THE ANTIDERIVATIVE UNDERSTANDING BY STUDENTS IN THE FIRST UNIVERSITY COURSES

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In this article, we present the results of a questionnaire designed to evaluate college students' understanding of the antiderivative. Specifically, by civil engineering students when answering the questionnaire' tasks, in order to identify and characterize the meanings on the antiderivative that are mobilized by them. In order to analyse the answers given, we used some theoretical and methodological notions provided by the theoretical model known as the Onto-Semiotic Approach (OSA) of mathematics cognition and instruction. The results show knowledge of antiderivative by the Civil Engineering students. Furthermore, the comparison between the mathematical activity of students provides information that allows concluding that the meanings that they mobilized might be shared among their communities.

BACKGROUND

In recent years, the mathematical education of engineering students has gained more attention from researchers in the field of mathematics education (Bingolbali, Monaghan & Roper, 2007). The reason lies in the fact that, nowadays, as pointed out by Gnedenko and Khalil (1979), mathematics has become more than just a calculus tool; it has become a powerful and flexible method for both science and engineering.

In this regard, there have been several studies that have dealt with the issue of how to address different mathematical notions in engineering contexts (Sonnert & Sadler, 2014). The suggestions given by these studies focus on the type of problems used to introduce mathematical notions, the impact of technological resources and textbooks for the teaching of mathematics to engineers, and even motivational factors. Other studies, focus on the study of the differences in the way of thinking mathematics between mathematics and engineering students (Jones, 2015).

This article aims at identifying and characterizing the meanings that civil engineering students, mobilize in their mathematical practices in connection to certain tasks assigned to them. For this purpose, we applied a questionnaire to two groups of civil engineering students, one from a Mexican University and another from a Colombian University. The questionnaire was designed as part of another study (Gordillo, Pino-Fan, Font & Ponce-Campuzano, 2015), to assess the aspects of comprehension that university students have of such mathematical object. The analysis of the answers to the questionnaire show the meanings and preferences that future civil engineers assign to the antiderivative, and how these relate to the partial meaning that make up the holistic meaning of this mathematical notion (Gordillo & Pino-Fan, 2016).

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THEORETICAL AND METHODOLOGICAL ASPECTS

In order to conduct this study, we considered the theoretical model known as the Onto-Semiotic Approach (OSA) of mathematical cognition and instruction. This theoretical approach arises in the field of the research of Mathematics Education in order to articulate the diverse dimensions that are present in the processes of teaching and learning of mathematics (Godino, Batanero & Font, 2007). In OSA, the notion of *systems of practices* (or *mathematical practices*) plays and important role in the teaching and learning of mathematics. Godino and Batanero (1994) define a system of practices as “any performance or manifestation (linguistic or not) done by someone in order to solve mathematical problems, communicate the solution to others, validate the solution and generalize it to other contexts and problems” (p. 334). These practices can be personal or institutional, depending on whether these are done by one person or shared within the core of an institution.

Besides, OSA assumes certain pragmatism when considering mathematical objects as entities that emerge from the systems of practices conducted in a field of problems (Godino & Batanero, 1994). In OSA, the meaning of mathematical objects is conceived from a pragmatic-anthropological perspective which considers the relativity of the context in which these are used. In other words, the meaning of a mathematical object can be defined as the system of operative and discursive practices that a person (or an institution) develops in order to solve certain type of situations-problems in which such object intervenes (Godino & Batanero, 1994). Thus, the meaning of a mathematical object can also be considered from two perspectives, institutional and personal.

In order to conduct a ‘finer’ and more systematic analysis of the mathematical practices developed regarding certain problems, OSA introduces a typology of primary mathematical entities (or primary mathematical objects), that intervene in the systems of practices: situations-problems, linguistic elements, concepts/definitions, propositions/properties, procedures and arguments. These primary mathematical objects are related among themselves forming nets of intervening objects that emerge from the systems of practices, which in OSA are known as *configurations*. These configurations can be epistemic (nets of institutional objects) or cognitive (nets of personal objects).

In this document, we use the notion of *cognitive configuration* to analyse the mathematical practices performed by civil engineering students regarding the solutions to the tasks of the questionnaire.

