lambda_{u_1}^{vw} u_1 + \lambda_{u_2}^{vw} u_2 + \lambda_{u_3}^{vw} u_3 + \lambda_{u_4}^{vw} u_4 + \lambda_b^{vw} b$ for some scalars λ_s^{vw}.

We begin by proving $\widetilde{x}_n \cdot \widetilde{v}_1 = \widetilde{y}_n \cdot \widetilde{v}_2 = 0$, for $n = 1, 2$, $\widetilde{x}_1 \cdot \widetilde{v}_2 = -\widetilde{y}_2$, $\widetilde{x}_2 \cdot \widetilde{v}_2 = \widetilde{y}_1$, $\widetilde{x}_2 \cdot \widetilde{v}_3 = \widetilde{x}_1$ and $\widetilde{x}_1 \cdot \widetilde{v}_4 = \widetilde{y}_1$.

Let us consider $a_i = \widetilde{x}_n$, $a_j = \widetilde{e}$ and $a_k = a_l = \widetilde{v}_1$ in equation (2), thus we have

$$((\widetilde{x}_n \cdot \widetilde{e}) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 + ((\widetilde{x}_n \cdot \widetilde{v}_1) \cdot \widetilde{v}_1) \cdot \widetilde{e} = 2((\widetilde{x}_n \cdot \widetilde{v}_1) \cdot \widetilde{v}_1.$$

replacing $\widetilde{x}_n \cdot \widetilde{e} = \dfrac{1}{2}\widetilde{x}_1$ and $\widetilde{x}_n \cdot \widetilde{v}_1 = \Lambda_{v_1}^{x_n}$ in the above equality and noting that $(\Lambda_{v_1}^{x_n} \cdot \widetilde{v}_1) \cdot \widetilde{e} = \Lambda_{v_1}^{x_n} \cdot \widetilde{v}_1$ we obtain $\dfrac{1}{2}\Lambda_{v_1}^{x_n} \cdot \widetilde{v}_1 = 0$, therefore $\lambda_a^{v_1 x_n} = \lambda_{u_2}^{v_1 x_n} = 0$.

If we take $a_i = \widetilde{x}_n$, $a_j = a_l = \widetilde{v}_1$ and $a_k = \widetilde{v}_3$ in equation (2), then $((\widetilde{x}_n \cdot \widetilde{v}_1) \cdot \widetilde{v}_3) \cdot \widetilde{v}_1 = 0$ and therefore $\lambda_{u_4}^{v_1 x_n} = 0$. Note that if $a_i = \widetilde{x}_n$, $a_j = a_l = \widetilde{v}_1$ and $a_k = \widetilde{v}_4$, then $\lambda_{u_3}^{v_1 x_n} = 0$.

Now, let $a_i = \widetilde{y}_2$, $a_j = a_k = \widetilde{v}_1$ and $a_l = e$ in equation (2). Using $\widetilde{y}_2 \cdot \widetilde{e} = \frac{1}{2}\widetilde{y}_2$, it follows that $2((\widetilde{y}_2 \cdot \widetilde{v}_1) \cdot \widetilde{v}_1) \cdot \widetilde{e} = 3((\widetilde{y}_2 \cdot \widetilde{v}_1) \cdot \widetilde{v}_1$. Since $\widetilde{y}_2 \cdot \widetilde{v}_1 = -\widetilde{x}_1 + \Lambda_{v_1}^{y_2}$ and $\widetilde{x}_1 \cdot \widetilde{v}_1 = \Lambda_{v_1}^{x_1}$ we obtain $3\Lambda_{v_1}^{x_1} - 2\Lambda_{v_1}^{x_1} \cdot \widetilde{e} - \Lambda_{v_1}^{y_2} \cdot \widetilde{v}_1 = 0$, hence $\lambda_b^{v_1 x_1} = \lambda_{u_2}^{v_1 y_2} = 0$, and $\lambda_{u_1}^{v_1 x_1} = \lambda_a^{v_1 y_2}$ Thus, we have $\widetilde{x}_1 \cdot \widetilde{v}_1 = \lambda_{u_1}^{v_1 x_1} u_1$. By a similar argument we could show that, $\widetilde{x}_2 \cdot \widetilde{v}_1 = \lambda_{u_1}^{v_1 x_2} u_1$, and $\widetilde{y}_n \cdot \widetilde{v}_2 = \lambda_{u_2}^{v_2 y_n} u_2$.

Considering $a_i = \widetilde{x}_1$, $a_j = a_l = \widetilde{v}_2$ and $a_k = \widetilde{v}_1$, in equation (2), we obtain $((\widetilde{x}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 = 2(\widetilde{x}_1 \cdot \widetilde{v}_2) \cdot \widetilde{e}$, therefore $\Lambda_{v_2}^{x_1} - 2\Lambda_{v_2}^{x_1} \cdot \widetilde{e} + (\Lambda_{v_2}^{x_1}\widetilde{v}_1) \cdot \widetilde{v}_2 - \Lambda_{v_1}^{y_2} \cdot \widetilde{v}_2 = 0$, it follows that $\lambda_b^{v_2 x_1} = \lambda_{u_1}^{v_2 x_1} = \lambda_{u_3}^{v_2 x_1} = \lambda_{u_4}^{v_2 x_1} = 0$, $\lambda_a^{v_2 x_1} = 2\lambda_{u_1}^{v_1 y_2}$ and $\lambda_{u_2}^{v_2 x_1} = \lambda_a^{v_1 y_2}$, hence $\widetilde{x}_1 \cdot \widetilde{v}_2 = -\widetilde{y}_2 + \lambda_a^{v_2x_1} a + \lambda_{u_2}^{v_2x_1} u_2$. (Similar arguments apply to the case $\widetilde{x}_2 \cdot \widetilde{v}_2 = \widetilde{y}_1 + \lambda_a^{v_2x_2} a + \lambda_{u_2}^{v_2x_2} u_2$).

Analogously, we obtain $\widetilde{y}_2 \cdot \widetilde{v}_1 = -\widetilde{x}_1 + \lambda_a^{v_1y_2} a + \lambda_{u_1}^{v_1y_2} u_1$, $(\widetilde{y}_1 \cdot \widetilde{v}_1 = \widetilde{x}_2 + \lambda_a^{v_1y_1} a + \lambda_{u_1}^{v_1y_1} u_1)$ moreover $\lambda_a^{v_1y_2} = 2\lambda_{u_2}^{v_2x_1}$ and $\lambda_a^{v_2x_1} = \lambda_{u_1}^{v_1y_2}$, thus $\lambda_a^{v_2x_1} = \lambda_{u_1}^{v_1y_2} = 2\lambda_{u_1}^{v_1y_2}$ and $\lambda_{u_2}^{v_2x_1} = \lambda_a^{v_1y_2} = 2\lambda_{u_2}^{v_2x_1}$, therefore $\lambda_a^{v_2x_1} = \lambda_{u_1}^{v_1y_2} = \lambda_{u_2}^{v_2x_1} = \lambda_a^{v_1y_2} = 0$ hence $\widetilde{x}_1 \cdot \widetilde{v}_2 = -\widetilde{y}_2$ and $\widetilde{y}_2 \cdot \widetilde{v}_1 = -\widetilde{x}_1$, $(\widetilde{x}_2 \cdot \widetilde{v}_2 = \widetilde{y}_1$ and $\widetilde{y}_1 \cdot \widetilde{v}_1 = \widetilde{x}_2)$

If we choosing $a_i = \widetilde{y}_2$, $a_j = a_k = \widetilde{v}_1$ and $a_l = \widetilde{v}_2$ in equation (2), we obtain $((\widetilde{y}_2 \cdot \widetilde{v}_1) \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 + ((\widetilde{y}_2 \cdot \widetilde{v}_2) \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 + 2\widetilde{y}_2 \cdot \widetilde{v}_1 = 4(\widetilde{y}_2 \cdot \widetilde{v}_1) \cdot \widetilde{e}$, if follows that $-\Lambda_{v_1}^{x_1} \cdot \widetilde{v}_2 + (\Lambda_{v_2}^{y_2} \cdot \widetilde{v}_1) \cdot \widetilde{v}_1 = 0$, later $\lambda_{u_1}^{v_1x_1} = \lambda_{u_2}^{v_2y_2} = 0$. (Analogously we obtain $\lambda_{u_1}^{v_1x_2} = \lambda_{u_2}^{v_2y_1} = 0$).

Note that if we take $a_i = \widetilde{x}_2$, $a_j = a_l = \widetilde{v}_3$ and $a_k = \widetilde{v}_4$ in equation (2), then $\widetilde{x}_2 \cdot \widetilde{v}_3 = \widetilde{x}_1 + \lambda_a^{v_3x_2} a + \lambda_{u_3}^{v_3x_2} u_3$, and $\lambda_a^{v_3x_2} = -2\lambda_{u_4}^{v_4x_1}$ and $\lambda_{u_3}^{v_3x_2} = -\lambda_a^{v_4x_1}$. Similarly we obtain $\widetilde{x}_1 \cdot \widetilde{v}_4 = \widetilde{x}_2 + \lambda_a^{v_4x_1} a + \lambda_{u_4}^{v_4x_1} u_4$, and $\lambda_a^{v_4x_1} = -2\lambda_{u_3}^{v_3x_2}$ and $\lambda_{u_4}^{v_4x_1} = -\lambda_a^{v_3x_2}$. Hence $\widetilde{x}_2 \cdot \widetilde{v}_3 = \widetilde{x}_1$ and $\widetilde{x}_1 \cdot \widetilde{v}_4 = \widetilde{x}_2$. Analogously, we can see that $\widetilde{y}_1 \cdot \widetilde{v}_3 = \widetilde{y}_2$ and $\widetilde{y}_2 \cdot \widetilde{v}_4 = \widetilde{y}_1$. □

Lemma 8.

$$\begin{array}{ll} \textbf{(i)}\ \widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 & \textbf{(ii)}\ \widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2 \\ \textbf{(iii)}\ \widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3, & \textbf{(iv)}\ \widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4 \\ \textbf{(v)}\ \widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{e} - 3\widetilde{f} & \textbf{(vi)}\ \widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} \end{array} \tag{14}$$

Proof. First we prove that $\widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1$. Assume that there exists $\Lambda_x^{12} \in \mathcal{N}_0$ such that $\widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 + \Lambda_x^{12}$. We note that $\mathcal{N}_0$ is spanned by $\langle m_1, m_2, n_1, n_2 \rangle$, therefore, we can take $\lambda_i \in \mathbb{F}$ for $i = 1, 2, 3, 4$ such that $\Lambda_x^{12} = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 m_3 + \lambda_4 m_4$.

