Eurasia Journal of Mathematics, Science & Technology Education, 2015, **11**(6), 1429-1456

Towards a Methodology for the Characterization of Teachers' Didactic-Mathematical Knowledge

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•*Received 20 December 2014*•*Revised 27 March 2015* •*Accepted 27 May 2015*

This research study aims at exploring the use of some dimensions and theoretical-methodological tools suggested by the model of *Didactic-Mathematical Knowledge* (DMK) for the analysis, characterization and development of knowledge that teachers should have in order to efficiently develop within their practice. For this purpose, we analyzed the activity performed by five high school teachers, in relation to an activity about patterns suggested in the framework of the Master of Mathematics Education Program at University of Los Lagos, Chile. As a result of the analysis, it becomes evident that teachers can indeed solve items related to the common content knowledge, but have certain difficulties when they face items that aim at exploring other dimensions of their knowledge, for example, about extended content knowledge, of resources and means, or of the affective state of students.

Keywords: teacher training, teacher's knowledge, patterns, didactic-mathematical knowledge

BACKGROUND

Introduction

The didactical and mathematical knowledge of mathematics teachers has been a subject of intense research activity (Ball, Lubienski, & Mewborn, 2001; Ball, Thames & Phelps, 2008). Hill, Rowan and Ball (2005) have provided valid evidence for the purported link between teacher knowledge and student achievement in mathematics. Schoenfeld and Kilpatrick (2008) referred to the importance of teachers knowing school mathematics in depth and breadth, with the general consensus being that this knowledge in turn impacts upon PCK and therefore upon the effectiveness of instruction. The literature informs, however, that many elementary teachers lack conceptual understanding of mathematics (Mewborn,

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doi: 10.12973/eurasia.2015.1403a

ISSN: 1305-8223

2001), and that both in-service and pre-service teachers' limited mathematical content knowledge and confidence in doing mathematics is of particular concern (Ryan & Williams, 2007; Lange & Meaney, 2011).

This article informs on a research conducted with inservice teachers when solving a didactic-mathematical activity about patterns, which was suggested in the framework of the course 'Didactical Analysis' that is taught in the Master of Mathematics Education Program at University of Los Lagos–Chile. The model of mathematics teachers' knowledge known as "model of Didactic-Mathematical Knowledge (DMK)", is the frame used to conduct the research. The following section presents the background, and surveys relevant literature on models of teacher knowledge; the remaining sections discuss the theoretical background and methodology; the analysis of the teacher knowledge on a patterning task, and finally the main findings and conclusions are presented.

State of the literature

- The knowledge of mathematics teachers has been a subject of intense research activity. As a result, there are several models that have contributed significantly to its characterization, through the identification of categories and subcategories of it.
- Such models do not offer theoretical-methodological tools that allow a more detailed analysis of each of the types of knowledge that can be utilized in an effective teaching of mathematics.
- The models proposed in the literature exhibit a multifaceted vision of the identification of the knowledge required for teaching and there is no universal agreement on a theoretical framework for describing the teachers' knowledge.

Contribution of this paper to the literature

- This research study aims at exploring the use of some dimensions and theoretical-methodological tools suggested by the model of Didactic-Mathematical Knowledge (DMK).
- The paper contributes in the analysis, characterization and development of knowledge that teachers should have in order to efficiently develop within their practice.
- The analysis tools proposed by DMK can be seen as theoretical and methodological tools that allow carrying out detailed analysis of the teachers' knowledge, involved in each of the dimensions and subcategories of knowledge.

Teacher's didactic-mathematical knowledge

The study of the knowledge that a mathematics teacher should have in order to perform an appropriate management of the student's learning is a subject that has recently been gaining more attention. Evidence of this are the focus groups that discuss about the teacher's training and knowledge, which are held at the most important international mathematics teaching congresses –The Annual Conference of the International Group for the Psychology of Mathematics Education (PME), International Congress on Mathematics Education (ICME), Congress of European Research in Mathematics Education (CERME), Interamerican Conference on Mathematics Education (CIAEM-IACME), among others–, and in the publications of handbooks and specialized magazines such as the Journal of Mathematics Teacher Education.

In relation to the mathematics teacher's knowledge, there are several models that have contributed significantly to its characterization, through the identification of categories and subcategories of it –e.g., Shulman's PCK (1986, 1987); the Mathematical Knowledge for Teaching (MKT) of Ball et al., (Ball, Thames & Phelps, 2008; Hill, Ball & Schilling, 2008); the theory of 'proficiency' in the teaching of mathematics (Schoenfeld & Kilpatrick, 2008), the knowledge quartet of Rowland et al., (Rowland, Huckstep & Thwaites, 2005)–. These scientific works, in which the many models of mathematics teacher's knowledge are developed, show a multifaceted vision of the identification of the knowledge required for teaching. More recent research (for example, the works presented in the last two PME and the latest CERME-8), show that, as Rowland and Ruthven (2011) pointed out, there is no universal agreement on a theoretical framework for describing the teacher's mathematical knowledge. And, despite of the many important advances regarding the characterization of the complex structure of knowledge that teachers should have for their mathematics teaching practice to be effective, in general, as mentioned by Godino (2009):

> The models of mathematical knowledge for teaching created from the research on Mathematics Education, include categories that are too global and disjointed, so it would be useful to have models that allow a more detailed analysis of each of the types of knowledge that can be utilized in an effective teaching of mathematics. Furthermore, it would allow orientating the design of formative actions and the elaboration of teachers' knowledge evaluation instruments. (p. 19)

In this sense, in Godino (2009) a system of categories for the analysis of the mathematics teacher's knowledge is proposed, referred to as "didactic-mathematical knowledge" by considering the Didactics of Mathematics as the discipline that systematically articulates the different aspects implied in the processes of teaching and learning of mathematics. The categories proposed in such work are related to the type of analysis tools elaborated in the core of the theoretical framework known as the Onto-Semiotic Approach (OSA), assuming that the use of each tool brings into play didactic-mathematical knowledge. In that way, the system formed by the different theoretical-methodological tools of the OSA, provides a system of categories and subcategories of knowledge that the teacher must know, comprehend and know how to apply. In several works (Godino & Pino-Fan, 2013; Pino-Fan, Godino & Font, 2013; Pino-Fan & Godino, 2014; Pino-Fan, Godino & Font, 2015) the system of categories mentioned above, has been refined, therefore constituting, the model of didactic-mathematical knowledge (DMK) of the teacher.

The DMK model proposes three large dimensions for interpreting and characterizing the teachers' knowledge (Pino-Fan, Godino & Font, 2015): 1) Mathematical; 2) Didactical; and 3) Meta Didactic-Mathematical. Each of these dimensions considers subcategories of knowledge, which, in turn, also include theoretical and methodological tools that allow operationalizing knowledge analysis regarding each subcategory. Furthermore, these dimensions, with their corresponding analysis tools, are involved in each of the phases proposed for the elaboration of Instructional Designs: preliminary study, design, implementation and evaluation.

The objective of this work is, precisely, exemplifying the use of such theoretical and methodological tools, through the analysis of the knowledge demonstrated by five teachers when solving an activity (of a didactic-mathematical nature) about patterns, which was suggested in the framework of the course 'Didactical Analysis' that is taught in the Master of Mathematics Education Program at University of Los Lagos–Chile. As a result, the analysis tools proposed by DMK can be foreseen as theoretical and methodological tools that allow carrying out detailed analysis of the teachers' knowledge, involved in each of the dimensions and subcategories of knowledge. Likewise, as a parallel result, the difficulties that teachers have when facing items that require knowledge that is different from common content knowledge, in order to be solved, become evident.

Why the study of knowledge of patterns?

In the past few years, patterns and algebra have become a part of the elementary level curriculum in many countries. Patterns offer both, a powerful vehicle for the comprehension of the relations of dependency among quantities that lie beneath mathematical functions, as well as a concrete and transparent way for young students to start working with notions of abstraction and generalization (Moss & Beatty, 2006). In general, the importance of patterns in mathematics has been pointed out by many authors. Zazkis and Liljedahl (2002), for example, state that "patterns are the heart and soul of mathematics" (p. 379).

Likewise, the 'Principles and Standards for School Mathematics' (NCTM, 2000), states that patterns constitute the basis of algebraic thought, therefore its exploration involves students in the identification of relations and in the establishment of generalizations, proposing the knowledge of patterns, functions and relations, as objectives for all teaching levels. Specifically, NCTM (2000) suggests that patterns should be taught from the first years of school, with the expectation of having students that, in second grade, are capable of analyzing how repetitive patterns are generated, and thus, by the end of fifth grade, are capable of representing patterns and functions through words, tables and graphs.

Other authors (e.g. Zazkis & Liljedahl, 2002; Souza & Diniz, 2003; Ponte, 2005), claim that patterns should be addressed, for the introduction of the concept of variable, arguing that, traditionally, variables are introduced as unknowns in equations where such unknowns do not possess a variable nature. Likewise, they point out that such addressing provides students with the opportunity to observe and verbalize their generalizations and then express them symbolically. However, the "step" from the strict utilization of numbers to the utilization of symbols, is not an automatic process and, it does not occur by chance, constitutes one of the main obstacles for the comprehension of school algebra, since the 'sense of symbols' (Arcavi, 2007) has to be constructed. In other words, the capacity of interpreting and using mathematical symbols creatively in the description of situations and problem-solving has to be encouraged (Ponte, 2006).

In this sense, the many approaches of early algebra, aim at facilitating the abrupt transition from arithmetic to algebra. These approaches have signaled a change in the research of early algebra over time, moving from the solving of equations as the main teaching and learning activity in algebra, to transition activities such as generalization, numeric patterns, variables and functions (Carraher & Schliemann, 2007). With regard to the above, Roig and Llinares (2008) point out that the problems of generalization, independently of the context in which these are presented, is obtaining a 'rule' that defines the sequence pattern. In order to achieve such purpose, usually, the following tasks are used:

1. Describing the next term of the sequence, writing or drawing it.
2. Determining a close term, that might be found through the continuation of the drawing or by writing the sequence before getting to the required term. Stacey (1989) calls it near-term generalization.
3. Determining a far term in order to make students, given the difficulty of continuing with the sequence, search a general rule. Stacey (1989) calls it far-term generalization.
4. Finding a pattern or rule that allows determining different terms of the sequence.
5. Expressing the pattern or rule found, symbolically (nth term of the sequence).