METHOD

This study uses the methodology of the mixed methods research (Creswell, 2009), since it is an exploratory study that considers the observation of quantitative variables (answers’ degree of accuracy: correct answers, partially correct answers and incorrect answers) and qualitative variables (the type of cognitive configuration connected to the practices on antiderivative). For the study of the qualitative variable we adopted a technique of analysis known as *semiotic analysis* (Godino, 2002), which allows to describe in a systematic way the mathematical practices of students as well as the elements of cognitive configuration (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) which are activated in such practices, and their respective meanings.

The questionnaire

The questionnaire that we used to gather data was designed to evaluate the comprehension of the notion of antiderivative of university students and is composed of five tasks (Gordillo, et al., 2015). Each of these tasks is closely related to one of the four partial meanings of the antiderivative that were identified through a historic-epistemological study that aimed at reconstructing the 'holistic meaning of reference' for such mathematical object (Gordillo & Pino-Fan, 2016). Chart 1 shows a summary of the characteristics and goals pursued by each of the tasks.

Chart 1. Summary of the characteristics of the tasks of the questionnaire

Task	Objective	Representation activated	Partial meaning activated
Task 1: Meanings of the antiderivative	To explore personal meanings and definitions given to the antiderivative	Verbal/Written	Global
Task 2: Graphic exploration of the antiderivative	Treatment of the graphic representation of the antiderivative	Graphic	Tangent- squaring
Task 3: Calculation of the primitive function (parts A and B)	Construction of a family of functions from a derived function	Symbolic, graphic and tabular	Differential-sum
Task 4: Difference integral-derivative	To explore if there are conceptual differences between the notions of integral and derivative	Verbal, Written and symbolic	Elementary functions
Task 5: Solving of ordinary differential equations	Use of the antiderivative for solving differential equations	Verbal, Written and symbolic	Fluents- Fluxions

The questionnaire was applied to two groups of Civil Engineering students. The first group was composed by 23 students of the Civil Engineering of the Universidad Distrital in Colombia. The second group was composed by 23 students of the Civil Engineering of the Universidad Autónoma de Querétaro in Mexico. An essential requisite for the selection of the students that participated in the study was that, at the moment of taking the questionnaire, they had taken Integral Calculus courses.

ANALYSIS OF DATA

In this section, we present the analysis of the answers given by the students of the two groups, Mexican and Colombian. For the analysis of the quantitative variable ('answers' level of accuracy). The first study that we conducted with the variable level of accuracy was to determine if there were significant differences between the Colombian group and the Mexican group.

For the analysis of the qualitative variable we used the notion of *cognitive configuration*, which allowed us to describe in a systematic way the primary mathematical objects (linguistic elements, concepts/definitions, propositions/properties, procedures and arguments) that form the mathematical practices of the students, in connection to the tasks of the questionnaire.

Analysis of the answers of the Mexican and Colombian engineering students

In this section, we present the results of the quantitative and qualitative analysis of each of the tasks of the questionnaire.

Task 1: Meanings of the antiderivative

Given the general nature of this first task, only correct answers (answers in which at least one of the partial meanings of the antiderivative was expressed in verbal/written form) and incorrect answers (answers in which any of the partial meanings of the antiderivative were enunciated) were considered. The students did not have difficulties for solving the task, answering 82,6% correctly.

A high percentage of Mexican students (13) as well as Colombian (11), answered that the antiderivative is "the inverse process of derivation". This first general approach to the conceptions that students have of the antiderivative show that more than half of them (52,2%) think of the antiderivative as a *procedure* (operation) that allows to find the "original function" from which certain derived function comes from. Out of the 46 students, only one student from Mexico answered that the antiderivative is a "family of functions". The solutions that we have labelled as 'absence of meaning', that refer to incorrect answers from the point of view of the level of accuracy, are answers in which the students did not give any meaning to the antiderivative, providing answers of the type "the antiderivative is the area below the curve", "the antiderivative is obtained from the fundamental theorem of calculus", "the antiderivative is a function f of $f=f'$ ", "the antiderivative is a mathematical form through which some real life problems can be solved".