Let $a_i = \widetilde{x}_1$, $a_j = \widetilde{x}_2$, $a_k = \widetilde{v}_1$ and $a_l = \widetilde{v}_2$ in equation (2). Substituting $\widetilde{x}_i \cdot \widetilde{v}_1 = 0$ we obtain

$$((\widetilde{x}_1 \cdot \widetilde{x}_2) \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 + ((\widetilde{x}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_1) \cdot \widetilde{x}_2 - ((\widetilde{x}_2 \cdot \widetilde{v}_2) \cdot \widetilde{v}_1) \cdot \widetilde{x}_1 = 2(\widetilde{x}_1 \cdot \widetilde{x}_2) \cdot \widetilde{e}.$$

Using Lemma 7 and noting that $\Lambda_x^{12} \cdot \widetilde{e} = \frac{1}{2}\Lambda_x^{12}$, it is clear that $(\Lambda_x^{12} \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 + \Lambda_x^{12} = 0$, therefore $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Similarly, we can show that $\widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2$, $\widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3$, and $\widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4$

Assume that $\widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{e} - 3\widetilde{f} + \Lambda_{xy}$ and $\widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} + \Delta_{xy}$. For $i = 1, 2, 3, 4$ let λ_i and β_i be some scalars such that $\Lambda_{xy} = \lambda_1 m_1 + \lambda_2 m_2 + \lambda_3 n_1 + \lambda_4 n_2$ and $\Delta_{xy} = \beta_1 m_1 + \beta_2 m_2 + \beta_3 n_1 + \beta_4 n_2$.

Considering $a_i = \widetilde{x}_1$, $a_j = \widetilde{y}_1$, $a_k = \widetilde{v}_1$ and $a_l = \widetilde{v}_2$ in equation (2). and substituting $\widetilde{y}_1 \cdot \widetilde{v}_2 = 0$ we obtain

$$((\widetilde{x}_1 \cdot \widetilde{y}_1) \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 + ((\widetilde{x}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_1) \cdot \widetilde{y}_1 - ((\widetilde{y}_1 \cdot \widetilde{v}_2) \cdot \widetilde{v}_1) \cdot \widetilde{x}_1 = 2(\widetilde{x}_1 \cdot \widetilde{y}_1) \cdot \widetilde{e} + (\widetilde{x}_1 \cdot \widetilde{v}_2) \cdot (\widetilde{y}_1 \cdot \widetilde{v}_1).$$

Using Lemma 7 and combining the different products we have $(\Lambda_{xy} \cdot \widetilde{v}_1) \cdot \widetilde{v}_2 + \frac{1}{2}\Lambda_{xy} = \Delta_{xy}$, hence the following equalities hold

$$\frac{1}{2}\lambda_1 = \beta_1, \quad \frac{1}{2}\lambda_2 = \beta_2, \quad \frac{3}{2}\lambda_3 = \beta_3, \text{ and } \frac{3}{2}\lambda_4 = \beta_4. \tag{15}$$

We now apply this argument again, with $a_i = \widetilde{x}_1$, $a_j = \widetilde{y}_1$, $a_k = \widetilde{v}_2$ and $a_l = \widetilde{v}_1$ to obtain

$$\frac{3}{2}\lambda_1 = \beta_1, \quad \frac{3}{2}\lambda_2 = \beta_2, \quad \frac{1}{2}\lambda_3 = \beta_3, \text{ and } \frac{1}{2}\lambda_4 = \beta_4. \tag{16}$$

Combining (15) and (16) it follows that $\Lambda_{xy} = \Delta_{xy} = 0$. $\square$

Let us prove the following theorem

Theorem 9. *Let $\mathcal{A}$ be a finite dimensional Jordan superalgebra with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N} \cong \mathcal{K}_{10}$. Then there exists a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{K}_{10}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.*

Proof. Recall that A. S. Shtern [13] proved that any irreducible Jordan bimodule over $\mathcal{K}_{10}$ is isomorphic to $\mathcal{R}\text{eg}\,(\mathcal{K}_{10})$ or its opposite. By Theorem 3, we only need to consider this cases.

If $\mathcal{N}$ is isomorphic to regular bimodule, then by Lemma 5, we can assume that there exists $\alpha \in \mathcal{N}$ such that

$$\begin{array}{ll} \textbf{(i)}\ \widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1 + \alpha u_1, & \textbf{(ii)}\ \widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2 + \alpha u_2 \\ \textbf{(iii)}\ \widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3 + \alpha u_3, & \textbf{(iv)}\ \widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4 + \alpha u_4 \\ \textbf{(v)}\ \widetilde{x}_1 \cdot \widetilde{y}_1 = \widetilde{e} - 3\widetilde{f} + \alpha a - 3\alpha b & \textbf{(vi)}\ \widetilde{x}_2 \cdot \widetilde{y}_2 = \widetilde{e} - 3\widetilde{f} + \alpha a - 3\alpha b \end{array} \tag{17}$$

By Lemma 6, there is a $\beta \in \mathbb{F}$ such that $\widetilde{x}_i = x_i + \beta\, m_i$ and $\widetilde{y}_i = y_i + \beta\, n_i$.
It is easy to verifies the following equalities

$$\begin{array}{rr} \widetilde{x}_1 \cdot \widetilde{y}_1 = x_1 \cdot y_1 + 2\beta(a - 3b), & \widetilde{x}_2 \cdot \widetilde{y}_2 = x_2 \cdot y_2 + 2\beta(a - 3b), \\ \widetilde{x}_1 \cdot \widetilde{x}_2 = x_1 \cdot x_2 + 2\beta u_1, & \widetilde{y}_1 \cdot \widetilde{y}_2 = y_1 \cdot y_2 + 2\beta u_2, \\ \widetilde{x}_1 \cdot \widetilde{y}_2 = x_1 \cdot y_2 + 2\beta u_3, & \widetilde{x}_2 \cdot \widetilde{y}_1 = x_2 \cdot y_2 + 2\beta u_3. \end{array} \tag{18}$$

Using (17) and (18), we get $\widetilde{x}_i \cdot \widetilde{y}_i = \widetilde{e} - 3\widetilde{f}$, $\widetilde{x}_1 \cdot \widetilde{x}_2 = \widetilde{v}_1$, $\widetilde{x}_1 \cdot \widetilde{y}_2 = \widetilde{v}_3$, $\widetilde{x}_2 \cdot \widetilde{y}_1 = \widetilde{v}_4$ and $\widetilde{y}_1 \cdot \widetilde{y}_2 = \widetilde{v}_2$ if and only if, $2\beta = \alpha$. This equality has always a solution.

If $\mathcal{N}$ is isomorphic to opposite regular bimodule, using Lemmas 7 and 8 we note that $\widetilde{x}_1$, $\widetilde{x}_2$, $\widetilde{y}_1$ and $\widetilde{y}_2$ satisfies the conditions.

Therefore, it is clear that an analogue to WPT holds in the case under consideration. $\square$

5. Jordan superalgebra of superform

In this section we use the classification of irreducible $\mathfrak{J}$-bimodules obtained by E. Zelmanov and C. Martinez in [17], where $\mathfrak{J} = \mathfrak{J}(\mathcal{V}, f) = (\mathbb{F} \cdot 1 \oplus \mathcal{V}_0) \dotplus \mathcal{V}_1$ is a Jordan superalgebra of nondegenerate super-symmetric superform f on a superspace $\mathcal{V} = \mathcal{V}_0 \dotplus \mathcal{V}_1$.

We may assume that $\dim \mathcal{V}_1 > 1$. Let $v_1, \ldots, v_n$ be an f-orthonormal basis of $\mathcal{V}_0$, i.e. $f(v_i, v_i) = 1, f(v_i, v_j) = 0$ for $i \neq j, \quad i, j = 1 \ldots, n$. Let $w_1, \ldots, w_{2m}$ be a basis of $\mathcal{V}_1$ such that $f(w_{2p-1}, w_{2p}) = 1$, $1 \leq p \leq m$, and all the other products of basis elements are zero.

Since [17] we have that all products

$$v_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}}$$

form a basis of the Clifford superalgebra $\mathcal{C}$ of $\mathcal{V}$, where $i_1, \ldots, i_n \in \{0, 1\}$ and $k_1, \ldots, k_{2m}$ are nonnegative integers and $\mathcal{C}$ denotes the Clifford superalgebra of $\mathcal{V}$. Let $\mathcal{C}_r$ be the subspace in $\mathcal{C}$ spanned by the products of basis elements of length at most r, and let $\mathfrak{J} = (\mathbb{F} \cdot 1 + \mathcal{V}_0) \dotplus \mathcal{V}_1$ be the Jordan superalgebra of superform f. Let a be an even vector such that $\mathcal{V}' = \mathcal{V} \oplus \mathbb{F} \cdot a$. We extend the superform f to $\mathcal{V}'$ so that $f(a, a) = 1, \ f(a, \mathcal{V}) = 0$. Denote by $\mathcal{C}'_r$ the subspace in $\mathcal{C}'$ defined in the same way as $\mathcal{C}_r$ in $\mathcal{C}$.

In this section, we put into correspondence to every element

$$v_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}}$$

a pair (I, K), where $I = (i_1, \ldots, i_n)$ is a n-tuple and $K = (k_1, \ldots, k_{2m})$ is a $2m$-tuple such that i_s, k_t satisfies the above conditions.

We write $\eta_{I,K} = v_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}} = V_I W_K$. Note that for any pair of elements $\eta_{I,K}, \eta_{I',K'} \in \mathcal{C}$, the following relation holds $\eta_{I,K} = \eta_{I',K'}$ if and only if $I = I'$, and $K = K'$. Thus every element of the basis of $\mathcal{C}$ has a unique representation in terms of (I, K). We denote $V_{(0)} = 1$, $V_{(1)} = v_1 v_2 \cdots v_n$.

Let $\mathcal{I}$, $\mathcal{K}$ be the following sets

$$\mathcal{I} = \{I = (i_1, \ldots, i_n), i_j = 0 \text{ or } 1, \ j = 1, \ldots, n\},$$
$$\mathcal{K} = \{K = (k_1, \ldots, k_{2m}), k_j \in \mathbb{Z}^+ \cup \{0\}, \ j = 1, \ldots, 2m\}.$$

For $I \in \mathcal{I}$, $K \in \mathcal{K}$, we denote $|I| = i_1 + \cdots + i_n$, $|K| = k_1 + \cdots + k_{2m}$ and $|\eta_{I,K}| = |I| + |K|$.

Some relations in $\mathcal{C}^{(+)}$.

Using the supersymmetric product in the Jordan superalgebra $\mathcal{C}^+$ we obtain $w_{2p-1}w_{2p} - w_{2p}w_{2p-1} = 2$, $w_{2p-1}w_q = w_q w_{2p-1}$ if $q \neq 2p$ and $v_i w_p = -w_p v_i$. It follows easy that

$$V_I W_K \circ v_j = \Big(-\frac{1}{2}\Big)^{i_1 + \cdots + i_{j-1}} V_{(i_1, \ldots, i_j + 1, \ldots, i_n)} W_K (1 + (-1)^{|\eta_{I,K}| + i_j}), \tag{19}$$

$$V_I W_K \circ w_{2p-1} = \frac{1}{2} V_I W_{(k_1, \ldots, k_{2p-1}+1, \ldots, k_{2m})} (1 + (-1)^{|\eta_{I,K}|}) - k_{2p} V_I W_{(k_1, \ldots, k_{2p}-1, \ldots, k_{2m})} \tag{20}$$

and

$$V_I W_K \circ w_{2p} = \frac{1}{2} V_I W_{(k_1,\dots,k_{2p}+1,\dots,k_{2m})}(1 + (-1)^{|K|}) + (-1)^{|K|+1} k_{2p-1} V_I W_{(k_1,\dots,k_{2p-1}-1,\dots,k_{2m})} \tag{21}$$

for $j = 1, \dots, n$ and $p = 1, 2, \dots, m$.