In this study we analyzed an activity about patterns, which we implemented with practicing elementary and high school teachers. This activity involves the five tasks described above. In the next section, as part of the theoretical and methodological framework of our study, we will describe the activity in detail.

THEORETICAL FRAMEWORK AND METHODOLOGY

In this study, we have used the model of mathematics teachers' knowledge known as DMK model, which is based upon theoretical assumptions and theoretical-methodological tools of the theoretical framework known as Onto-Semiotic Approach (OSA) to cognition and mathematical instruction (Godino, Batanero & Font, 2007). Based on some of the assumptions of DMK, the activity described below in the corresponding section is presented to a group of practicing mathematics teachers. The DMK model, as well as the subjects' characteristics and the context in which the activity is presented, are described in the following two sections, respectively.

The model of Didactic-Mathematical Knowledge (DMK)

In this study we used the DMK model, which was suggested by taking into consideration: 1) the contribution and development of the theoretical framework known as Onto-Semiotic Approach (OSA) to cognition and mathematical instruction, which has been developed in several research studies by Godino et al., (Godino & Batanero, 1994; Godino, Batanero & Font, 2007; Font, Godino & Gallardo, 2013); 2) the development and contribution of the research by Godino (2009) where the foundations and basis of DMK are presented; 3) the findings and contribution of the several models that currently exist in the field of research of Mathematics Education –Shulman (1986, 1987); Grossman (1990); Ball, Thames & Phelps (2008); Hill, Ball & Schilling, (2008); Schoenfeld & Kilpatrick (2008); Rowland, Huckstep & Thwaites (2005)–; and 4) the results obtained in several empiric studies that we have conducted (Pino-Fan, Godino & Font, 2011; Pino-Fan, Godino, Font & Castro, 2012; Pino-Fan, Godino, Font & Castro, 2013; Pino-Fan, Godino & Font, 2013; Pino-Fan, Godino & Font, 2015).

The DMK model interprets and characterizes the teacher's knowledge from three dimensions: mathematical dimension, didactical dimension and meta didactic-mathematical dimension (Figure 1).

DMK's mathematical dimension makes reference to the knowledge that allows the teacher to solve the problem or mathematical activity that is to be implemented in the classroom and link it with mathematical objects that can later be found in the school mathematics curriculum. It includes two subcategories of knowledge: common content knowledge and extended content knowledge. The first subcategory, common content knowledge, is the knowledge of a specific mathematical object, which is considered as sufficient to solve problems and tasks

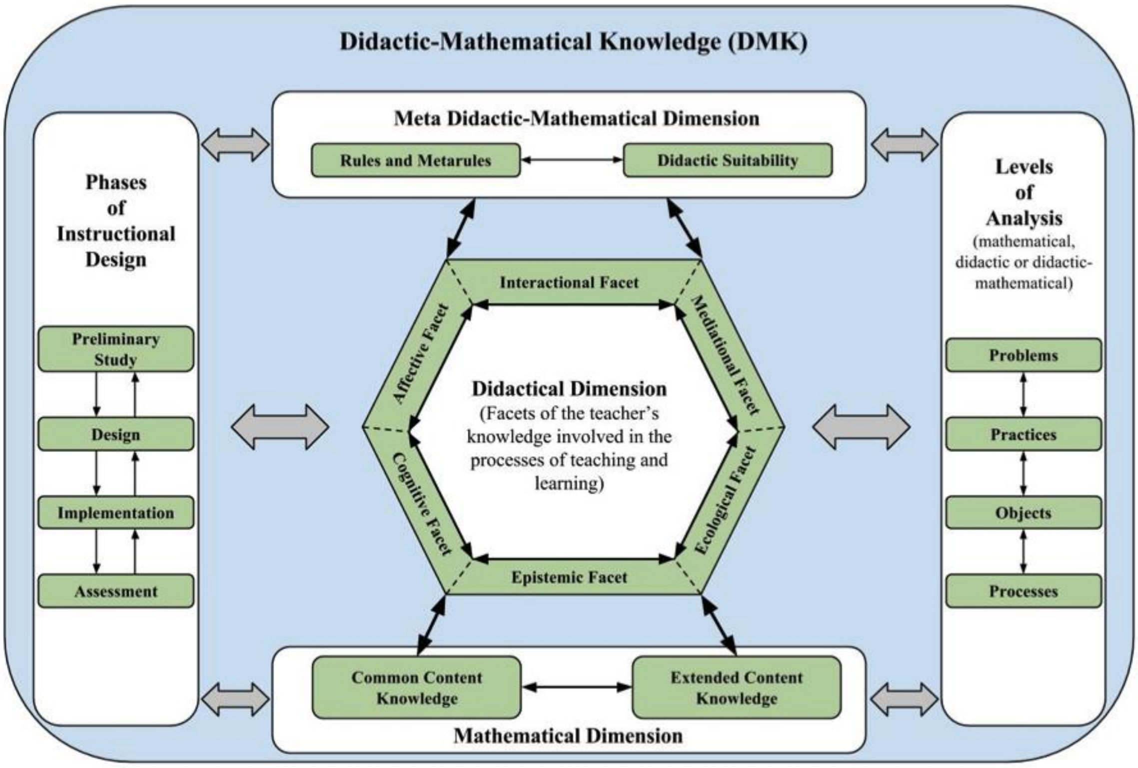


Figure 1. Dimensions and components of Didactic-Mathematical Knowledge

proposed in the mathematics curriculum and in the textbooks of a certain educational level; it is a shared knowledge between the teacher and the students. The second subcategory, extended knowledge, refers to the knowledge that the teacher must have about mathematical notions that, taking the mathematical notions that are being studied at a certain time as a reference (for example, derivatives), come ahead in the curriculum of the educational level in question or in the next level (for example, integers in high school or the fundamental theorem of calculus in college). Extended content knowledge provides the teacher with the necessary mathematical foundations to suggest new mathematical challenges in the classroom, to link a certain mathematical object being studied with other mathematical notions and to guide students to the study of subsequent mathematical notions to the notion that is being studied. According to Pino-Fan and Godino (2014), these two subcategories that include the mathematical dimension of DMK, are reinterpretations of both the common content knowledge (Hill, Ball & Schilling, 2008; Ball, Thames & Phelps, 2008) and the horizon knowledge (Ball & Bass, 2009), respectively. According to these authors, this interpretation is based on the need to settle the knowledge that a mathematics teacher should possess on specific topics to be taught at some specific school grades.

It is clear that the DMK mathematical dimension, that enables the teacher to solve mathematical tasks and problems, is not enough to the practice of teaching. The authors of manifold models cited previously, agree that more than mathematical knowledge is needed, for instance, the knowledge of some features that affects the class planning and management of a specific subject. In this sense, the didactical dimension of DMK considers six subcategories (Pino-Fan & Godino, 2014; Pino-Fan, Godino & Font, 2015):

1. *Epistemic facet*, which refers to specialized knowledge of the mathematical dimension. The teacher, apart from the mathematics that allow him solving problems which require him mobilize his common and extended knowledge, must have a certain amount of mathematical knowledge "shaped" for teaching; that is to say, the teacher must be able to mobilize several representations of a mathematical object, to solve a task through different procedures, to link mathematical objects with other mathematical objects taught at a certain educational level or from previous or upcoming levels, to comprehend and mobilize the diversity of partial meanings for a single mathematical object –that are part of the holistic meaning for such object (Pino-Fan, Godino & Font, 2011)–, to provide several justifications and argumentations, and to identify the knowledge at play during the process of solving a mathematical task. Thus it is clear that this DMK's subcategory includes not only the notions proposed in the model of the proficiency in teaching mathematics of Schoenfeld and Kilpatrick (2008, p. 322) on "knowing school mathematics profoundly and thoroughly" but also the notions of Hill, Ball and Schilling (2008, p. 377-378) on "the mathematical specialized content knowledge".
2. *Cognitive facet*, that refers to the knowledge about the students' cognitive aspects. This subcategory considers the necessary knowledge to 'reflect and evaluate' the proximity or degree of adjustment of personal meanings (students' knowledge) regarding institutional meanings (knowledge from

the point of view of the educational center). To this end, the teacher must be able to foresee (during the planning/design stage) and trying (during the implementation stage), from the students' pieces of work, or expected pieces of work, possible answers to a certain problem, misconceptions, conflicts or mistakes that arise from the process of solving the problem, links (mathematically correct or incorrect) between the mathematical object that is being studied and other mathematical objects which are required to solve the problem.

3. *Affective facet,* that refers to the knowledge about the students' affective, emotional and behavioral aspects. It is about the knowledge required to comprehend and deal with the students' mood changes, the aspects that motivate them to solve a certain problem or not. In general, it refers to the knowledge that helps describing the students' experiences and sensations in a specific class or with a certain mathematical problem, at a specific educational level, keeping in mind the aspects that are related to the ecological facet. The cognitive and affective facets such as are defined by the EOS (Godino, Batanero & Font, 2007; Godino, 2009), together provide a better approximation and understanding of the knowledge that the mathematics teachers should possess on the features and aspects that are connected to the way students think, know, act and feel in the class while solving a mathematics problem. Thus, according to Pino-Fan and Godino (2014) these two facets (cognitive and affective) includes and broaden Shulman's ideas (Shulman, 1987, p. 8) –on the "knowledge of students and their characteristics"–, Schoenfeld and Kilpatrick (2008) –on "knowing the students as persons who think and learn"–, Grossman (1990, p. 8) –on the "comprehension of students, their beliefs and mistakes about specific topics"–, and Hill, Ball and Schilling (2008, p. 375) –on the "knowledge of content and students"–.
4. *Interactional facet.* The study of the required features to appropriately manage the students learning on specific mathematics topics, have considered to the interactions as a fundamental component in the learning and teaching process (Coll & Sánchez, 2008; Planas & Iranzo, 2009). In this sense, and having in mind the ideas of Schoenfeld and Kilpatrick (2008) on constructing relationships that support the learning process, the interactional facet refers to the knowledge of the interactions that occur within a classroom. This subcategory involves the required knowledge to foresee, implement and evaluate sequences of interaction, among the agents that participate of the process of teaching and learning, oriented towards the fixation and negotiation of meanings (learning) of students. These interactions do not only occur among the teacher and the students (teacher-student), but also can occur between students (student-student), student-resources, and teacher-resources-students.