Task 2: Graphic exploration of the antiderivative

For this task, we only considered correct answers (in which the elements that belong to the family of the antiderivative were correctly identified and the way of finding them was justified), and incorrect answers (in which the graph provided did not correspond with the elements of the family of antiderivative for the function provided graphically). Task 3 has a higher level of difficulty for the students, with only 41,3% answering correctly. Among the mathematical practices that the students performed as part of their answers, we could identify three types of cognitive configurations.

Of the three configurations identified, the most used by the students was the 'particular function' (34,8%), in which a symbolic expression for the function is obtained from the graph of the function, and through algebraic procedures, it is possible to identify (or try to identify) which are the graphs of the elements of the family of antiderivatives. The second more used type of configuration was the 'tabular interpretation of the graph' (30,4%), which refers to the answers in which a table of values that describe the function given originally is constructed from the graph of the function provided; from the table constructed (and the relations and properties that are established with it) it is possible to try to identify the elements that belong to the family of antiderivatives. The configuration that we have identified as 'advanced' was activated in answers which were characterized by the use of procedures and justifications centred on the properties/propositions of derivation, specifically the criterion for the analysis of the characteristics and construction of graphs of functions, in order to identify graphically the member that belongs to the family of antiderivatives of the function provided.

Task 3: Calculation of the primitive function

Task three was composed of two parts. For the first part, part A, we considered as correct all the answers in which a valid symbolic expression was provided for $f(x)$; while incorrect answers were all the answers that did not provide valid symbolic expressions for $f(x)$. For part B, all the answers which provided a second expression for $f(x)$, different from the one given in part A and with valid justifications, were considered as correct. All the answers in which it was explicitly or implicitly mentioned that it was not possible to find a second expression for $f(x)$ were considered incorrect.

The students did not have problems to provide a symbolic expression for $f(x)$ in part A of the task, with 87% of them giving a correct answer. However, the students had more difficulties to answer part B of the task, with 50% (23) of them giving a second valid expression for $f(x)$ different to the one provided in part A.

We could identify two types of cognitive configurations from the answers provided by the students to part A of the task. The first type 'graphic-technical', refers to the answers in which, from the data given in the table, a graphic representation is provided from which the algebraic expression is obtained (graphic and symbolic linguistic elements, respectively) for the derived function. Subsequently, an expression for $f(x)$ is found from the argumentations and procedures centred on the "rules" (properties/propositions) of derivation. The second type of cognitive configuration, "numeric-technical", refers to the answers in which a pattern (property) that allows establishing the rule of correspondence that defines the derived function (concept/definition) is determined from the combination of the data provided in the table. Later, from the argumentations and procedures centred on the "rules" of derivation, an expression for $f(x)$ is found.

Regarding the cognitive configurations connected to the answers in part B of the task, we found three types. The first type, 'wrong interpretation of the uniqueness of the derivative', are answers in which the students show a wrong conception about the uniqueness of the derivative at a point and the derived function, providing answers of the type "it is not possible to find another expression for $f(x)$ because for $f'(x)$ there is one and only one $f(x)$, and vice versa". The second type of configuration, 'equivalent functions' is related to the answers in which, explicitly or implicitly, by means of the use of equivalent functions (concept/definition), some algebraic operations are developed (procedures that serve as arguments) to show that it is not possible to find another different function. The third type of cognitive configuration, 'advanced solution', was activated in answers in which the procedures and their justifications explicitly establish a connection among concepts such as antiderivative, the fundamental theorem of calculus, rules of integration, etc., to point out with the proposition "another expression for $f(x)$ can be any member of the family of functions $f(x) = x^2 + C$", that it is, indeed, possible to find another expression for $f(x)$. As we can observe, 50% of the students (12 Colombian and 11 Mexican), mobilized the third type of configuration to provide their answers. Regarding the antiderivative, the third type of configuration brings associated the meaning of inverse process of derivation.

Task 4: Difference between integral and antiderivative

Task 4 aimed at exploring whether the students conceived the integral and the antiderivative as different notions or not.