Assume that $\mathcal{S}_0 = \mathbb{F} \cdot 1 + \mathbb{F} \cdot \widetilde{v}_1 + \dots + \mathbb{F} \cdot \widetilde{v}_n$, $\mathfrak{J}_0 \cong \mathcal{S}_0$, $\mathcal{A}_0 = \mathcal{S}_0 \oplus \mathcal{N}_0$ and $\mathcal{A}_1/\mathcal{N}_1 = \mathbb{F} \cdot \bar{w}_1 + \mathbb{F} \cdot \bar{w}_2 + \dots + \mathbb{F} \cdot \bar{w}_{2m}$. We consider two cases for $\mathcal{N}$.

5.1. $\mathcal{N}$ is isomorphic to $\mathcal{C}_r/\mathcal{C}_{r-2}$

Without loss of generality, we can take

$$\mathcal{N}_0 = \mathbf{vect}_{\mathbb{F}} \langle \eta_{I,K}, |\eta_{I,K}| = r, r-1 \text{ and } |K| \text{ is even} \rangle,$$
$$\mathcal{N}_1 = \mathbf{vect}_{\mathbb{F}} \langle \eta_{I,K}, |\eta_{I,K}| = r, r-1 \text{ and } |K| \text{ is odd} \rangle.$$

Using the notation introduced above, due to equations (19) and (20), we have the following products:

$$\eta_{I,K} \cdot \widetilde{v}_j = \begin{cases} \pm V_{(i_1,\dots,i_{j-1},0,i_{j+1},\dots,i_n)} W_K & \text{if} \quad |\eta_{I,K}| = r,\ i_j = 1, \\ \pm V_{(i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n)} W_K & \text{if} \quad |\eta_{I,K}| = r-1,\ i_j = 0. \end{cases} \tag{22}$$

$$\eta_{I,K} \cdot \widetilde{w}_p = \begin{cases} \pm k_{p\pm1} V_I W_{(k_1,\dots,k_{p\pm1}-1,\dots,k_{2m})} & \text{if} \quad |\eta_{I,K}| = r, \\ V_I W_{(k_1,\dots,k_p+1,\dots,k_{2m})} & \text{if} \quad |\eta_{I,K}| = r-1. \end{cases} \tag{23}$$

First, we prove three lemmas.

Lemma 10. $\widetilde{v}_j \cdot \widetilde{w}_s = 0$.

Proof. Setting $a_i = \widetilde{w}_s$, $a_j = a_l = \widetilde{v}_i$ and $a_k = \widetilde{v}_j$ in (2), we have

$$((\widetilde{w}_s \cdot \widetilde{v}_i) \cdot \widetilde{v}_j) \cdot \widetilde{v}_i = 0. \tag{24}$$

We may assume that there exist some scalars $\xi^k_{(I,K)}$ such that

$$\widetilde{v}_k \cdot \widetilde{w}_s = \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r \\ |K| \text{ odd}}} \xi^k_{(I,K)} V_I W_K + \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} V_I W_K.$$

Let $\xi^k_{(I,k)} \eta_{I,K} \in \mathcal{N}_1$ be a nonzero element and $j \neq k$, $j, k \in \{1, 2, \dots, n\}$. Using (22) and (24), we obtain the following relations:

- **(a)** If $|\eta_{I,K}| = r-1$ and $i_j = 1$, then $(\xi^k_{(I,K)} \eta_{I,K} \cdot \widetilde{v}_j) \cdot \widetilde{v}_k = 0$.
- **(b)** If $|\eta_{I,K}| = r-1$, $i_j = i_k = 0$, then $(\xi^k_{(I,K)} \eta_{I,K} \cdot \widetilde{v}_j) \cdot \widetilde{v}_k = 0$.

(c) If $|\eta_{I,K}| = r$, $i_j = 0$, then $(\xi^k_{(I,K)}\eta_{I,K} \cdot \widetilde{v}_j) \cdot \widetilde{v}_k = 0$.
(d) If $|\eta_{I,K}| = r$, $i_j = i_k = 1$, then $(\xi^k_{(I,K)}\eta_{I,K} \cdot \widetilde{v}_j) \cdot \widetilde{v}_k = 0$.

From **(a)**–**(d)** and (24), we have

$$\begin{aligned}\widetilde{v}_k \cdot \widetilde{w}_s = & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} V_{(i_1,\dots,i_{j-1},0,i_{j+1},\dots,i_n)} W_K + \\ & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r \\ |K| \text{ odd}}} \xi^k_{(I,K)} V_{(i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n)} W_K + \\ & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} V_{(i_1,\dots,i_{j-1},0,i_{j+1},\dots,i_{k-1},0,i_{k+1},\dots,i_n)} W_K + \\ & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r \\ |K| \text{ odd}}} \xi^k_{(I,K)} V_{(i_1,\dots,i_{j-1},0,i_{j+1},\dots,i_{k-1},0,i_{k+1},\dots,i_n)} W_K = 0.\end{aligned}$$

Thus,

$$\begin{aligned}\widetilde{v}_k \cdot \widetilde{w}_s = & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} v_1^{i_1} \cdots v_j \cdots v_n^{i_n} W_K + \\ & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} v_1^{i_1} \cdots v_j \cdots v_k \cdots v_n^{i_n} W_K + \\ & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r \\ |K| \text{ odd}}} \xi^k_{(I,K)} v_1^{i_1} \cdots v_{j-1}^{i_{j-1}} v_{j+1}^{i_{j+1}} \cdots v_n^{i_n} W_K + \\ & \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r \\ |K| \text{ odd}}} \xi^k_{(I,K)} v_1^{i_1} \cdots v_j \cdots v_k \cdots v_n^{i_n} W_K = 0.\end{aligned}$$

If we apply (24) to the obtained above equation for all $j \neq k$, we get

$$\widetilde{v}_k \cdot \widetilde{w}_s = \sum_{\substack{I,\,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} v_1 \cdots v_k^{i_k} \cdots v_n W_K + \sum_{\substack{K \\ |K|=r \\ |K| \text{ odd}}} \xi^k_{(0,K)} W_K + \sum_{\substack{K \\ |K|=r-n \\ |K| \text{ odd}}} \xi^k_{(1,K)} V_{(1)} W_K. \tag{25}$$

Substituting a_i by $\widetilde{w}_s$ and $a_j = a_k = a_l$ by $\widetilde{v}_k$ respectively in (2), we obtain

$$((\widetilde{w}_s \cdot \widetilde{v}_k) \cdot \widetilde{v}_k) \cdot \widetilde{v}_k = z_s \cdot \widetilde{v}_k. \tag{26}$$

Applying (26) to (25), we get

$$\widetilde{v}_k \cdot \widetilde{w}_s = \sum_{\substack{I,\, K \\ |I|+|K|=r-1 \\ |K| \text{ odd}}} \xi^k_{(I,K)} v_1 \cdots v_k^{i_k} \cdots v_n W_K. \tag{27}$$

Substituting a_i , a_j, a_k, and a_l by $\widetilde{w}_s$, $\widetilde{v}_k$, $\widetilde{v}_j$ and $\widetilde{v}_j$ respectively in (2), we have

$$((\widetilde{w}_s \cdot \widetilde{v}_k) \cdot \widetilde{v}_j) \cdot \widetilde{v}_j + ((\widetilde{w}_s \cdot \widetilde{v}_j) \cdot \widetilde{v}_j) \cdot \widetilde{v}_k = \widetilde{w}_s \cdot \widetilde{v}_k$$

Applying the obtained equality to (27), we have $\widetilde{v}_k \cdot \widetilde{w}_s = 0$. □

Lemma 11.

$$\widetilde{w}_p \cdot \widetilde{w}_q = \alpha_0^{p,q} + \sum_{\substack{K \\ |K|=r-1}} \alpha^{p,q}_{(0,K)} W_K + \sum_{\substack{K \\ |K|=r-n \\ n \text{ odd}}} \alpha^{p,q}_{(1,K)} V_{(1)} W_K, \text{ where } \alpha_0^{p,q} \in \{0,1\}$$

Proof. Since $\widetilde{w}_p \cdot \widetilde{w}_q \in \mathcal{A}_0$, we can assume that there exist some scalars $\alpha^{p,q}_{(0,K)}$, $\alpha^{p,q}_{(I,K)}$ such that

$$\widetilde{w}_p \cdot \widetilde{w}_q = \alpha_0^{p,q} + \sum_{\substack{I,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ even}}} \alpha^{p,q}_{(I,K)} V_I W_K + \sum_{\substack{I,K \\ \eta_{I,K}=r-1 \\ |K| \text{ even}}} \alpha^{p,q}_{(I,K)} V_I W_K, \tag{28}$$

where $\alpha_0^{p,q}$ is 0 or 1.

If we take $a_i = \widetilde{w}_p$, $a_j = \widetilde{w}_q$ and $a_k = a_l = \widetilde{v}_i$ in (2), and use Lemma 10, then we obtain $((\widetilde{w}_p \cdot \widetilde{w}_q) \cdot \widetilde{v}_i) \cdot \widetilde{v}_i = \widetilde{w}_p \cdot \widetilde{w}_q$. Combining (28) in the stated before equality, we have

$$\sum_{\substack{I,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ even}}} \alpha^{p,q}_{(I,K)} (\eta_{I,K} \cdot \widetilde{u}_i) \cdot \widetilde{v}_i + \sum_{\substack{I,K \\ \eta_{I,K}=r-1 \\ |K| \text{ even}}} \alpha^{p,q}_{(I,K)} (\eta_{I,K} \cdot \widetilde{u}_i) \cdot \widetilde{v}_i = \sum_{\substack{I,K \\ |\eta_{I,K}|=r-1 \\ |K| \text{ even}}} \alpha^{p,q}_{(I,K)} \eta_{I,K} + \sum_{\substack{I,K \\ \eta_{I,K}=r-1 \\ |K| \text{ even}}} \alpha^{p,q}_{(I,K)} \eta_{I,K}. \tag{29}$$

Let $\eta = \eta_{I,K}$ be a nonzero element in $\mathcal{N}_0$. Using equality (22), one can easily prove the following statements

(i) If $|\eta_{I,K}| = r - 1$, and $i_j = 0$, then $(\eta \cdot \widetilde{v}_j) \cdot \widetilde{v}_j = \eta$.

(ii) If $|\eta_{I,K}| = r-1$, and $i_j = 1$, then $\eta \cdot \widetilde{v}_j = 0$ therefore, $(\eta \cdot \widetilde{v}_j) \cdot \widetilde{v}_j = 0$.
(iii) If $|\eta_{I,K}| = r$, and $i_j = 0$, then $(\eta \cdot \widetilde{v}_j) = 0$, thus $(\eta \cdot \widetilde{v}_j) \cdot \widetilde{v}_j = 0$.
(iv) If $|\eta_{I,K}| = r$, and $i_j = 1$, then $(\eta \cdot \widetilde{v}_j) \cdot \widetilde{v}_j = \eta$.

Using statements **(i)**–**(iv)**, we note that if $|\eta| = r-1$ and $i_j = 1$ for some j then the left part of (29) is equal to zero and, consequently, the right part is zero. Thus, in the right part of (29) the only terms of length $r-1$ are of type $w_1^{k_1} \cdots w_{2m}^{k_{2m}}$. Now, if $|\eta| = r$ and $i_j = 0$ for some j then, $\widetilde{v}_j \cdot \eta = 0$. Hence, every term of length r on the right hand side of (29) must contain every v_j, but this is only possible if n is an odd integer.