5. *Mediational facet*. In relation to the resources and means used to manage the learning, the proposed models by Shulman (1987) and Grossman (1990) consider the knowledge of classroom materials as part of the curriculum knowledge. Nonetheless, due to the actual mathematics curriculum tendencies, these acquire an important role in the organization and management of learning. For this reason, the meditational facet refers to the knowledge of resources and means which might foster the students' learning process. It deals with the knowledge that a teacher should have to assess the pertinence of the use of materials and technological resources to foster the learning of a specific mathematical object, and also the assigning of time for the diverse learning actions and processes. According to Pino-Fan and Godino (2014), the link between the interactional and mediational facets develop and enrich the notion of "knowledge of content and teaching" proposed by Ball, Thames and Phelps (2008, p. 401).
6. *Ecological facet*, which refers to the knowledge of curricular, contextual, social political, economic… aspects that have an influence on the management of the students' learning. In other words, teachers should have knowledge of the mathematics curriculum of the level that considers the study of a mathematical object, the links that might exist with other curricula, the relations that such curriculum has with social, political and economic aspects that support and condition the teaching and learning process. The features considered in this knowledge facet take into account the ideas of Shulman (1987, p. 8) –on the "curriculum knowledge", "knowledge of educational ends, purposes and values"– and Grossman (1990, p.8) –on the "knowledge about horizontal and vertical curriculum for a specific topic", and the "knowledge of context"–.

Pino-Fan & Godino (2014) point out that the six facets that compose the didactical dimension of didactic-mathematical knowledge, along with the mathematical dimension of DMK, can be considered when it comes to analyze, describe and develop the teacher's knowledge –or future teachers'– involved in the different phases of the design of processes of teaching and learning of specific mathematical topics: preliminary study, planning or design, implementation and assessment. Furthermore, they suggest that the reflection as well as the evaluation and detection of potential improvements to the practice, are immerse within the assessment phase, thus the required knowledge for teachers to reflect on their own performance, to assess and detect potential improvements in the teaching and learning processes, and also the knowledge of rules and metarules that regulate such processes and about the contextual conditions and restrictions, are part of the meta didactic-mathematical dimension of DMK. According to Schoenfeld and Kilpatrick (2008, p. 348), "Once it is habitual, reflection can become the principal mechanism for improving one's teaching practice". Other features of DMK that make up this dimension "meta", are the knowledge about the norms and metanorms (epistemic, ecological, cognitive, interactional, mediational and affective), the conditions and contextual restrictions (Pino-Fan & Godino, 2014; Assis, Godino & Frade, 2012).

Furthermore, for each of the dimensions that DMK contemplates, four levels of analysis are anticipated, for which, in time, theoretical and methodological tools that help operationalize such levels of analysis, are considered. This levels allow: a) identifying the features of the problems or sequence of problems that the teachers propose for their implementation; b) describing mathematical or didactical practices that the teachers and students perform regarding the problems involved; c) identifying and systematically describing the mathematical or didactic-mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) immersed in the development of such practices; and d) studying the processes that teachers and students perform, and which leads to the emergence of such mathematical objects. In the Analysis of the Teachers' Knowledge section, we will exemplify the use of the tool known as "onto-semiotic configuration", which allowed us characterizing the knowledge of our sample of teachers, from the identification and description of the practices, objects and processes, which they mobilize in connection to the activity proposed.

Subjects and context

In the framework of the course 'Didactic Analysis of Teaching and Learning Processes of Mathematics' that is taught in the fourth semester of the Master of Mathematics Education Program at University of Los Lagos, Chile, the activity that is described below was proposed to the students, who were all practicing teachers with experience (5 to 20 years) in the teaching of mathematics at elementary and high school levels (i.e., they teach mathematics to students whose ages range from 6 to 18 years old). It is necessary to highlight that, since it is a 4-semester long disciplinary Master's Program, the teachers, apart from their experience in the classroom, had built the necessary basis to start researching, during the Program.

It is important to clarify that teachers, to resolve the ten items that make up the activity, received no previous training neither on the DMK model nor on patterning tasks. This was done because our research objective was double: the first was to explore teachers' 'spontaneous' knowledge to solve the items; the second was to highlight the DMK's tools. We have taken, considering the objectives of this study, the answers provided by 5 teachers (three women and two men) regarding the activity proposed. For the purpose of the results that we presented in this study, we considered that it was not necessary to distinguish the teachers by their alias, thus we refer to them as Teacher A, Teacher B, Teacher C, Teacher D and Teacher E.

Data sources

For this research data sources included: discussions that took place within the context of the methods course of inservice teachers attending a master's course, students' work on a series of classroom problems and students' free-write responses to various writing prompts. The data collection was guided by Bullough, Knowles and Crow´s (1991) suggestion that case study methodology is a responsive methodology. Data was gathered on a weekly basis as discussion took place. Students were informed that their participation was voluntary and their refusal to offer interviews will not affect their final grade.

In regard to data collection, the data were collected over a period of an academic term. The items were designed taking into account the findings both on algebraic reasoning (Roig & Linares, 2008; Stacey, 1989), and on teachers' beliefs (Van Dooren, Verschaffel & Onghema, 2002; 2003). The ten items are in correspondence to the DMK model used on this research.

The activity

The activity proposed was taken from Moss & Beatty (2006), and falls within a functional perspective for the development of algebraic reasoning. The teachers must be able to identify patterns, find the underlying regularity and expressing it as an explicit function or rule through a generalization process. The task has been adapted in order to explore the aspects of the mathematical and didactical dimensions of DMK of teachers, so that each one of the items that compose the activity, are linked to one subcategory of the teachers' knowledge, belonging to the didactical and mathematical dimensions of DMK. Thus, the items 1 and 2 are related to common content knowledge; items 3, 4 and 5 are related to the epistemic facet; item 6 is related to the ecological facet; items 7 and 8 to the cognitive facet; item 9 explores the affective facet, and item 10 is related to the interactional facet.

> A company manufactures color bars by linking cubes in a straight line. The company uses a machine to label and put stickers of smiley faces on the bars. The machine puts exactly one sticker on each face, in other words, each external side of each cube must have a sticker on. For example, a bar of length 2 (two cubes) would need ten stickers (Figure 2).
>
> Item 1. How many stickers would be needed for a bar of length: a) three; b) four; c) ten; and d) twenty?
>
> Item 2. Based on your answers to the previous question, determine what is the rule to calculate the number of stickers for a bar of any length.
>
> Item 3. Is there any other way to answer the previous questions, apart from the solution you provided? If so, write the solution down. If not, justify why it is not possible.
>
> Item 4. What knowledge (algebraic or any other) does come into play in order to solve this problem?
>
> Item 5. How would you explain the solution of this problem to a student who has not been able to solve it?
>
> Item 6. Which educational level do you consider suitable for this problem to be implemented at and why?
>
> Item 7. What are the main difficulties that the students might encounter when solving this problem?
>
> Item 8. What kind of mistakes could the students make when solving this problem?
>
> Item 9. What measures would you implement in the classroom in order to motivate students to solve this problem?
>
> Item 10. What strategy or strategies do you consider pertinent for the implementation of this activity, considering the school level that you suggested in item 6?

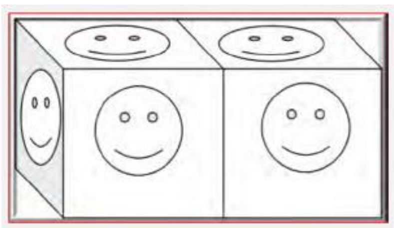


Figure 2. Stickers required for a bar that is two cubes long

In section three below, the answers provided by the teachers for each of the items, are analyzed. Now we want to discuss how each one of the items listed before put on display mathematical and didactical dimensions of DMK.

Data analysis

Data analysis was influenced by models of qualitative research advocating a systematic and ongoing breaking of the data, leading to an identification of core themes (Wolcott, 1993). Initially open coding procedures were applied to the data in order to identify core themes around which more detailed findings could be extracted. Based on the results obtained, further tasks were designed to obtain more information, which in turn leads to a more detailed coding procedure intended to understand the subtleties of teachers' answers to the items proposed to them.

ANALYSIS OF THE TEACHERS' KNOWLEDGE

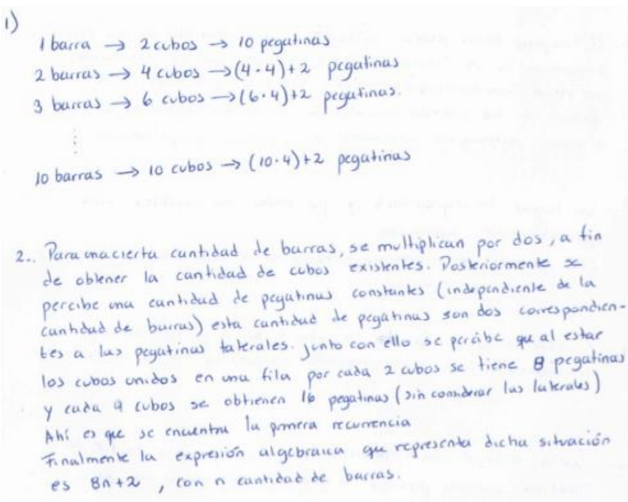
Mathematical dimension of DMK

Common knowledge

The first two items of the activity were proposed in connection to this subcategory of the mathematical dimension of DMK. For item 1 it is expected that teachers observe and describe the behavior of a pattern, by means of particular cases. Likewise, with this first item, it was expected that, with the largest number of cubes, teachers would feel motivated to find a pattern that determines the number of stickers for bars of any length. It was possible to anticipate the next answer by the teachers: a) for 3 cubes, 14 stickers are needed; b) for 4 cubes, 18 stickers are needed; c) for 10 cubes, 42 stickers are needed; and d) for 20 cubes, 82 stickers are needed.