The correct answers were those in which the students pointed out and justified which were the differences between both notions. Partially correct answers were those in which the students mentioned that there were differences, but, the differences were not pointed out, or no justification was given, or the justification was not valid (from the institutional point of view). Only 26,1% of the students pointed out that the antiderivative and the integral were the same notion and that the terms were synonyms (Hall, 2010).

As shown above, the most activated cognitive configuration in the answers was 'definitions for the notions', used by 67,4% of the students. Such configuration was activated in answers in which there were arguments regarding the difference between the concepts of antiderivative and integral, providing definitions (personal or institutional) for both notions. For example, "…are different because the integral is a number, while the antiderivative is another function". The configuration 'examples of use' was the second most activated configuration (2 Colombian students and 6 Mexican), and was activated in answers in which there were arguments regarding the difference between both notions by means of concrete examples (situations/problems) of their use or application, for example, "the integral serves to calculate the area below the curve while the antiderivative serves to obtain a function". It is important to point out that the examples of use that were provided in this second configuration, made reference to the notions involved as process (or procedure) and not from a conceptual point of view. The third type of configuration activated was 'particular-general' (4 Colombian and 3 Mexican students), in answers in which the arguments were oriented towards the distinction of the antiderivative as a general case of the definite integral, in other words, the antiderivative was seen as indefinite integral, which is similar to what was found by Hall (2010).

Task 5: Solution of ordinary differential equations

The main objective of this task was to explore the process followed by the students in order to find the antiderivative, by means of a problem in which they needed to describe how they obtain the solution of a first order differential equation. Additionally, by means of the descriptions of the students, it was also intended to explore the meaning that they give to the constant *C*, known as constant of integration, in order to see if they comprehend the "inverse process" that finding an antiderivative implies.

Needless to say that the students had serious difficulties to solve the task presented. Only 5 of them were able to describe, from a correct mathematical point of view, the process that they follow in order to find the solution to the differential equation presented. Twelve of them (26,1%) omitted the constant of "integration" in their solutions, so we labelled their answers as partially correct. 63% of the students did not answer or answered something 'incongruent' (not valid or senseless from a mathematical point of view). The main cause mentioned by this 63% of the students, either orally at the moment that the questionnaire was given or written in the box intended for the answer to the task, was that they did not remember or did not know how to solve a differential equation.

Regarding the types of cognitive configuration activated in the answers, these were of 3 types, and were classified according to the type of linguistic element used in their arguments. The first, 'verbal', is a configuration that was activated in answers in which the verbal-descriptive language to narrate the procedure that they had to follow in order to solve a differential equation, but without "developing" such procedures symbolically, in other words, there is a description of what should be done, but it is not actually performed. Only one student who activated this type of configuration gave a correct answer.

The second type of configuration, 'symbolic' was activated in answers that centred their arguments on the procedure itself of calculation of the solution, in other words, they solved the differential equation symbolically without describing with words the process they followed. The third configuration activated was a mixture of the two previous configurations. Four students (two Colombian and two Mexican) described the procedure and the properties/propositions used in the calculation of the solution, verbally. Three of the students, who mobilized the third configuration, 'verbal-symbolic', answered the task correctly.

FINAL REFLECTIONS

Partial meanings of the antiderivative such as *tangents-squarings* and *elementary functions* (Gordillo & Pino-Fan, 2016), were not activated in the answers of the students. Now the questions would be, why did the engineering students of our study activate, with difficulties, one of the four partial meanings of the antiderivative? The answer to this question leads us, on the one hand, to face one of the limitations of our study, the type of problems suggested, were they appropriate for engineers, for their practices and interests? Although the questionnaire was designed to activate the different partial meanings of the antiderivative, and it aimed at exploring the comprehension that university students have of such notion (Gordillo, et al., 2015). On the other hand, the question brings to our mind the role of the educator of engineers. For this purpose, the educator of future engineers should be aware, first of all, of the diversity of partial meanings of the mathematical object under study, in our case, the antiderivative (Gordillo & Pino-Fan, 2016). By comprehending the use of such partial meanings in the context in which he works, the educator would have opportunities to pose problems that mobilize such meanings and, at the same time, adjust to the real needs of the engineers in training.

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