We have thus proved

$$\widetilde{w}_p \cdot \widetilde{w}_q = \alpha_0^{p,q} + \sum_{\substack{K \\ |K|=r-1}} \alpha_{(0,K)}^{p,q} W_K \sum_{\substack{K \\ |K|=r-n \\ n \text{ odd}}} \alpha_{(1,K)}^{p,q} V_{(1)} W_K. \quad \square$$

Lemma 12.

$$\widetilde{w}_p = w_p + \sum_{\substack{K \\ |K|=r}} \xi_{(0,K)}^{p} W_K + \sum_{\substack{K \\ |K|=r-n-1 \\ n \text{ odd}}} \xi_{(1,K)}^{p} V_{(1)} W_K.$$

Proof. Let $p \in \{1, \ldots, 2m\}$ be a fixed integer. We assume that there exist some scalars $\xi_{(I,K)}^p$ such that

$$\widetilde{w}_p = w_p + \sum_{\substack{I,K \\ |\eta_{I,K}|=r-1 \\ K \text{ odd}}} \xi_{(I,K)}^{p} \eta_{I,K} + \sum_{\substack{I,K \\ |\eta_{I,K}|=r \\ K \text{ odd}}} \xi_{(I,K)}^{p} \eta_{I,K}. \tag{30}$$

Using Lemma 10, we have that $\widetilde{w}_p \cdot \widetilde{v}_j = 0$, and therefore,

$$0 = \sum_{\substack{I,K \\ |\eta_{I,K}|=r-1 \\ K \text{ odd}}} \xi_{(I,K)}^{p} \eta_{I,K} \cdot \widetilde{v}_j + \sum_{\substack{I,K \\ |\eta_{I,K}|=r \\ K \text{ odd}}} \xi_{(I,K)}^{p} \eta_{I,K} \cdot \widetilde{v}_j. \tag{31}$$

Let $\eta = \xi_{I,K}\eta_{I,K}$ be a nonzero element in (31). We shall use statements **(i)**–**(iv)** from Lemma 11.

We note that if $|\eta| = r-1$ and $i_j = 1$, then $\eta \cdot \widetilde{v}_j = 0$. Using (31) one can easily verify that if $I \neq (1)$ then $\xi_{(I,K)} = 0$. Therefore, we see that the only elements in (30) of length $r-1$ are of type $V_{(1)}W_K$.

Let $|\eta| = r$ and $i_j = 1$, then $(\eta \cdot \widetilde{v}_j) \cdot \widetilde{v}_j = \eta$ and therefore, if $V_I \neq 1$, then $\xi_{(I,K)} = 0$. Thus, the only elements of length r that are not zero on the right part of (31) are precisely those where $i_j = 0$. As this is valid for every j, we have that the only elements of length r that appear in (30) are of type $w_1^{k_1} \cdots w_{2m}^{k_{2m}}$, with $|K| = r$. Thus, we have proved that

$$\widetilde{w}_p = w_p + \sum_{\substack{K \\ |K|=r}} \xi^p_{(0,K)} W_K + \sum_{\substack{K \\ |K|=r-n-1 \\ n \text{ odd}}} \xi^p_{(1,K)} V_{(1)} W_K. \quad \square$$

5.2. $\mathcal{N}$ is isomorphic to $u\,\mathcal{C}_r/u\,\mathcal{C}_{r-2}$, where r is an even integer and u is an even vector

Without loss of generality we can take

$$\mathcal{N}_0 = \mathbf{vect}_{\mathbb{F}} \langle uV_I W_K \, |\eta_{I,K}| = r, r-1 \text{ and } |K| \text{ even} \rangle,$$
$$\mathcal{N}_1 = \mathbf{vect}_{\mathbb{F}} \langle uV_I W_K, |\eta_{I,K}| = r, r-1 \text{ and } |K| \text{ odd} \rangle.$$

As in above case, one can easily verify that

$$uV_I W_K \circ \widetilde{v}_j = (-\frac{1}{2})^{i_1+\cdots+i_{j-1}} uV_{(i_1,\ldots,i_j+1,\ldots,i_n)} W_K((-1)^{|\eta_{I,K}|-i_j} - 1).$$

Moreover, we have

$$uV_I W_K \cdot \widetilde{v}_j = \begin{cases} \pm uV_{(i_1,\ldots,i_{j-1},1,\ldots,i_n)} W_K & \text{if} \quad |\eta_{I,K}| = r-1,\, i_j = 0, \\ \pm uV_{(i_1,\ldots,i_{j-1},1,\ldots,i_n)} W_K & \text{if} \quad |\eta_{I,K}| = r,\, i_j = 1, \end{cases}$$
$$uV_I W_K \cdot \widetilde{w}_p = \begin{cases} \mp k_{p\pm 1} uV_I W_{(k_1,\ldots,k_{p\pm1}-1,\ldots,k_{2m})} & \text{if} \quad |\eta_{I,K}| = r, \\ uV_I W_{(k_1,\ldots,k_p+1,\ldots,k_{2m})} & \text{if} \quad |\eta_{I,K}| = r-1. \end{cases}$$

Now, we note one can prove analogues to Lemmas 10, 11 and 12, implying the following equalities

$$\widetilde{w}_p \cdot \widetilde{w}_q = \delta_{p+1,q} + \sum_{\substack{K \\ |K|=r}} \alpha^{p,q}_{(0,K)} uW_{(k_1,\ldots,k_{2m})} + \sum_{\substack{K \\ |K|=r-n \\ n \text{ even}}} \alpha^{p,q}_{(1,K)} uV_{(1)} W_{(k_1,\ldots,k_{2m})},$$
$$\widetilde{w}_p = w_p + \sum_{\substack{K \\ |K|=r-1}} \xi^i_{(0,K)} uW_{(k_1,\ldots,k_{2m})} + \sum_{\substack{K \\ |K|=r-n-1 \\ n \text{ even}}} \xi^i_{(1,K)} uV_{(1)} W_{(k_1,\ldots,k_{2m})}.$$

We shall prove the following theorem

Theorem 13. *Let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with a solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$, $\mathcal{A}/\mathcal{N}$ is isomorphic to the Jordan superalgebra of superform $\mathfrak{J}$ and $\mathcal{N} \in \overline{\mathfrak{M}}(\mathfrak{J}; J^{(k)})$, where $J^{(k)} = C_{2k+1}/C_{2k-1}$ if $\dim V_0 = 2k+1$ or $J^{(k)} = aC_{2k}/C_{2k-2}$ if $\dim V_0 = 2k$. Then there exists a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathfrak{J}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.*

Proof. By Theorem 3 it suffices to prove the theorem when $\mathcal{N}$ is irreducible. So, by Theorem 7.7 in [17] we only need to consider the two cases.

Using Lemma 10, we have $\widetilde{w}_p \cdot \widetilde{v}_i = 0$ for $i = 1, \ldots, n$; $p = 1, \ldots, 2m$.

Let p be an odd integer. Due to Lemmas 11 and 12, we can assume that

$$\widetilde{w}_p \cdot \widetilde{w}_q = \delta_{p+1,q} + \sum_{\substack{K \\ |K|=r-1}} \alpha^{p,q}_{(0,K)} W_K + \sum_{\substack{K \\ |K|=r-n \\ n \text{ odd}}} \alpha^{p,q}_{(1,K)} V_{(1)} W_K, \tag{32}$$

$$\widetilde{w}_p = w_p + \sum_{\substack{K \\ |K|=r}} \xi^{p}_{(0,K)} W_K + \sum_{\substack{K \\ |K|=r-n-1 \\ n \text{ odd}}} \xi^{p}_{(1,K)} V_{(1)} W_K. \tag{33}$$

Thus

$$\begin{aligned} \widetilde{w}_p \cdot \widetilde{w}_q =& w_p \cdot w_q + \sum_{\substack{K \\ |K|=r}} \xi^{p}_{(0,K)} W_K \cdot w_q + \sum_{\substack{K \\ |K|=r-n-1 \\ n \text{ odd}}} \xi^{p}_{(1,K)} V_{(1)} W_k \cdot w_q + \\ & \sum_{\substack{K \\ |K|=r}} \xi^{p}_{(0,K)} w_p \cdot W_K + \sum_{\substack{K \\ |K|=r-n-1 \\ n \text{ odd}}} \xi^{p}_{(1,K)} w_p \cdot V_{(1)} W_K. \end{aligned} \tag{34}$$

Using (20), (32) and (34), we have that $\widetilde{w}_p \cdot \widetilde{w}_q = \delta_{p+1,q}$ if and only if,

$$\begin{aligned} 0 =& \sum_{\substack{K \\ |K|=r-1}} \alpha^{p,q}_{(0,K)} W_K + \sum_{\substack{K \\ |K|=r-1}} (-k_{p+1} \xi^{p}_{(0,K)}) W_{(k_1,\dots,k_{p+1}-1,\dots,k_{2m})} + \\ & \sum_{\substack{K \\ |K|=r-1}} k_{p-1} \xi^{q}_{(0,K)} W_{(k_1,\dots,k_{p-1}-1,\dots,k_{2m})} + \sum_{\substack{K \\ |K|=r-n \\ n \text{ odd}}} \alpha^{p,q}_{(1,K)} V_{(1)} W_K + \\ & \sum_{\substack{K \\ |K|=r-n \\ n \text{ odd}}} \xi^{p}_{(1,K)} V_{(1)} W_{(k_1,\dots,k_{p+1}-1,\dots,k_{2m})} + \sum_{\substack{K \\ |K|=r-n \\ n \text{ odd}}} \xi^{q}_{(1,K)} V_{(1)} W_{(k_1,\dots,k_p+1,\dots,k_{2m})}. \end{aligned} \tag{35}$$

Combining the above equality with a linear independence property of the elements $V_I W_K$ we have the following relations:

$$\begin{aligned} \sum_{\substack{K \\ |K|=r-1}} \alpha^{p,q}_{(0,K)} W_K =& \sum_{\substack{K \\ |K|=r-1}} k_{p+1} \xi^{p}_{(0,K)} W_{(k_1\dots,k_{p+1}-1,\dots,k_{2m})} - \\ & \sum_{\substack{K \\ |K|=r-1}} (k_{p-1} \xi^{q}_{(0,K)}) W_{(k_1\dots,k_{p-1}-1,\dots,k_{2m})}, \end{aligned} \tag{36}$$

$$\sum_{\substack{K\\|K|=r-n\\n\text{ odd}}} \alpha^{p,q}_{(1,K)} V_{(1)} W_K = -\sum_{\substack{K\\|K|=r-n\\n\text{ odd}}} \xi^{p}_{(1,K)} V_{(1)} W_{(k_1\dots,k_p+1,\dots,k_{2m})} - \sum_{\substack{K\\|K|=r-n\\n\text{ odd}}} \xi^{q}_{(1,K)} V_{(1)} W_{(k_1\dots,k_p+1,\dots,k_{2m})}. \tag{37}$$

Let $\alpha^{p,q}_{(0,S_t)} W_{S_t}$ be a nonzero element at the left part of (36), such that $S_t = (s_1, \dots, s_{p-1}, s_p, s_{p+1}, \dots, s_n)$ is a 2m-tupla, with $|S_t| = r-1$. We shall find a 2m-tupla S_{p+1} and S_{p-1} on the right part of (36), such that $S_{p+1} = (s_1, \dots, s_{p-1}, s_p, s_{p+1}+1, \dots, s_n)$ and $S_{p-1} = (s_1, \dots, s_{p-1}+1, s_p, s_{p+1}, \dots, s_n)$. We observe that $|S_{p-1}| = |S_{p+1}| = r$.