Regarding item 2, it was expected that the teachers would find a relation between the length of the bar and the number of stickers, and therefore, they would find a mathematical formula that would make it possible to calculate the number of stickers for a bar of any length. Thus, a possible solution expected from the teachers was P(c) = 4c + 2, where P is the number of stickers and c the number of cubes that composed the length of the bar.

Figure 3 shows the answers provided by Teacher A for the two items. It is important to point out that, in order to analyze the answers of the two teachers, we used the notion of onto-semiotic configuration (Godino, Batanero & Font, 2007;


1)

1 barra → 2 cubos → 10 pegatinas
2 barras → 4 cubos → (4·4)+2 pegatinas
3 barras → 6 cubos → (6·4)+2 pegatinas.
⋮
10 barras → 10 cubos → (10·4)+2 pegatinas

2.. Para una cierta cantidad de barras, se multiplican por dos, a fin de obtener la cantidad de cubos existentes. Posteriormente se percibe una cantidad de pegatinas constantes (independiente de la cantidad de barras) esta cantidad de pegatinas son dos correspondientes a las pegatinas laterales. Junto con ello se percibe que al estar los cubos unidos en una fila por cada 2 cubos se tiene 8 pegatinas y cada 9 cubos se obtienen 16 pegatinas (sin considerar las laterales) Ahí es que se encuentra la primera recurrencia
Finalmente la expresión algebraica que representa dicha situación es 8n+2, con n cantidad de barras.

1)
1 bar →2 cubes→10 stickers
2 bars→4 cubes→(4*4)+2 stickers
3 bars→6 cubes→(6*4)+2 stickers
…
10 bars→10 cubes→(10*4)+2 stickers

2. For a certain amount of bars, it is multiplied by 2, in order to obtain the current amount of cubes. After that, there is a perceived constant amount of stickers (independently from the amount of bars). This amount of stickers is 'two', which correspond to the lateral stickers. Also, it is perceived that, since the cubes are linked in a row, for every 2 cubes there are 8 stickers and for every 9 cubes, there are 16 stickers (without considering the lateral stickers). That is the first recurrence.
Finally, the algebraic expression that represents such situation is 8n+2, n being the number of bars.

Figure 3. Answers of Teacher A to items 1 and 2 of the activity

Pino-Fan, Godino & Font, 2015), which allows us to identify and to describe, in a detailed manner, the primary mathematical objects (linguistic elements, concepts/definitions, properties/propositions, procedures and arguments), their meanings, and the processes involved in the institutional (epistemic configuration) or personal (cognitive configuration) practices.

In Figure 3, we can observe that Teacher A uses linguistic elements, mostly verbal, to establish relations; likewise, the teacher utilizes arithmetic and algebraic expressions [e.g., (4*4)+2; 8n+2], n as a variable, natural numbers, signs to refer to additions and multiplications, grouping signs (e.g., parenthesis). Among the concepts/definitions that the teacher uses, we mention constant amount and algebraic expression. Regarding propositions/properties, for item 1, the four that are established as a relation among the bars, the cubes and the stickers (e.g., 1 bar → 2 cubes → 10 stickers); while for item 2, Teacher A considers the proposition "1 bar are 2 cubes" –maybe 'inspired' by the drawing in the task–. For this reason, the teacher indicates that the number of stickers for a bar of any length is twice the number of bars (to indicate the number of cubes); also, she enunciates the propositions "for every two cubes, there are 8 stickers" and "for every 4 cubes there are 16 stickers", from where the final proposition is obtained and which refers to the answer to item 2 "the algebraic expression that represents such situation is 8n+2". Regarding the procedure that Teacher A uses, there is the recognition of regularities (the teacher calls it "recurrences"), through which, despite the teacher's confusion between lengths-bars-cubes (the teacher considers length as the number of bars, and each bar composed by two cubes), achieves a process of generalization through which she obtains a symbolic expression to determine the nth term, which, according to her conception, refers to the 'number of bars'. In relation to the arguments of her procedures, and answers, through processes of enunciation, the teacher points out two: 1) for a certain amount of bars, multiply by 2, in order to obtain the current amount of cubes. Later, the two lateral stickers are considered as constant; and 2) since the cubes are linked in a straight line, for every 2 cubes there are 8 stickers and for every 4 cubes there are 16 stickers (without considering the lateral stickers). Due to recurrence, the expression 8n+2 is obtained. In general terms, teacher A solution is quite descriptive and ends up with an algebraic expression.

Regarding the answer provided by Teacher B (Figure 4), it was possible to identify that the linguistic elements utilized are mainly symbolic elements –natural numbers, symbolic expression 4n+2, where n is the number of cubes–, which are

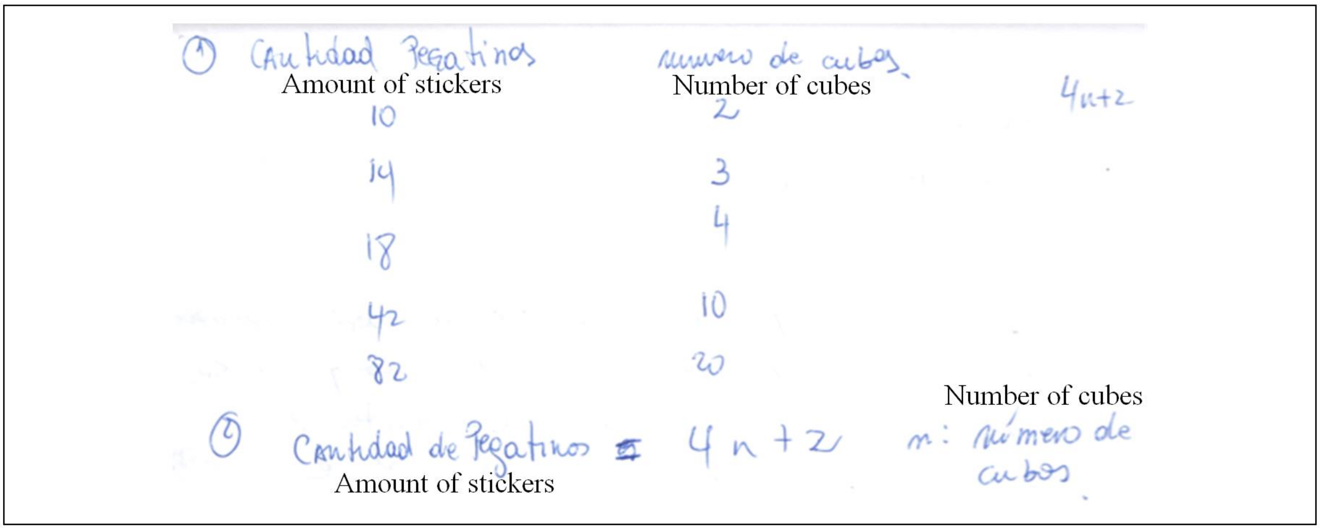


Figure 4. Answers of teacher B to items 1 and 2 of the activity

presented (through a process of enunciation and representation) in tabular form, for item 1. On teacher B's answers to items 1 and 2, he does not make the use of certain concepts/definitions explicit, which is why these are not mentioned here because we would be speculating on Teacher B's answer. However, the concepts/definitions are explicit in his answer to item 4. Regarding properties/propositions we can point out the ones he enunciates in tabular form –relation established for particular cases of "amount of stickers and number of cubes"–, and the generalization he reaches when pointing out "amount of stickers = 4n+2". Teacher B's procedure was the identification or recognition of the pattern or regularity through induction, though this argument was not made explicit by him. The solution proposed basically is numeric and leaps into an algebraic solution, in this regard coincides with teacher A's solution.

Figure 5 shows the answer provided by Teacher C. It can be observed that this teacher utilizes linguistic elements of a 'graphic-visual' type (the drawings of the cubes) in her solution, which are used to support her propositions; this teacher also utilizes the same verbal and symbolic language to convey her propositions, which in time, make reference to the answers of the sections of both items. Teacher C, like teacher B, does not explicitly mention the use of certain concept/definitions, but we can infer that she utilizes the notions of dependent and independent variable when considering, respectively, 'a' as the number of stickers, and 'b' as the length of the bar. Regarding propositions, we can observe that there are verbal propositions, such as, "For a bar of length two, 10 stickers are needed", "For a bar of length 3, 15 stickers are needed", etc. Likewise, we can observe propositions expressed.

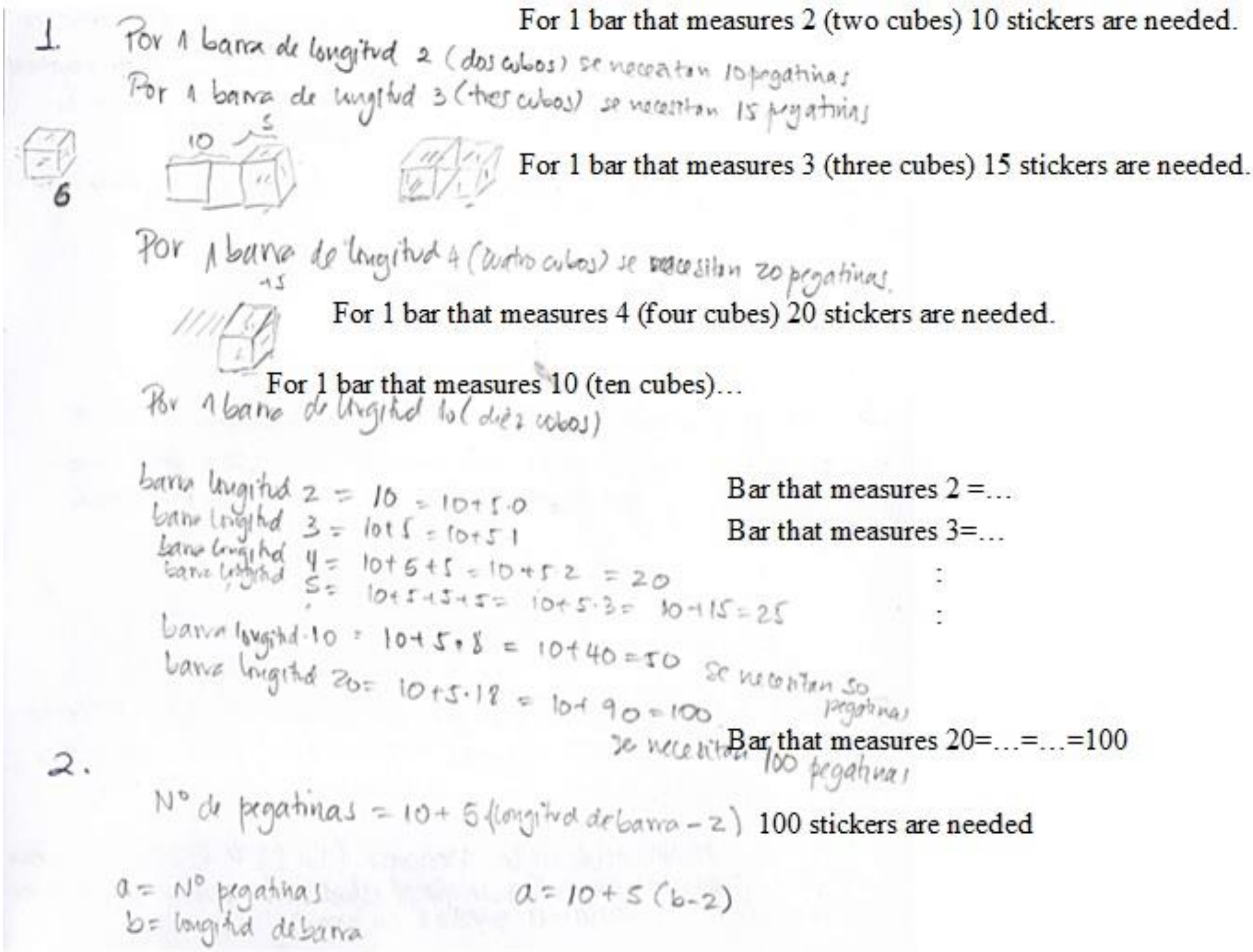


Figure 5. Answers by Teacher C to items 1 and 2 of the activity

symbolically, for example, "bar of length 5 = 10 + 5 + 5 + 5 = 10 + 5*3 = 10 + 15 = 25", or the proposition that she reaches to, through a process of generalization "a = 10 + 5 (b – 2)". Regarding her procedure, we can see that Teacher C initially considers a bar with two cubes and points out that, for these two cubes, 10 stickers are needed (an aspect that is pointed out through a process of verbal enunciation, and through a process of iconic representation). From such proposition (for a bar that measures 2, 10 stickers are needed), for each cube that is added to the bar, this teacher considers that 5 stickers are needed (instead of 4, as in fact occurs, if one considers that the cubes are added in the 'middle', which would lead to eliminate the stickers in the 'ends'). The teacher develops the answers in a numeric setting, and uses it to propose the algebraic rule. It is interesting how the teacher constructs the symbolic expression combining the equal sign with words and numbers in a gradual way. Van Dooren, Verschaffel and Onghema (2002) reports that the teachers in their study proceeds from the numerical field to the algebraic field, using the numbers to validate the letters. The general rule formulated by Teacher C, is therefore: a = 10 + 5 (b – 2). In this sense, the main argument of Teacher C is that for a bar of a certain length, one considers the number (constant) of stickers in 2 cubes, in other words, ten (which would correspond to the 10 stickers that are needed for the cubes that would be at the ends of the bar). Subsequently, she adds the result of the multiplication of five stickers (only four should have been considered) by the length of the bar minus two, to this constant number of stickers. Solution provided by teacher C includes four levels of representation: verbal, graphic, numeric and algebraic. It is interesting the way the teacher uses the equal sign. According to Carpenter, Frankle and Levi (2003) it is not advisable to work with such expression that could induce a wrong use of the equal sign. Compare to solutions by Teachers A and B, this solution includes more representation systems, what is desirable, according to Duval (2006).

On the other hand, the answer of Teacher D (Figure 6) could be summarized in the identification that, for a bar of any length, the cubes at the ends require five stickers, while the rest of the cubes (in the middle) require four stickers. In such answer , we identified linguistic elements of 'pictorial or visual' type –the boxes that

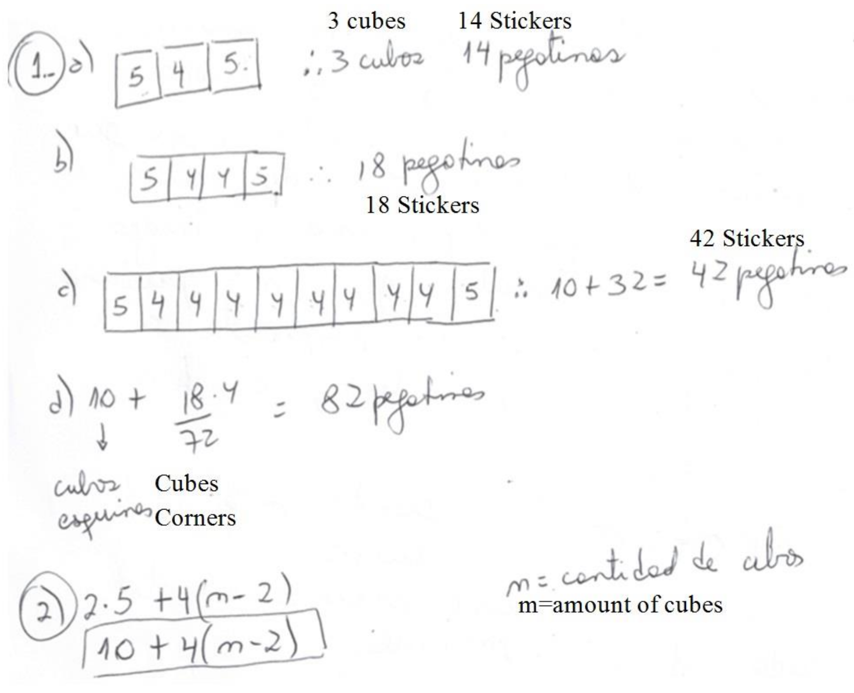


Figure 6. Answers by Teacher D to items 1 and 2 of the activity

represent the cubes and in which Teacher D writes down the number of stickers that are needed for every cube–, symbolic – e.g., the numbers that are written inside the boxes, or the expression '10 + 4 (m – 2)'–, and verbal expressions –'n = amount of cubes' –. Although Teacher D does not explicitly mention the concepts/definitions used, it is possible to infer through propositions such as "2*5 + 4 (m – 2), m is the amount of cubes", from which most of her procedure can be deduced. Other propositions are the ones of symbolic-pictorial type: "$\boxed{5}\boxed{4}\boxed{5}$ ∴ 3 cubes 14 stickers". Regarding the procedure of Teacher D, we observe that she manages to recognize regularities, and answer what is the number of stickers for specific cases of different lengths of the bar (induction); then, through a process of generalization, she symbolically expresses (through processes of enunciation and representation) the general rule: "10 + 4 (m – 2), where m = number of cubes". Regarding the arguments that are provided by Teacher D, we firstly identify a pictorial argument where the number of faces exposed of each cube 'can be read': five in the ones at the 'corner' and four in the ones in the middle. Then, Teacher D considers that the cubes at the ends leave 5 faces exposed each, which is pointed out in the next-to-last expression as '2*5'. For the remaining cubes (the 'amount of cubes' determined by the length of bar minus two) only four faces are exposed. Solution provided by teacher D resembles teacher's E solution, both use a combination of graphic and numeric representation, finally they both propose an algebraic rule.

Figure 7 corresponds to the answer provided by Teacher E for the first two items of the activity. In this answer, we can observe that the teacher uses linguistic elements that are predominantly symbolic –natural numbers, symbols for additions and multiplications–, which he used to convey his propositions "faces = 4 cubes + 2", "y = 4 x + 2", "4*3 + 2 = 12 + 2 = 14", and so on. We observe that Teacher E does not explicitly state neither the procedure nor the argument that allows him to find the general rule. The teacher must forge links among numerical representation, pictorial representations and mathematical symbols (Hill & Ball, 2009). This teacher still has to forge such relations.

What can be seen is that, in order to find the number of stickers required for specific cases of length of the bar, the teacher uses processes of particularization, employing the general rule enunciated from the beginning.