Applying similar arguments to above stated, and using (37), we have that for each $K_p = (k_1, \dots, k_p, \dots, k_n)$ we should take $\widetilde{K}_p = (k_1, \dots, k_p - 1, \dots, k_n)$. Moreover, if $|K_p| = r-n$, then $|\widetilde{K}_p| = r-n-1$.

It is easy to see that equations (36) and (37) are respectively equivalent to

$$\begin{aligned} &\sum_{\substack{K\\|K|=r-1}} \Big(\alpha^{p,q}_{(0,S)} - (s_{p+1}+1)\xi^{p}_{(0,S_{p+1})} + (s_{p-1}+1)\xi^{q}_{(0,S_{p-1})}\Big) W_{S_t} = 0,\\ &\sum_{\substack{K\\|K|=r-n\\n\text{ odd}}} \Big(\alpha^{p,q}_{(1,K_p)} + \xi^{q}_{1,\widetilde{K}_p} + \xi^{p}_{1,\widetilde{K}_p}\Big) V_{(1)} W_{K_t} = 0. \end{aligned} \tag{38}$$

Using the linear independence of W_K, $V_{(1)}W_K$ and (38), for each $t \in \{1, \dots, 2m\}$, we have

$$\begin{aligned} \alpha^{p,q}_{(0,S_t)} - (s_{p+1}+1)\xi^{p}_{(0,S_{t+1})} + (s_{p-1}+1)\xi^{q}_{(0,S_{t-1})} &= 0,\\ \alpha^{p,q}_{(1,K_t)} + \xi^{q}_{(1,\widetilde{K}_t)} + \xi^{p}_{(1,\widetilde{K}_t)} &= 0. \end{aligned} \tag{39}$$

Hence, we have a solvable linear equation system if $r \neq n$. We note that a similar procedure is valid if r is an even integer. $\square$

Remark 1. By Lemmas 11 and 12, if n is an odd integer and $r = n$, then

$$\widetilde{w}_p = w_p + \sum_{\substack{K\\|K|=r}} \xi^{p}_{(0,K)} W_K, \quad \text{and} \quad \widetilde{w}_p \cdot \widetilde{w}_q = \delta_{i+1,j} + \sum_{\substack{K\\|K|=n-1}} \alpha^{p,q}_{(0,K)} W_K + \alpha^{p,q}_{(1,0)} V_{(1)}.$$

We see that the system $\alpha^{p,q}_{(0,S_t)} - \xi^{p}_{(0,S_{t+1})} + \xi^{q}_{(0,S_{t-1})} = 0$, and $\alpha^{p,q}_{(1,0)} = 0$ has no solution when $\alpha^{p,q}_{(1,0)} \neq 0$.

5.3. Counter-examples to WPT for Jordan superalgebras of superform with radical $\mathcal{C}_r/\mathcal{C}_{r-2}$ and $\dim \mathfrak{J}_0 = r$

Now we will show that the restrictions imposed in Theorem 13 are essential, and we have two cases to consider:

Case 1. Let n be an odd integer. Consider the superalgebra

$$\mathfrak{J} = (\mathbb{F}\cdot 1 + \mathbb{F}\cdot v_1 + \cdots + \mathbb{F}\cdot v_n + \mathcal{N}_0) \dotplus (\mathbb{F}\cdot w_1 + \cdots + \mathbb{F}\cdot w_{2m} + \mathcal{N}_1),$$

where

$$\mathcal{N}_0 = \mathrm{Spann}\langle\, v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}}, \quad |K| \text{ is even}, |\, I\,| + |\, K\,| = n-1 \text{ or } n\,\rangle,$$
$$\mathcal{N}_1 = \mathrm{Spann}\langle\, v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}}, \quad |K| \text{ is odd }, |\, I\,| + |\, K\,| = n-1 \text{ or } n\,\rangle,$$

and $i_1,\ldots,i_n$ are 0 or 1, and k_i are nonnegative integers, $|K| = k_1 + \cdots + k_{2m}$, $|I| = i_1 + \cdots + i_n$. All nonzero products of the basis elements of $\mathfrak{J}$ are defined as follows

$$v_i^2 = 1, w_1\cdot w_2 = 1 + v_1\cdots v_n = -w_2\cdot w_1,$$
$$w_{2s-1}\cdot w_{2s} = -w_{2s}\cdot w_{2s-1} = 1 \text{ for } s\in\{2,3,\ldots,m\},$$
$$v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}}\cdot w_p = \frac{1}{2} v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_p^{k_p+1}\cdots w_{2m}^{k_{2m}}(1+(-1)^{|I|+|K|}) -$$
$$k_{p+1} v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{p+1}^{k_p-1}\cdots w_{2m}^{k_{2m}} \text{ if } p = 2s-1,\ s\in\{1,\ldots,m\},$$
$$v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}}\cdot w_p = \frac{1}{2} v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_p^{k_p+1}\cdots w_{2m}^{k_{2m}}(1+(-1)^{|I|+|K|}) +$$
$$k_{p-1} v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{p-1}^{k_{p-1}-1}\cdots w_{2m}^{k_{2m}} \text{ if } p = 2s,\ s\in\{1,\ldots,m\},$$
$$v_1^{i_1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}}\cdot v_j =$$
$$(-\frac{1}{2})^{i_1+\cdots+i_{j-1}} v_1^{i_1}\cdots v_j^{i_j+1}\cdots v_n^{i_n} w_1^{k_1}\cdots w_{2m}^{k_{2m}}(1+(-1)^{|I|+|K|-i_j}).$$

We note that $\mathfrak{J}/\mathcal{N} = (\mathbb{F}\cdot 1 + \mathbb{F}\cdot v_1 + \cdots + \mathbb{F}\cdot v_n) \dotplus (\mathbb{F}\cdot w_1 + \cdots + \mathbb{F}\cdot w_{2m})$ is a Jordan superalgebra isomorphic to Jordan superalgebra of superform, $\mathcal{N}$ is isomorphic to $\mathcal{C}_n/\mathcal{C}_{n-2}$.

If we assume that the WPT is valid for $\mathfrak{J}$, then, for $i = 1,\ldots,2m$ there exists $\widetilde{w}_i \in \mathfrak{J}_1$ such that $\widetilde{w}_i \equiv w_i (\mathrm{mod} \mathcal{N}_1)$, and $\widetilde{w}_{2i-1}\cdot\widetilde{w}_{2i} = 1$.

By Lemma 12 there exist β_K, ξ_K, α, $\lambda \in \mathbb{F}$ such that $\widetilde{w}_1 = w_1 + \sum\limits_{|K|=n} \beta_K w_1^{k_1}\cdots w_{2m}^{k_{2m}}$ and $\widetilde{w}_2 = w_2 + \sum\limits_{|K|=n} \xi_K w_1^{k_1}\cdots w_{2m}^{k_{2m}}$. Hence,

$$\widetilde{w}_1\cdot\widetilde{w}_2 = w_1\cdot w_2 + \sum_{|K|=n} \xi_k w_1\cdot w_1^{k_1}\cdots w_{2m}^{k_{2m}} + \sum_{|K|=n} \beta_K w_1^{k_1}\cdots w_{2m}^{k_{2m}}\cdot w_2.$$

We observe that $\widetilde{w}_1 \cdot \widetilde{w}_2 = 1$ if and only if $v_1 \cdots v_n + \sum_{|K|=n} \omega_K w_1^{t_1} \cdots w_{2m}^{t_{2m}} = 0$. Using the fact that $v_1 \cdots v_n$ and $w_1^{k_1} \cdots w_{2m}^{k_{2m}}$ are linearly independent, we have a contradiction.

Case 2. Let n be an even integer. Consider the superalgebra

$$\mathfrak{J} = (\mathbb{F} \cdot 1 + \mathbb{F} \cdot v_1 + \cdots + \mathbb{F} \cdot v_n + \mathcal{N}_0) \dot{+} (\mathbb{F} \cdot w_1 + \cdots + \mathbb{F} \cdot w_{2m} + \mathcal{N}_1).$$

$\mathcal{N}_0$ is spanned by $\langle\, uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}}, \quad |K| \text{ is even}\,\rangle$ and $\mathcal{N}_1$ is spanned by $\langle\, uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}}, \quad |K| \text{ is odd}\,\rangle$, where $i_1, \ldots, i_n$ are 0 or 1 and k_i are nonnegative integers, $|K| = k_1 + \cdots + k_{2m}$, $|I| = i_1 + \cdots + i_n$ and $|K| + |I| = n$ or $|K| + |I| = n-1$.

All nonzero products of the basis elements of $\mathfrak{J}$ are defined as follows

$$v_i^2 = 1, \quad w_1 \cdot w_2 = 1 + uv_1 \cdots v_n = -w_2 \cdot w_1,$$

$$w_{2s-1} \cdot w_{2s} = -w_{2i} \cdot w_{2i-1} = 1 \text{ for } s \in \{2, 3, \ldots, m\},$$

$$uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}} \cdot w_p = \frac{1}{2} uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_p^{k_p+1} \cdots w_{2m}^{k_{2m}} (1 + (-1)^{|I|+|K|}) +$$
$$k_{p+1} uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{p+1}^{k_{p+1}-1} \cdots w_{2m}^{k_{2m}}, \quad \text{if } p = 2s-1,\ s \in \{1, \ldots, m\},$$

$$uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}} \cdot w_p = \frac{1}{2} uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_p^{k_p+1} \cdots w_{2m}^{k_{2m}} (1 + (-1)^{|I|+|K|}) -$$
$$k_{p-1} uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{p-1}^{k_{p-1}-1} \cdots w_{2m}^{k_{2m}}, \quad \text{if } p = 2s,\ s \in \{1, \ldots, m\},$$

$$uv_1^{i_1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}} \cdot v_j =$$
$$(-\frac{1}{2})^{i_1 + \cdots + i_{j-1} + 1} uv_1^{i_1} \cdots v_j^{i_j+1} \cdots v_n^{i_n} w_1^{k_1} \cdots w_{2m}^{k_{2m}} (1 + (-1)^{|I|+|K|-i_j}).$$

It is easy to verify that $\mathfrak{J}/\mathcal{N}$ is a Jordan superalgebra of superform and $\mathcal{N} \cong u\,\mathcal{C}_n / u\,\mathcal{C}_{n-2}$.

If we assume that the WPT is valid for $\mathfrak{J}$, then, for $i = 1, \ldots, 2m$ there exists $\widetilde{w}_i \in \mathfrak{J}_1$ such that $\widetilde{w}_i \equiv w_i (\mathrm{mod} \mathcal{N}_1)$, and $\widetilde{w}_{2i-1} \cdot \widetilde{w}_{2i} = 1$, $i \geq 2$.

By an analogous to Lemma 12, we have that $\widetilde{w}_1 = w_1 + \sum_{|K|=n-1} \beta_K uw_1^{k_1} \cdots w_{2m}^{k_{2m}}$ and $\widetilde{w}_2 = w_2 + \sum_{|K|=n-1} \xi_K uw_1^{k_1} \cdots w_{2m}^{k_{2m}}$ for some β_K, ξ_K, α, $\lambda \in \mathbb{F}$.