Teacher E does not explicitly state the concepts/definitions that he uses in his practice. Nevertheless, we can infer the use of the notions of dependent variable, particularized as "y= number of faces", and independent variable, represented with the proposition "x = number of cubes". Anticipating the analysis of the answers that this teacher provided for other items of the activity, the argument utilized in his answer for the first two items is explicit until item 3. Such argument can be

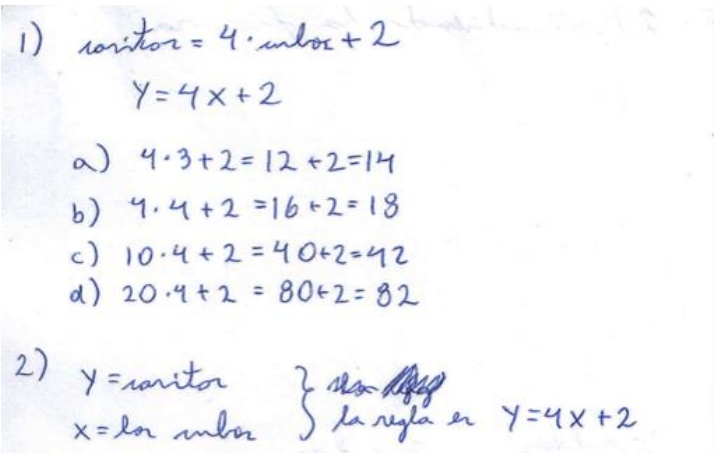


Figure 7. Answers by Teacher E to items 1 and 2 of the activity

summarized as follows: The teacher considers that each cube has six faces (6x), but two faces on each cube are covered because these are connected in a straight bar (6x – 2x = 4x), except the two cubes at the ends, which only have one face covered, each (4x+2). All the teachers uses linguistic elements, concepts/definitions, properties/propositions, procedures and arguments to express their solutions, but in a different way. It offers teachers educators the opportunity to compare how this basic elements are being used by teachers to proposed a solution.

In general, as we can observe, Teachers B, D and E, manage to provide satisfactory answers for items 1 and 2, so it can be stated that these teachers possess a good mastering of common content knowledge, to solve problems such as the one presented here. It is not so for Teacher A, who had difficulties comprehending the formulation of the problem, by confusing length-bars-cubes, as mentioned above. However, despite this confusion, the use of the mathematical objects that compose the cognitive configuration that she mobilizes is adequate. The inservice teachers manage to move from the verbal expression to the numerical and then to the algebraic representation. Anghileri (1995) suggests the close relationship between mathematics contexts, procedures and the words used to describe them. The preservice teachers recognized that some language structures could affect the comprehension and solution of certain mathematical activities, and this recognition could be a sign of their own evolution in the comprehension of algebraic notation (Castro & Godino, 2014). For MacGregor and Price (1999), "…conscious awareness of language structures and the ability to manipulate those structures may be the manifestation of deeper cognitive process that also underlies the understanding of algebraic notation" (p. 462).

Extended Knowledge

Since there is not an item that is directly linked to this type of knowledge, we analyze whether in their answers the teachers established connections with mathematical objects that were more advanced in the mathematics curriculum of the educational level that they pointed out on item 6. We obtained as a result that, while Teacher A (who selected 7th grade to apply this activity) linked the notion of patterns to notions of algebra, Teacher B (who mentioned 9th grade as the level to apply this activity) linked the patterns to the notion of function. In this sense, even though we cannot be certain regarding the teachers' knowledge in this subcategory of DMK, Teacher B would have more extended knowledge than teachers A, C, D and E. Yet, all of them have a low level of extended content knowledge. It would be ideal to apply, in future research studies, concrete items that would help exploring this subcategory of the mathematical dimension of DMK in an explicit manner.

Didactical Dimension of DMK

As mentioned in the theoretical framework section, this dimension of didactic-mathematical knowledge considers six subcategories of knowledge. Below, we will analyze the knowledge of the teachers in connection to each subcategory, through the analysis of the answers they provided to the different items of the activity.

Epistemic facet

Items 3, 4 and 5 of the activity, seek to explore the knowledge of the teachers, regarding this subcategory of DMK. On the one hand, item three aims at having the teacher reflect on other procedures to solve the first two items of the activity, procedures which involve the use of representations, concepts/definitions, propositions/properties, different from the ones utilized in their answers to items 1 and 2. Item 4 seeks to explore the knowledge that teachers have of the identification of the knowledge that is mobilized when solving the task (specifically, items 1 and 2 of the activity), which would contribute, in time, to the development of competences and skills to determine whether it is possible to evaluate or help their students' learning with a specific activity. On the other hand, item 5 aims at reflecting on the several explanations or argumentations to the solution provided for the task. Item 5 is closely related to item 3, and in a cycle of teacher training, questions such as these can contribute to the development of competences and knowledge of the teachers, towards a proper management of the students' learning.

According to Ball (2009) the epistemic facet or specialized knowledge is crucial to teaching and is related to: interpret and analyze student work; provide a mathematical explanation and forge links between mathematical symbols and pictorial representations (p.70). This knowledge is difficult to acquire and requires a lot of reflection by the teachers, but it is important because is related to students' outcomes (Hill, Rowan, and Ball, 2005).

Regarding the answers provided by Teacher A we can point out: 1) she does not indicate another solution for item 3 –"For the time being, I cannot see another more proper way to solve the activity"–; 2) for item 4, she limits to point out one concept/definition –"algebraic expression"–, a procedure –"algebraic regularities"– and one general and ambiguous sentence –comprehending algebra as a generalization of a problematic situation–; and 3) for item 4, she suggests a designed practice with key words that would serve as 'clues', to help students reach and achieve the generalization. The teacher points out, indirectly, the need of allowing students to build their own knowledge, by mentioning that they must be provided with 'clues' that would facilitate the analysis of the situation. She describes as follows: "I would try to promote a practice like the one in question 1, trying to make students observe regularities in the construction of the process. I would design key questions that provide clues and would facilitate the analysis of the situation". This shows that the knowledge connected to the epistemic facet of DMK, has yet to be improved by Teacher A.

Regarding Teacher B, for item 3, we can see in Figure 8a that once again he 'explains' his solution for items 1 and 2 by utilizing iconic elements (little balls). He explains his 'new procedure' as follows: "the difference among the cubes is of 4 stickers [referring to the bars of different but consecutive length], therefore, we multiply 4 by the number of the cubes plus two".

As we can observe in the Figure 8, Teacher B does not provide a new procedure, let alone finds a new symbolic expression to determine the number of stickers of a bar of any length. Some teachers prefer using the arithmetic knowledge instead of algebra because they consider arithmetic appear first in primary curriculum. Regarding his answer to item 4, this teacher only points out two concepts/definitions –"sequences (patterns), functions"– and one procedure –"analysis of variables (dependent and independent)"–. Like Teacher A, Teacher B mentions that, for item 5, he would ask the students for the number of stickers for bars of a specific length, then he would try to "join logical thinking (empirical) with algebraic thinking", in order to find a general rule. This Teacher also points out that he would conduct a formalization process by asking, "what happens with zero (when there is not any cube)". Teachers A and B find it hard to provide alternative ways to present the solution, may be this could be explained on the grounds of lack of experience teaching the subject.

Teacher C, on the other hand, suggests the use of a table (two columns, where the first one would refer to the number of stickers, while the other would refer to the number of cubes) as an answer for item 3. This suggests the use of the procedure of induction and the rescaling of the problem in order to shape the mathematics knowledge to student's needs. She points it out as: "The results obtained in the figures [of the cubes] can be ordered in a table. The general relation that corresponds to the number of stickers in connection to the number of cubes is 5n, where n is the number of bars [in a later interview, she says that n is the number of cubes]". It is important to mention that the result of this procedure is different from the results of items 1 and 2, however, she continues considering that each cube (whether in the middle or at the ends of the bar) has five faces/sides exposed. Regarding item 4, Teacher C mentions a concept/definition –"regularities that involve variables"–, a property/proposition –"numerical relations"–, and a process – "generalization"–. For item 5, she suggests asking questions, starting with questions that would invite students to observe "the behavior of the numbers" (the teacher suggests that these are organized in a table) and "the relations of the variables at play" (through drawings of the cubes), then asking questions directed to the search of "a general rule to avoid drawing all the cubes of the bar", and identifying that with each cube that is added to the bar, the number of stickers increases in five units. She puts it this way: "(...) then, when observing the behavior of the numbers, it can be noticed that, when the length of the bar increases, the number of stickers increases in five units, therefore, it is necessary to find a general way to avoid drawing all the cubes".

In connection to the solution proposed by Teacher D to solve item 3 of the activity, we can observe in Figure 8b that her approach is a little different from the one she establishes in items 1 and 2. The solution is built on numeric grounds. Teacher identifies an algebraic rule that can be traced back to the numeric examples provided.

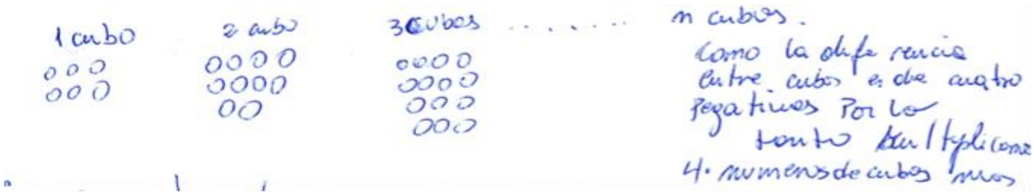


Figure 8a. Teacher B response

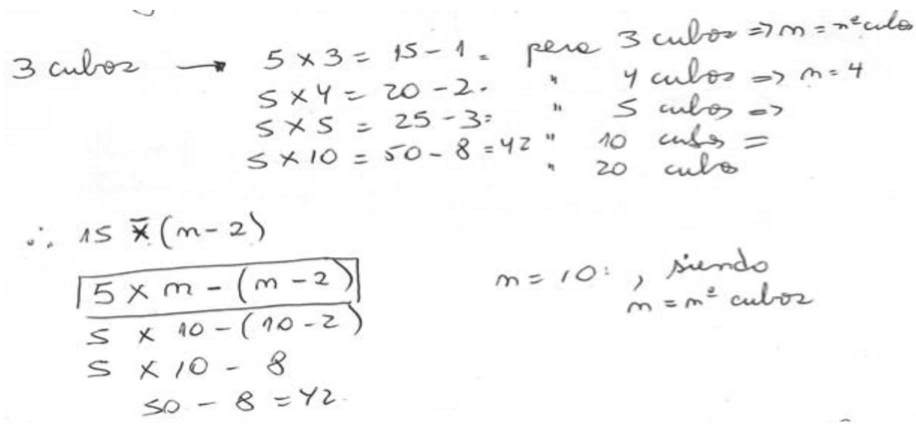


Figure 8b. Teacher D response

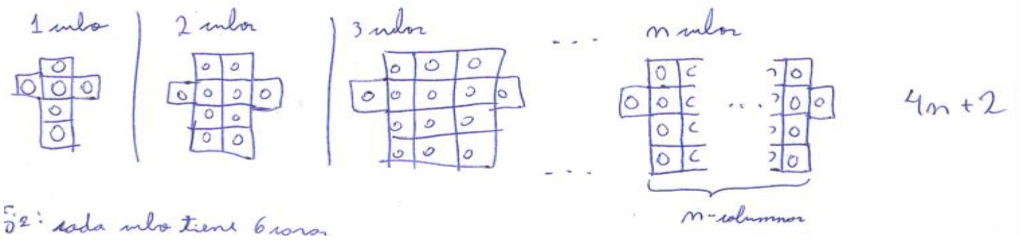


Figure 8c. Teacher E response

Figure 8. Answers from Teachers B, D and E to item 3 of the activity

Teacher D points out that it is possible to provide another solution. Such solution, as we can observe in Figure 8b, consists on considering that, for every cube that is added to the bar, the number of stickers increases in five, which is expressed with propositions "$5 * 3 = \cdots; 5 * 4 = \cdots; \ldots 5 * \mathrm{m} \cdots$, [where] m = number of cubes". Then, she subtracts two cubes to the total number of cubes (that correspond to the cubes that are at the ends of the bar), and obtains the expression "m – 2". With this last expression, the teacher obtains, on the one side, the total cubes that are in the middle of the bar (the length of the bar minus the cubes that are at the ends), but on the other hand, obtains the number of additional stickers that she must reduce for each cube that is in the middle, the teacher expresses it as "$5 * \mathrm{m} - (\mathrm{m} - 2)$", proposition that she enunciates through a process of generalization and that helps her to determine the number of stickers. This answer to item 3 of the activity constitutes a new procedure for Teacher D, where she mobilizes a language that is mainly symbolic and, from her point of view, provides a new reasoning for the solution of the problem. Some teachers believe that algebra deal with letters, and then letters must be used along an "algebraic solution" (Mac Gregor & Stacey, 1997).

In regard to item 4, Teacher D points out that the knowledge at play is "geometry of third grade, algebra, patterns, sequences or series using tangible materials until 6th and 7th grade elementary school, tables (2nd and 3rd grade and through the use of concrete materials)". While for item 5, this teacher pointed out that if a student is not able to solve the problem, he would explain the solution to the student by using a method that she calls "COPISY method", which she explains as: "…the work must begin with activities that utilize COncrete materials, followed by PIctorial resources, and then, keep on working trying to reach the SYmbolic language from the organization (for example, in tables) of the data obtained".

Regarding the solution that Teacher E provides for item 3 (Figure 8c), he suggests, "imagining the cubes as little square boxes …and disassemble or dismantle those boxes as shown in the figure [Figure 8c]". He explains his figure as follows: "First, for cube 1, there are two units that would be fixed in all positions, and then a group of four little balls that represent the sides/faces. For two cubes, two groups of four little balls, and two fixed balls that correspond to the sides/faces that will stay at the two ends of the bars of cubes. For three cubes, there are, then, three groups of four little balls, plus the two fixed balls, and so on. Therefore, in the nth position there are n groups of four little balls". This enables the teacher to obtain the proposition or general rule "4n+2", that determines the number of stickers for a bar of any length. We can observe how this teacher uses mathematical induction as procedure, by using a linguistic resource of an 'iconic-pictorial-visual' type (the crosses that represent the boxes disassembled) which shows the necessary stickers with each cube that is added to the bar. This is a good example of how, mathematics knowledge has to be transformed in order to suit students' knowledge; math induction is not a good strategy to explain the problem due to its complexity. As an answer to item 4, this teacher mentions a process (generalization), and two concepts/definitions (regularity and patterns). Likewise, he adds that the activity can be solving without using algebra, although he does not specifies how. Finally, for item 5 the teacher suggests the "drawings of diagrams, separating the two faces at the ends and adding 4 faces to each cube that is added in the middle. See my drawing [referring to Figure 8c]". Stacey and Mac Gregor (2000) stated that the students tend to use different methods to solve algebraic problems, and not many students solve them using an exclusively algebraic way. Sometimes their methods tend to be lengthier.

In general, we have seen how teachers suggest activities that, from their point of view, can help students to experience the process of generalization. The pattern exploration offers opportunities to pupils to observe and verbalize (English &

Warren, 1998) that in association with generalization are considered suitable mathematical activities to introduce children to algebraic aspects (Mason, Graham, Pimm & Gower, 1985). However, the way they approach such activities, suggests that generalizing from regularities is not something that could be directly taught by merely indicating specific procedures. Using the words of two teachers: "(…) students manage to reach generalizations or find general rules for tasks like the one presented to us, doing them, and trying until they do it … and by providing them with material that allow them to perceive and touch". All of the above shows that teachers have a low level of knowledge regarding the epistemic facet of DMK in regard to these mathematic tasks.

Ecological facet

Item 6, aims at exploring some relevant aspects, specifically curricular aspects, of the teachers' knowledge of ecological issues involved in the teaching and learning of mathematics. This item specifically tries to enquire whether the teachers know the relations among the elementary school mathematics curriculum (6-12 years old students), secondary school mathematics curriculum (12-15 years old students) and high school mathematics curriculum (15-18 years old students), which are the levels they teach, focusing their attention on the links among such mathematics curricula and the mathematical objects that are mobilized in activities like the one presented in this paper (items 1and 2).

As an answer to item 6, Teacher A points out: "I think that a proper level would be 7th grade [students between 12 and 13 years old], because they already have some algebraic notions. Besides, at this level one can perceive students who are more committed to their student duties. Also, students already have cognitive concerns and conduct more thorough analysis of their practices". The teacher does not consider that this task can be proposed to students of any age. He considers that the students should have learned some algebraic notions in order to provide a solution. In this sense, the teacher suggests that this type of activity can allow the "construction of the concept of variable".

In connection to the same item, Teacher B answers the following: "9th grade [students between 15 and 16 years old] because in order to introduce the concept of function, it is necessary to start relating variables". On the other hand, Teacher C states "...I would only propose the first question [item 1] in 1st to 4th grade [students between 6 and 10 years old]; I would propose the first two questions [items 1 and 2] with 5th to 8th grade students [students between 10 and 15 years old], which is when the algebraic notions have already been introduced, so they are in conditions to reach a general rule".

Teacher D states that she would implement the activity with secondary students [students between 12 and 15 years old], which is when, according to her, "(...) students have developed the necessary background knowledge". Meanwhile, Teacher E suggests 6th grade (students between 11 and 12 years old), "because it is the time they are faced with algebra".

Once again, the suggestions that the teachers make regarding the educational level where they would implement the activity, indicate that they think that it is necessary to have algebraic notions, without considering that the activities of sequences and patterns are the ones that the experts remark as most interesting and rich for the introduction of algebra. The notes of most of the teachers, however, go in the opposite direction, since they state that it is necessary that "students already have algebraic notions" (Teacher A), "students had developed background knowledge" (Teacher D), "students already know how to relate variables" (Teacher B), or that they already "have had contact with algebra" (Teacher E), so that they can successfully work with patterns. In this sense, we can say that neither of the teachers, except Teacher C, perceives the potential of the activity to be developed, for example, elementary algebraic reasoning (Godino, Castro, Aké & Wilhelmi, 2012), so the activity could be presented to students between 10 and 18 years old, or even before, depending on the mathematical notion that is intended to address and foster. Some emphasis should be put on "unpacking" (Ball, Thames & Phelps, 2008) algebraic knowledge present in mathematic content for the teachers to foster and to recognized it. Furthermore, even inservice teachers "have little experience with the rich and connected aspects of algebraic reasoning" (Blanton & Kaput, 2005, p.414).

Cognitive facet

Items 7 and 8 of the activity seek to explore the teachers' knowledge related to this subcategory of the didactical dimension of DMK. In relation to item 7, Teacher A states that the main difficulty is translation from the arithmetic process into an algebraic expression. She justifies such difficulty arguing that the teaching and learning of algebra is initially limited to, almost all the time, algorithmic processes, creating an "obstacle" for the students' learning. For item 8, she points out that one of the main mistakes of students would be "not identifying the regularity and thus, do not reach a generalization", and mentions that the causes for this would be "...precisely, given the difficulties pointed out in the answer to item 7 (moving from arithmetic into algebra). The Teacher shows certain knowledge of how the study of algebra is commonly started, commenting on the way of prioritizing algorithmic processes at the expense of meaningful learning, in the current teaching system in her country.

Teacher B shows less mastering of the topic, when mentioning empirically, for both items 7 and 8, that the main difficulty and mistake would occur when it comes to generalizing, and explains: "because not everybody has the ability to merge logic with algebraic work".

Both teachers, A and B, consider that "generalization" is difficult for the students to understand, and Teacher C makes a similar comment for item 7, "The main difficulty is not reaching generalization"; while for item 8, she says, "...the main mistake would be to stick stickers to the faces that are joined together when a new cube is added to the bar".

Regarding Teacher D, she identifies some affective aspects, such as the fact that some students do not like mathematics or that they will not be motivated when implementing the task, as the main difficulties they would encounter. She also mentions the absence of necessary background knowledge in students, or the necessary abstraction level to deal with the task as difficulties, and suggests that the teacher should implement the activity using specific materials. For item 8, the teacher mentions as main mistake "the fact that the students do not realize that the sticker that are located where the cubes join should not be considered, or that they only consider the faces that are seen in the drawing, ignoring the ones in the back".

For his part, Teacher E bases his answer on the solution that he proposed for item 3, and states that the difficulty lies in "not considering the faces that are covered in the drawing... or well, not considering that each cube has two faces that will not be labeled, except for the two at the ends". In this sense, he points out that the main mistake of the students is "not considering the faces/sides that will not have a sticker, and then, state that the solution is 6n".