It is clear that $\widetilde{w}_1 \cdot \widetilde{w}_2 = w_1 \cdot w_2 + \sum_{|K|=n-1} \omega_K uw_1 \cdot w_1^{t_1} \cdots w_{2m}^{t_{2m}}$, $\omega_K \in \mathbb{F}$. Therefore $\widetilde{w}_1 \cdot \widetilde{w}_2 = 1$ if and only if $uv_1 \cdots v_n + \sum_{|K|=n-1} \omega_K uw_1 \cdot w_1^{k_1} \cdots\ w_{2m}^{k_{2m}} = 0$. Once again, $uv_1 \cdots v_n$ and $uw_1^{k_1} \cdots w_{2m}^{k_{2m}}$ are linearly independent, consequently, we have a contradiction.

6. Superalgebra $\mathcal{D}_t$ and Kaplansky $\mathcal{K}_3$

In this section, we consider the Jordan superalgebra $\mathcal{D}_t = (\mathbb{F}\cdot e_1+\mathbb{F}\cdot e_2)\dot{+}(\mathbb{F}\cdot x+\mathbb{F}\cdot y)$, and Kaplansky, $\mathcal{K}_3 = (\mathbb{F}\cdot e)\dot{+}(\mathbb{F}\cdot x+\mathbb{F}\cdot y)$.

We stress that $\mathcal{D}_t$ is a simple Jordan superalgebra if $t \neq 0$. If $t = 0$, then $\mathcal{D}_0$ contain $\mathcal{K}_3$. Unital irreducible bimodules over $\mathcal{D}_t$ and $\mathcal{K}_3$ were classified by C. Martinez and E. Zelmanov in [16] and by M. Trushina in [15]. In this section, we shall use the examples, notations and ideas introduced by M. Trushina.

Let sl_2 be a Lie algebra with the basis e, f, h and the multiplication given by $[f, h] = 2f$, $[e, h] = 2e$, $[e, f] = h$, where $[a, b] = ab - ba$.

We shall say that a module $\mathcal{L}$ with the basis $l_0, l_1, \dots, l_n$ is an irreducible sl_2-module with standard basis $l_0, l_1, \dots, l_n$ if

$$\begin{aligned} & & l_i\cdot h &= (n-2i)l_i, & \\ l_0\cdot e &= 0, & l_i\cdot e &= (-in+i(i-1))l_{i-1} & \text{for } i > 0,\\ l_n\cdot f &= 0, & l_i\cdot f &= l_{i+1} & \text{for } i < n. \end{aligned}$$

By R_a we denote the operator of right multiplication by a, we also denote it by the capital letter A. One can easily check that the operators $\frac{2}{1+t}X\circ Y$, $\frac{2}{1+t}X^2$, $\frac{2}{1+t}Y^2$ span the simple lie algebra sl_2. In terms of operators above, it is easy to see that a bimodule $\mathcal{L}$ with basis $l_0, l_1, \dots, l_n$ is an irreducible sl_2-module with the standard basis $l_0, l_1, \dots, l_n$ if

$$\begin{aligned} & l_i\, X\circ Y = \frac{1+t}{2}(n-2i)l_i,\\ & l_0\, X^2 = 0, \quad l_i\, X^2 = \frac{1+t}{2}(-in+i(i-1)l_{i-1}) \text{ for any } i > 0,\\ & l_n\, Y^2 = 0, \quad l_i\, Y^2 = \frac{1+t}{2}l_{i+1} \text{ for } i < n. \end{aligned} \tag{40}$$

In terms of right multiplication operators, equality (2) may be written as follows:

$$\begin{aligned} \mathrm{R}_{a_i}\mathrm{R}_{a_j}\mathrm{R}_{a_k}+(-1)^{ij+ik+jk}\mathrm{R}_{a_k}\mathrm{R}_{a_j}\mathrm{R}_{a_i} + (-1)^{jk}\mathrm{R}_{(a_ia_k)a_j} =&\\ \mathrm{R}_{a_i}\mathrm{R}_{a_ja_k} + (-1)^{ij+ik+jk}\mathrm{R}_{a_k}\mathrm{R}_{a_ja_i} + (-1)^{ij}\mathrm{R}_{a_j}\mathrm{R}_{a_ia_k}.& \end{aligned} \tag{41}$$

Substituting $a_i = a_k = x$ and $a_j = e_1$ (respectively $a_i = a_k = y$ and $a_j = e_1$) in (41), we obtain $[X^2, E] = 0$ (respectively $[Y^2, E] = 0$), where E denote R_{e_1}.

6.1. Jordan superalgebra $\mathcal{D}_t$

In this section, we shall prove the following theorem.

Theorem 14. *Let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with a solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$, $\mathcal{A}/\mathcal{N} \cong \mathcal{D}_t$, $t \neq -1$, and $\mathcal{N} \in \overline{\mathfrak{M}}(\mathcal{A}/\mathcal{N}; Reg\mathcal{D}_t)$, where $Reg\mathcal{D}_t$ is a* *regular $\mathcal{D}_t$-bimodule. Then there exists a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathcal{D}_t$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.*

Proof. Using 3 and Theorem 1.1 in [15], we need to consider three main cases. Here, $\mathcal{S}_0 = \mathbb{F} \cdot \widetilde{e}_1 + \mathbb{F} \cdot \widetilde{e}_2 \cong (\mathcal{D}_t)_0$ and $\mathcal{A}_1/\mathcal{N}_1 = \mathbb{F} \cdot \bar{x} + \mathbb{F} \cdot \bar{y} \cong (\mathcal{D}_t)_1$, $\mathcal{A}_0 = \mathcal{S}_0 \oplus \mathcal{N}_0$.

Case 1. Let n be a positive integer and suppose $t \in \mathbb{R}$, $t \neq 0, 1, -\frac{n+2}{n}$.

We assume that where $\mathcal{N}_0 = \mathcal{L}^1_{n+1} \oplus \mathcal{L}^2_{n+1}$, $\mathcal{N}_1 = \mathcal{L}_{n+2} \oplus \mathcal{L}_n$. Here, $\mathcal{L}^1_{n+1}$, $\mathcal{L}^2_{n+1}$, $\mathcal{L}_{n+2}$, $\mathcal{L}_n$ are the same as in Example 1 in [15].

It is easy to see that $E\mid_{\mathcal{N}_1} \equiv \frac{1}{2}$, therefore $\widetilde{e}_i \cdot \widetilde{x} = \frac{1}{2}\widetilde{x}$ and $\widetilde{e}_i \cdot \widetilde{y} = \frac{1}{2}\widetilde{y}$. Assume that there exist scalars $\beta^x_{3,i}$, $\beta^x_{4,i}$, $\beta^y_{3,i}$, $\beta^y_{4,i}$, $\xi^{x,y}_{1,j}$ and $\xi^{x,y}_{2,j}$ for $i = -1, 0, 1, \ldots, n$ and $j = 0, 1, \ldots, n$, such that

$$\begin{aligned}\widetilde{x} &= x + \beta^x_{3,-1}m + \beta^x_{3,n}mY^{2(n+1)} + \sum_{k=1}^{n} \gamma^x_{3,4,k}mY^{2k} - \sum_{k=1}^{n} \beta^x_{4,k}mY^{2k-1}EY, \\ \widetilde{y} &= y + \beta^y_{3,-1}m + \beta^y_{3,n}mY^{2(n+1)} + \sum_{k=1}^{n} \gamma^y_{3,4,k}mY^{2k} - \sum_{k=1}^{n} \beta^y_{4,k}mY^{2k-1}EY,\end{aligned} \tag{42}$$

$$x \cdot y = \widetilde{e}_1 + t\widetilde{e}_2 + \sum_{k=0}^{n} (\xi^{x,y}_{1,k} - \xi^{x,y}_{2,k})mY^{2k+1}E + \sum_{k=0}^{n} \xi^{x,y}_{2,k}mY^{2k+1}, \tag{43}$$

where $\gamma^x_{3,4,k} = \beta^x_{3,k-1} + \alpha\beta^x_{4,k}$ and $\gamma^y_{3,4,k} = \beta^y_{3,k-1} + \alpha\beta^y_{4,k}$.

Now, we have $\widetilde{x} \cdot \widetilde{y} = \widetilde{e}_1 + t\widetilde{e}_2$, if and only if

$$\begin{aligned}0 = &\Big(\beta^x_{3,-1}m + \beta^x_{3,n}mY^{2n+2} + \sum_{k=1}^{n} \gamma^x_{3,4,k}mY^{2k} - \sum_{k=1}^{n} \beta^x_{4,k}mY^{2k-1}EY\Big) \cdot y - \\ &\Big(\beta^y_{3,-1}m + \beta^y_{3,n}mY^{2n+2} + \sum_{k=1}^{n} \gamma^y_{3,4,k}mY^{2k} - \sum_{k=1}^{n} \beta^y_{4,k}mY^{2k-1}EY\Big) \cdot x + \\ &\sum_{k=0}^{n} (\xi^{x,y}_{1,k} - \xi^{x,y}_{2,k})mY^{2k+1}E + \sum_{k=0}^{n} \xi^{x,y}_{2,k}mY^{2k+1} = \\ &\beta^x_{3,-1}mY + \beta^x_{3,n}mY^{2n+3} + \sum_{k=1}^{n} \gamma^x_{3,4,k}mY^{2k+1} + \sum_{k=1}^{n} \beta^x_{4,k}mY^{2k+1}E + \\ &\beta^y_{3,n}\frac{(1+t)(n+1)}{2}mY^{2n+1} + \sum_{k=1}^{n} \frac{(1+t)(n-(k-1))}{2}\beta^y_{4,k}mY^{2k-1}E + \\ &\sum_{k=1}^{n} \Big(\frac{1+t}{2}k\gamma^y_{3,4,k} + (-1)^k\frac{(1+t)n+2}{4}\beta^y_{4,k}\Big)mY^{2k-1} + \\ &\sum_{k=1}^{n+1} (\xi^{x,y}_{1,k-1} - \xi^{x,y}_{2,k-1})mY^{2k-1}E + \sum_{k=1}^{n+1} \xi^{x,y}_{2,k-1}mY^{2k-1}.\end{aligned} \tag{44}$$

Since $\alpha = \frac{(1+t)n+2}{2(1+t)(n+1)}$, we have that

$$\left[\beta^x_{3,-1} + \frac{(1+t)\beta^y_{3,0}}{2} - \frac{n\alpha(1+t)\beta^y_{4,1}}{2} + \xi^{x,y}_{2,0}\right] mY + \beta^x_{3,n} mY^{2n+3} +$$

$$\left[\beta^x_{3,n-1} + \alpha\beta^x_{4,n} + \xi^{x,y}_{2,n} + \frac{(1+t)(n+1)\beta^y_{3,n}}{2}\right] mY^{2n+1} +$$

$$\left[\frac{(1+t)n\beta^y_{4,1}}{2} + \xi^{x,y}_{1,0} - \xi^{x,y}_{2,0}\right] mYE + \left[\xi^{x,y}_{1,n} - \xi^{x,y}_{2,n} + \beta^x_{4,n}\right] mY^{2n+1}E +$$

$$\sum_{k=2}^{n}\left[\beta^x_{3,k-2} + \alpha\beta^x_{4,k-1} + \frac{(1+t)}{2}[k\beta^y_{3,k-1} + \alpha(k+(-1)^k(n+1)\beta^y_{4,k}] + \xi^{x,y}_{2,k-1}\right] mY^{2k-1} +$$

$$\sum_{k=2}^{n}\left[\beta^x_{4,k-1} + \beta^y_{4,k}\frac{(1+t)(n-(k-1))}{2} + \xi^{x,y}_{1,k-1} - \xi^{x,y}_{2,k-1}\right] mY^{2k-1}E = 0. \tag{45}$$