Affective facet

This subcategory of knowledge complements the knowledge of the cognitive facet. The cognitive and affective facets, together, provide a good approach and understanding of the knowledge that mathematics teachers should have of the characteristics and aspects that are connected to the students' way of thinking, knowing, acting and feeling, inside a classroom. Item 9 seeks to explore this type of knowledge in teachers.

In regard to how to motivate students, Teacher A suggests an initial discussion of the elements proposed in the activity (item 1). We infer, from the answers to all the items, that this teacher thinks that most of the difficulties (which implies little motivation) are located in the process of interpreting the formulation of the problem (students do not comprehend what they are being asked), which by the way, the teacher itself experience by not comprehending what was being asked in the problem (items 1 and 2). On the other hand, Teacher B suggests the introduction of a specific material, "Excel sheets", as a strategy to motivate students. He explains, "…Excel sheets, because they allow to model situations, and [Excel] provides formulas and charts related to the problem". Teacher B suggests using a computer, which leads us to infer that he believes that its use is interesting for students and it could, therefore, motivate them.

Teacher C states that, in order to motivate students, it is necessary to present the activity as a "challenge", and implementing group work. Teacher D suggests that a motivating factor would be "using songs about geometrical shapes, especially the cube", and also suggests the use of concrete manipulative materials, such as "real cubes and stickers to be stuck on the faces of the cube". In this sense, Teacher E also mentioned the use of concrete materials that can be manipulated by students, as a motivating factor. All the teachers offer ways to deal with the difficulties they foresee with the students patterns' understanding.

From the results obtained with the items that are related to the cognitive and affective facet of DMK, we can point out that, in general, teachers do not have major problems to indicate, from their point of view, the main difficulties and mistakes that students might have or make when facing activities such as the one presented in this study. The real problem for teachers consists in knowing what to do to help students overcome their difficulties and mistakes. In this sense, Teacher B states that "not everybody has the ability" to deal with mathematics. The teacher believes that in order to learn mathematics, a gift from birth, or, using his own words "talent for mathematics" was necessary. According to Ferreira (2001), the teachers in his study seem to believe in the innate skill to learn mathematics, so the teacher's job would be just to develop the already existing mathematics skill in those gifted students. Teacher C points out as a strategy to motivate student, to refer to the activity as "a challenge", instead of a problem (we infer that it is so, in order to avoid the panic that some students might feel when we say that it is a mathematical problem). Teacher D, on the other hand, suggests, "to do something that students like… songs…" (a suggestion that, we infer, might have been proposed in favor of making mathematics look less boring). Both teachers D and E suggest the utilization of manipulative materials. Teacher B proposes the utilization of Excel sheets and software. These elements –manipulative materials or a computer– are considered by teachers as motivating resources that can potentially promote the discovering of regularities by the students.

Interactional facet

Item 10 of the activity seeks to explore the teachers' knowledge related to this subcategory of DMK. Thus, regarding the strategies for the implementation of the activity and considering the educational level pointed out on item 6, Teacher A suggests working with tangible materials, so that from the practice, students can perceive the recurrence that occurs in each situation. We can infer that by indicating the use of concrete materials, the teacher thinks that at the level she suggests, the students have difficulties to deal with abstraction, and therefore, need tangible materials for better comprehension.

Teacher B suggest something similar, by saying that, as a strategy, he would incorporate a software that would allow to work with different types of representation –he suggest tabular and algebraic–. This teacher also mentions that, prior to the introduction of the software, and for students to be successful when dealing with this type of problems, he "would start to work with numerical patterns, implementing several exercises where sequence work is involved and would also implement modeling from manual work". This talks about a strategy where, we interpret that, from the manipulation of concrete materials, students would be given the chance to identify what he calls 'sequences' (patterns), in order to reach the mathematical modeling of the proposed situation.

Like Teacher B, Teachers C, D and E, suggested some aspects similar to the ones proposed on item 9 of the activity, as teaching strategies. For example, Teacher C mentions as a strategy, the utilization of concrete materials and presenting the activity in way that students would perceive it as a "game", and she would organize groups of 4 people.

On the other hand, Teacher D mentions "...working with already described concrete materials...and teams of no more than three students...". While Teacher E, says that he "would start with other type of problems that leads children to the same direction, but inductively, because at that level [the level suggested by himself/herself on item 6] children basically have arithmetic knowledge". The answers of the two teachers (to all the items) suggest the indication of a pattern of interaction where the student is given certain autonomy to build his/her own knowledge. Teachers are well aware of the importance to activate students in the learning of mathematics using different artifacts -tangible material, software- or strategies -working in teams or by teaching inductively-.

This option of interaction indicates a knowledge (although tacit) of how our subjects think that students learn. But, what other problems could make students move from an arithmetic level into an algebraic level? These teachers do not point that out. They suggest working with concrete materials but at the same time, state that if the students do not have background knowledge of algebra, they will face great difficulties. In other words, the teachers do not perceive or do not believe that the work with patterns is a powerful vehicle for the comprehension of the relations of dependency between amounts that underlie mathematical functions as a concrete and transparent way for young students to start working with notions of abstraction and generalization (Moss & Beatty, 2006). The teachers manifest a conflict between their beliefs about secondary algebra, and elementary algebraic reasoning.

Mediational facet

With regard to this point, all the teachers suggested including, in a possible implementation of the activity, materials such as an Excel sheet or a software (Teacher B) and tangible materials –real cubes and stickers, maybe– (Teachers A, C, D and E). This kind of choice reveals that they know that working with the proper materials can facilitate students' learning. None of the teachers mentions the times that they would assign for the implementation of the task.

Meta didactic-mathematical dimension of DMK

Since it was an activity that could be classified as a "planning or design" of the activity, the aspects of the teachers' knowledge that make reference to the meta didactic-mathematical dimension, did not arise with the activity proposed. As a continuation of this research, the teachers' knowledge related to this dimension, when implementing instructional designs for the teaching and learning of specific mathematical topics, could be explored. For such analysis, the guidelines and criteria suggested by Assis, Godino & Frade (2012) for the analysis of the rules and metarules involved in the management of the students' learning, and also the criteria and suggestions by Godino (2011) for the analysis of didactical suitability of processes of studies, should be used.

FINAL REFLECTIONS

In this study we exemplified how the use of some dimensions and theoretical-methodological tools proposed by the model of didactic-mathematical knowledge (DMK) can be useful to analyze the teachers' knowledge. Specifically, we have exemplified in this document the use of the mathematical and didactical dimensions of DMK, and also the use of the "onto-semiotic configuration" tool in its version of cognitive configuration.

A literature review shows papers dealing with inservice teachers and algebra (Doerr, 2001; Dörfler, 2008; Wilkie, 2014) but none specifically on inservice teachers' knowledge and patterns. This research proposal builds on the teacher knowledge about patterns in relation to some components of the DMK knowledge, on specific areas, which is not taken into account in the above researcher papers.

From the answers that, at the beginning, seemed like 'did not have anything to say', and with the help of the dimensions and tools pointed out, we explored and described some features of the knowledge of five teachers who had some experience in the teaching the topic studied in this research study. Such tools act as 'lens' that directs the focus in order to make it possible to reveal some particularities that would go unnoticed otherwise.

In regard to the DMK model, it has links to the Ball's model (and other models) of teacher knowledge, nonetheless the DMK offers tools to analyze the teaching and learning activity that are not offered, for example, by the PCK of Ball et al., (e.g., interactional and mediational phases). The PCK model gives insight into the big areas of teacher knowledge that are required to offer, for instance, teacher training programs, but it lacks the tools that allows analyzing the teaching and learning processes. The Ball's model does not consider either the affective or the mediational facet, together they provide a good approach and understanding of the knowledge that mathematics teachers should have in order to motivate and to interact with students' way of acting and feeling, inside a classroom.

The DMK model proposes dimensions that offer a general look to the teaching and learning process, the common content knowledge and the extended content knowledge is seeing as a duo that connects the mathematical dimension to the knowledge that a teacher should have in order to perform his teaching in a proper way (Didactical and meta-didactic-mathematical dimensions). A more local perspective has to be assumed, and the interactional –identifying and answering to students' conflicts–, the mediational –choosing the best materials for the students to work the task–, the affective –reacting to anguish, indifference, anger, etc., manifested by students–, ecological –aligning tasks according to institutional mandated curriculum–, and cognitive –understanding student's solutions–, are all present while the teacher perform his classroom teaching duties.

The three dimensions proposed by DMK as well as the tools of analysis, can be seen in each of the four phases of the instructional design that are considered within the DMK: preliminary study, design, implementation and assessment (Pino-Fan & Godino, 2014). In this way, it could be stated that, the questions suggested in the activity that we presented, as well as the answers given by the teachers who participated in the study, correspond with the didactic-mathematical knowledge that they should possess for the preliminary study and design (or planning) of the activity, which, of course, is prior to the implementation phase. So, in this paper we have look into the DMK's potential for planning a mathematic task. We want to investigate the way teachers teach patterns in a normal class setting. We would like to investigate how teachers provide students tasks to promote the appropriate learning and how these tasks connect along grades in primary school.

As a result of the analysis, it becomes evident that the teachers can solve the items related to common content knowledge, but have certain problems when facing items that aim at exploring other dimensions of their knowledge, for example, with extended content knowledge, the resources and means or with the students' affective states. This discussion leads us to a teacher training style that often gives priority to theoretical disciplines at the expense of pedagogical/methodological disciplines. The results obtained indicate the need of fostering the aforementioned types of knowledge in the training of teachers (preservice and in service). Along with Doerr (2001, p.281) we conclude that it is important for teachers to develop a knowledge base for teaching algebra that includes not only the dynamics but the complexity of the various facets we have visited on this research.

ACKNOWLEDGEMENTS

This research has been performed in the framework of research projects EDU2012-32644 (Universitat de Barcelona), R02/15 (Universidad de Los Lagos) and CODI-Acta 643 (Universidad de Antioquia).

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