Note that fixing the ξ's in (43), we get

$$\begin{aligned}
&\beta^x_{3,n} = 0, \quad \beta^x_{4,n} = \xi^{x,y}_{2,n} - \xi^{x,y}_{1,n}, \quad \beta^y_{4,1} = \frac{2(\xi^{x,y}_{2,0} - \xi^{x,y}_{1,0})}{(t+1)n},\\
&\beta^x_{3,0} = \frac{n\alpha(1+t)\beta^y_{4,1} - 2\beta^x_{3,-1} - 2\xi^{x,y}_{2,0}}{(1+t)},\\
&\beta^y_{3,n} = \frac{-2(\beta^x_{3,n-1} + \alpha\beta^x_{4,n} + \xi^{x,y}_{2,n})}{(1+t)(n+1)},\\
&\beta^y_{4,k} = \frac{2}{(1+t)(n-(k-1))}\left[-\beta^x_{4,k-1} - \xi^{x,y}_{1,k-1} + \xi^{x,y}_{2,k-1}\right], \quad k = 2,\dots,n\\
&\beta^y_{3,k-1} = \frac{1}{k}\left[\frac{-2(\beta^x_{3,k-2} + \alpha\beta^x_{4,k-1} + \xi^{x,y}_{2,k-1})}{(1+t)} - \alpha(k+(-1)^k(n+1)\beta_{4,k}\right]
\end{aligned} \tag{46}$$

such that equality (45) holds.

Case 2. Let n be a positive integer, $\frac{1}{t} = -\frac{n}{n+2}$. Consider the following cases: $\mathcal{N} \cong \mathcal{M}(n+1, n+2)$, $\mathcal{N} \cong \mathcal{M}(n+1, n)$, $\mathcal{N} \cong \widetilde{\mathcal{M}}(n_1)$ and $n_1 \neq n$. (See example 2 in [15].)

(A) Assume that $\mathcal{N} \cong \mathcal{M}(n+1, n+2)$ where $\mathcal{N}_0$ is the irreducible sl_2-module with the standard basis $l_0, \dots, l_n$ and $\mathcal{N}_1$ is spanned by $l_0x, l_0y, l_1y, \dots, l_ny$.

Since $E\mid_{\mathcal{M}_1} \equiv \frac{1}{2}$, then $\widetilde{e}_i \cdot \widetilde{x} = \frac{1}{2}\widetilde{x}$ and $\widetilde{e}_i \cdot \widetilde{y} = \frac{1}{2}\widetilde{y}$. Assume that there exist $\xi^{u,z}_0, \dots, \xi^{u,z}_n$, $\beta^u_{0,x}$, $\beta^u_0, \dots$, β^u_n, $\beta^z_{0,x}$, β^z_0, …, and β^z_n scalars such that

$$x \cdot y = \widetilde{e}_1 + t\widetilde{e}_2 + \sum_{k=0}^{n} \xi_k^{x,y} l_k,$$

$$\widetilde{x} = x + \beta_{0,x}^{x} l_0 x + \sum_{k=0}^{n} \beta_k^{x} l_k y, \quad \widetilde{y} = y + \beta_{0,x}^{y} l_0 x + \sum_{k=0}^{n} \beta_k^{y} l_k y$$

We observe that $\widetilde{x} \cdot \widetilde{y} = \widetilde{e}_1 + t\widetilde{e}_2$ if and only if

$$0 = \sum_{k=0}^{n} \xi_k^{x,y} l_k + \beta_{0,x}^{x} \left(\frac{1-t}{2}\right) l_0 + \beta_n^{y} \left(\frac{(n+1)t}{n+2}\right) l_n$$
$$+ \sum_{k=0}^{n-1} \beta_k^{x} \left(\frac{1+t}{2}\right) l_{k+1} - \sum_{k=0}^{n-1} \beta_k^{y} \left(\frac{1+t}{2(n-k)}\right) (-(k+1)n + (k+1)k) l_k,$$

which gives rise to the system of equations

$$\begin{aligned} 0 &= \xi_0^{x,y} + \beta_{0,x}^{x} \left(\frac{1-t}{2}\right) + \beta_0^{y} \left(\frac{1+t}{2}\right), \\ 0 &= \xi_n^{x,y} + \beta_n^{y} \left(\frac{(n+1)t}{n+2}\right) + \beta_{n-1}^{x} \left(\frac{1+t}{2}\right), \\ 0 &= \xi_k^{x,y} + \beta_{k-1}^{x} \left(\frac{1+t}{2}\right) - \beta_k^{y} \left(\frac{1+t}{2(n-k)}\right) (-(k+1)n + (k+1)k) l_k, \end{aligned} \tag{47}$$

for $k = 1, 2, \ldots, n-1$. We note that the system (47) has always a solution.

(B) $\mathcal{N} \cong \mathcal{M}(n+1, n)$ where $\mathcal{N}_0$ is the irreducible sl_2-module with the standard basis $l_0, \ldots, l_n$ and $\mathcal{N}_1$ is spanned by $l_1 x, \ldots, l_n x$.

As in the case **(A)**, $\widetilde{e}_i \cdot \widetilde{x} = \frac{1}{2}\widetilde{x}$, $\widetilde{e}_i \cdot \widetilde{y} = \frac{1}{2}\widetilde{y}$ and there exist $\xi_0^{x,y}, \ldots, \xi_n^{x,y} \in \mathbb{F}$ such that $x \cdot y = \widetilde{e}_1 + t\widetilde{e}_2 + \sum_{k=0}^{n} \xi_k^{x,y} l_k$. Let us find β_k^x and β_k^y such that $\widetilde{x} = x + \sum_{k=1}^{n} \beta_k^x l_k x$, and $\widetilde{y} = y + \sum_{k=1}^{n} \beta_k^y l_k x$.

Now, we note that $\widetilde{x} \cdot \widetilde{y} = \widetilde{e}_1 + t\widetilde{e}_2$ if and only if

$$0 = \sum_{k=0}^{n} \xi_k^{x,y} l_k - \sum_{k=1}^{n} \beta_k^{x} \left(\frac{1+t}{2}\right) k\, l_k - \sum_{k=1}^{n} \beta_k^{y} \left(\frac{1+t}{2}\right) (-kn + k(k-1)) l_{k-1},$$

which gives rise to the system of equations

$$\begin{aligned} 0 = &\quad \xi_0^{x,y} - \beta_0^{y} \left(\frac{1-t}{2}\right) n = \xi_n^{x,y} - n\beta_n^{x} \left(\frac{1+t}{2}\right), \\ 0 = &\quad \xi_k^{x,y} - \beta_k^{x} k \left(\frac{1+t}{2}\right) - \beta_{k+1}^{y} \left(\frac{1+t}{2}\right) (-(k+1)n + (k+1)k), \end{aligned} \tag{48}$$

for $k = 1, 2, \ldots, n-1$.

System of equations (48) has always a solution.

(C) Finally, if $\mathcal{N} \cong \widetilde{\mathcal{M}}(n_1)$, $n_1 \neq n$. This case is similar to Case **(1)**, with the replacement of n by n_1.

Case 3. Let $t = 1$. In this case, $\mathcal{N}$ is isomorphic to $\widetilde{\mathcal{M}}(n)$ or to a 1-dimensional vector space with a generator m such that $mx = 0$, $me = \frac{1}{2}m$. (See example 3 in [15].)

It only remains to consider the case when $\mathcal{N}$ is isomorphic to a 1-dimensional vector space. The case when $\mathcal{N} \cong \widetilde{\mathcal{M}}(n)$ is similar to Case **(1)**. In particular we take $t = 1$ in equation (45). Now we shall consider two subcases:

(A) If m is an even vector, then $\mathcal{N}_0 = \mathbb{F} \cdot m$. Assume that $x \cdot y = \widetilde{e}_1 + t\widetilde{e}_2 + \eta m$, for some $\eta \in \mathbb{F}$ Since $\mathcal{N}_1 = 0$ we have $\widetilde{x} = x$, $\widetilde{y} = y$. Note that the equality $\widetilde{x} \cdot \widetilde{y} = \widetilde{e}_1 + t\widetilde{e}_2$ holds if and only if $\eta = 0$. If we take $a_i = x$, $a_j = y$, $a_k = e_1$ and $a_l = e_2$ in the equality (2) we obtained, $0 = ((\widetilde{x} \cdot \widetilde{y})e_1)e_2 = ((e_1 + te_2 + \eta m)e_1)e_2 = \frac{1}{4}\eta m$ and therefore, $\eta = 0$, thus the WPT is valid.

(B) If m is an odd vector, then $\mathcal{N}_0 = 0$ and $\mathcal{N}_1 = \mathbb{F} \cdot m$. Therefore, $x \cdot y = \widetilde{e}_1 + t\widetilde{e}_2$. Let $\widetilde{x} = x + \beta^u m$ and $\widetilde{y} = y + \beta^z m$, hence $\widetilde{x} \cdot \widetilde{y} = \widetilde{e}_1 + t\widetilde{e}_2$ is always solvable.

From Cases **(1)**–**(3)**, we conclude that it is possible to give some conditions for η_u and $\eta_z \in \mathcal{N}_1$, such that an analogue to WPT is valid under the Theorem conditions. □

6.2. Jordan superalgebra $\mathcal{K}_3$

We shall proof the following theorem

Theorem 15. *Let $\mathcal{A}$ be a finite-dimensional Jordan superalgebra with solvable radical $\mathcal{N}$, $\mathcal{N}^2 = 0$ and such that $\mathcal{A}/\mathcal{N} \cong \mathcal{K}_3$. Then there exists a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{A}/\mathcal{N} \cong \mathcal{S}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$.*

Proof. Since Theorem 3 and [15], we have to consider two cases, $\mathcal{N} \cong \mathcal{R}\mathrm{eg}\,\mathcal{K}_3$ and $\mathcal{N} \cong \widetilde{\mathcal{M}}(n)$. But the second cases is analogous to case **(1)**, for $\mathcal{D}_t$, one can obtain an analogue of equality (44) substituting $t = 0$. This gives rise to the system of equations equivalent to (46).

We consider $\mathcal{N} \cong \mathcal{R}\mathrm{eg}\,\mathcal{K}_3$. Assume that $(\mathcal{K}_3)_0 \cong \mathcal{S}_0 = \mathbb{F} \cdot \widetilde{e}_1$, $(\mathcal{K}_3)_1 \cong \mathcal{A}_1/\mathcal{N}_1 = \mathbb{F} \cdot \bar{x} + \mathbb{F} \cdot \bar{y}$, $\mathcal{N} = \mathbb{F} \cdot f \dotplus (\mathbb{F} \cdot u + \mathbb{F} \cdot z)$, where $f \leftrightarrow e$, $u \leftrightarrow x$, $z \leftrightarrow y$. Let $\widetilde{x}$ and $\widetilde{y}$ be some preimages of $\bar{x}$ and $\bar{y}$ respectively and suppose that $xy = \widetilde{e}_1 + \eta f$ for some $\eta \in \mathbb{F}$. Let α, β, γ and δ be scalars such that $\widetilde{x} = x + \alpha u + \beta z$ and $wy = y + \gamma u + \delta z$. We note that $\widetilde{x} \cdot \widetilde{y} = \widetilde{e}_1$ if and only if $\alpha + \delta = \eta$. The equality is always solvable. □

Remark 2. In the case of the Jordan superalgebra $\mathcal{K}_3 \oplus \mathbb{F} \cdot 1$ we have that the irreducible bimodules are the same as the ones for the Jordan superalgebra $\mathcal{K}_3$. In general, for any algebra $\mathcal{A}$ there exist an isomorphism of category of bimodules over $\mathcal{A}$, (Bimod $\mathcal{A}$) into category of unital bimodules over $\mathcal{A}^{\#}$, (Bimod $\mathcal{A}^{\#}$). Thus, the proof of the above theorem is also true if we substitute $\mathcal{K}_3$ by $\mathcal{K}_3 \oplus \mathbb{F} \cdot 1$.

6.3. *Counter-examples to WPT for Jordan superalgebras of type $\mathcal{D}_t$, $t \neq -1$*

Now we will show that restrictions imposed in Theorem 14 are essential.

Let $\mathcal{B} = \mathcal{A} \oplus \mathcal{N}$ be a superalgebra, where $\mathcal{A}_0 = \mathbb{F} \cdot e_1 + \mathbb{F} \cdot e_2 + \mathbb{F} \cdot a_1 + \mathbb{F} \cdot a_2$, $\mathcal{A}_1 = \mathbb{F} \cdot x + \mathbb{F} \cdot y + \mathbb{F} \cdot v + \mathbb{F} \cdot w$, $\mathcal{N}_0 = \mathbb{F} \cdot a_1 + \mathbb{F} \cdot a_2$ and $\mathcal{N}_1 = \mathbb{F} \cdot v + \mathbb{F} \cdot w$. All nonzero products of the basis elements of $\mathcal{B}$ are defined as follows:

$$\begin{aligned} &e_i^2 = e_i, \quad e_i a_j = \delta_{ij} a_j, \quad e_i x = \frac{1}{2} x, \quad e_i y = \frac{1}{2} y, \\ &a_i x = \frac{1}{2} v, \quad a_i y = \frac{1}{2} w, \quad e_i v = \frac{1}{2} v, \quad e_i w = \frac{1}{2} w, \end{aligned} \tag{49}$$

$$xw = vy = a_1 + t a_2, \quad xy = e_1 + t e_2 + a_1 + (-2 - t) a_2 \tag{50}$$

for $i = 0, 1$. The products in (49) and (50) commute and anticommute respectively and $t \neq -1$.

One easily verifies that $\mathcal{B}$ is a Jordan superalgebra, and $\mathcal{B}/\mathcal{N}$ is a Jordan superalgebra isomorphic to $\mathcal{D}_t$, $t \neq -1$, $\mathcal{N} \cong \mathcal{R}\mathrm{eg}\, \mathcal{D}_t$ and $\mathcal{N}^2 = 0$.

Consider the product $xy = e_1 + t e_2 + \alpha a_1 + \beta a_2$. Replacing $a_i = a_k = x$, $a_j = y$ and $a_l = e_1$ in (2) we obtain $((xy) \cdot x) \cdot e_1 - \frac{1}{2}(xy) \cdot x = 0$, thus we have $1 + t + \alpha + \beta = 0$, later on $\alpha + \beta = -1 - t$ and therefore, $\mathcal{B}$ is a Jordan superalgebra.

If we assume that the WPT is valid for $\mathcal{B}$, then there are $\widetilde{x}, \widetilde{y}$ such that $\widetilde{x} \equiv x$, $\widetilde{y} \equiv y \,(\mathrm{mod}\, \mathcal{N}_1)$ and $\widetilde{x}\widetilde{y} = e_1 + t e_2$, $e_i \widetilde{x} = \frac{1}{2} \widetilde{x}$, $e_i \widetilde{y} = \frac{1}{2} \widetilde{y}$.

We note that $\widetilde{x} = x + \sigma v$ and $\widetilde{y} = y + \omega w$. If $\widetilde{x} = x + \sigma v + \lambda w$, using $\widetilde{x}^2 = 0$, we obtain $\lambda x w = 0$ and therefore $\lambda = 0$. Now

$$\widetilde{x}\widetilde{y} = xy + \sigma y v + \omega x w = e_1 + t e_2 + a_1 + (-2 - t) a_2 - \sigma(a_1 + t a_2) + \omega(a_1 + t a_2)$$

Therefore, $1 - \sigma + \omega = 0$ and $(-2 - t) - \sigma t + \omega t = 0$, later on $\omega - \sigma = -1$ and $0 = (-2 - t) + t(\omega - \sigma) = -2 - 2t$, thus $t = -1$ and this is a contradiction.

7. Main theorem

Using Theorems 9, 13, 14 and 15, we have the following theorem:

Theorem 16. *Let $\mathcal{A}$ be a finite dimensional Jordan superalgebra with solvable radical $\mathcal{N}$ such that $\mathcal{N}^2 = 0$ and $\mathcal{A}/\mathcal{N} \cong \mathfrak{J}$ where $\mathfrak{J}$ is a simple Jordan superalgebra. We set $\mathfrak{M}(\mathfrak{J}; \mathcal{N}_1, \ldots, \mathcal{N}_t) = \{ V/V$ is a $\mathfrak{J}$-bimodule such that homomorphic images of V do not contain subbimodules isomorphic to $\mathcal{N}_i$ for $i = 1, 2, \ldots, t\}$. If one of the following conditions holds:*

i) $\mathfrak{J} \cong \mathcal{K}_{10}$;
ii) $\mathfrak{J} \cong \mathcal{K}_3$;

iii) *$\mathfrak{J}$ is a superalgebra of a superform with even part of dimension n such that $\mathcal{N} \in \mathfrak{M}(\mathfrak{J}; \mathcal{C}_n/\mathcal{C}_{n-2}\,(n \text{ is odd}),\, u \cdot \mathcal{C}_n/u \cdot \mathcal{C}_{n-2}\,(n \text{ is even}))$;*

iv) *$\mathfrak{J} \cong \mathcal{D}_t$, $t \neq -1$, $\mathcal{N} \in \mathfrak{M}(\mathcal{D}_t; \mathcal{R}\text{eg } \mathcal{D}_t)$;*

then there is a subsuperalgebra $\mathcal{S} \subseteq \mathcal{A}$ such that $\mathcal{S} \cong \mathfrak{J}$ and $\mathcal{A} = \mathcal{S} \oplus \mathcal{N}$. The restrictions of items iii) and iv) are essential.

Acknowledgments

The author is thankful to Prof. Ivan Shestakov for his suggestion to solve the problem considered in this paper as a part of the author's Doctoral Thesis and for other valuable advises.

The author wishes to express his thanks to referee for several helpful comments.

References

[1] Th. Molien, On systems of higher complex numbers (Ueber Systeme höherer complexer Zahlen), Math. Ann. XLI (1893) 83–156.

[2] Th. Molien, Correction to the article "On systems of higher complex numbers" (Berichtigung zu dem Aufsatze "Ueber Systeme höherer complexer Zahlen"), Math. Ann. XLII (1893) 308–312.

[3] S. Epsteen, J.H. Maclagan-Wedderburn, On the structure of hypercomplex number systems, Trans. Amer. Math. Soc. 6 (1905) 172–178.

[4] R.D. Schafer, The Wedderburn principal theorem for alternative algebras, Bull. Amer. Math. Soc. 55 (1949) 604–614.

[5] A.A. Albert, The Wedderburn principal theorem for Jordan algebras, Ann. of Math. 48 (1) (1947) 1–7.

[6] A.J. Penico, The Wedderburn principal theorem for Jordan algebras, Trans. Amer. Math. Soc. 70 (1951) 404–420.

[7] V.G. Askinuze, A theorem on the splittability of J-algebras (Russian), Ukrainian Math. J. 3 (1951) 381–398.

[8] E.J. Taft, Invariant Wedderburn factors, Illinois J. Math. 1 (1957) 565–573.

[9] N.A. Pisarenko, The Wedderburn decomposition in finite dimensional alternative superalgebras, Algebra Logic 32 (4) (1993) 231–238, translation from Algebra Logika 32 (4) (1993) 428–440.

[10] V.G. Kac, Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras, Comm. Algebra 5 (13) (1977) 1375–1400.

[11] I.L. Kantor, Jordan and Lie superalgebras determined by a Poisson algebra, in: Algebra and Analysis, Tomsk, Russia, 1989, in: Amer. Math. Soc. Transl. Ser. 2, vol. 151, 1992, pp. 55–80.

[12] E. Zelmanov, Semisimple finite dimensional Jordan superalgebras, in: Yuen Fong, et al. (Eds.), Lie Algebras, Rings and Related Topics, Papers of the 2nd Tainan–Moscow International Algebra Workshop 97, Taiwan, January 11–17, 1997, Springer, Hong Kong, ISBN 9624301107, 2000, pp. 227–243 (pbk).

[13] A.S. Shtern, Representations of an exceptional Jordan superalgebra, Funct. Anal. Appl. 21 (1–3) (1987) 253–254, translation from Funktsional. Anal. i Prilozhen. 21 (3) (1987) 93–94.

[14] C.T.C. Wall, Graded Brauer groups, J. Reine Angew. Math. 213 (1964) 187–199.

[15] M. Trushina, Irreducible representations of a certain Jordan superalgebra, J. Algebra Appl. 4 (1) (2005) 1–14.

[16] C. Martínez, E. Zelmanov, Unital bimodules over the simple Jordan superalgebra $\mathcal{D}_t$, Trans. Amer. Math. Soc. 358 (8) (2006) 3637–3649.

[17] C. Martínez, E. Zelmanov, Representation theory of Jordan superalgebras. I, Trans. Amer. Math. Soc. 362 (2) (2010) 815–846.

[18] C. Martínez, I. Shestakov, E. Zelmanov, Jordan bimodules over the superalgebras $\mathcal{P}(n)$ and $\mathcal{Q}(n)$, Trans. Amer. Math. Soc. 362 (4) (2010) 2037–2051.

[19] C. Martinez, I. Shestakov, Unital irreducible bimodules over $\mathcal{M}_{1|1}(\mathbb{F})^+$, preprint.

[20] F.A. Gómez González, Jordan superalgebras of type $\mathcal{M}_{n|m}(\mathbb{F})^{(+)}$ and the Wedderburn principal theorem (WPT), Comm. Algebra 44 (7) (2016) 2867–2886.
[21] F.A. Gómez González, R. Velásquez, Wedderburn principal theorem for orthosymplectic Jordan superalgebras $\mathrm{Josp}_{n|2m}(\mathbb{F})$, Algebra Discrete Math. (2018), in press.
[22] F.A. Gómez González, Jordan superalgebras of type JP_n, $n \geq 3$ and the Wedderburn principal theorem, preprint.