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# Self-organization and pattern formation in coupled Lorenz oscillators under a discrete symmetric transformation

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**Abstract.** We present a spatial array of Lorenz oscillators, with each cell lattice in the chaotic regime. This system shows spatial ordering due to self-organization of chaos synchronization after a bifurcation. It is shown that an array of such oscillators transformed under a discrete symmetry group, does not maintain the global dynamics, although each transformed unit cell is locally identical to its precursor. Alternatively, it is shown that in a 1-dimensional lattice, the coupling destroy the chaotic behavior but there are similar global behaviors between both coupled arrays, suggesting that is the local equivariance which controls the dynamics.

## 1. Introduction

A covering dynamical system is one with a discrete symmetry, and the equations of motion that describe it are left unchanged under the action of a discrete symmetry group. By means of a local diffeomorphism, is possible to map it to a locally equivalent dynamical system without symmetry [1]. The dynamical properties of this *new* system, such as Lyapunov exponents and fractal dimensions are left unchanged [2], although the resulting system (*image*) is topologically different from the former (*cover*). In this order of ideas, a system without symmetry is more easy to study, since there is a reduction of the branched manifolds in the *asymmetry* space, and this allow us to study a transformed asymmetric attractor, gather conclusions about its dynamics, and then make the inverse transformation without loss of information. For a more extense discussion about lost topological information in transformed spaces see [3].

Recently, the pattern formation in systems of chaotic coupled attractors has become of main importance [4, 5, 6]. However, in general, is difficult to anticipate that chaotic units can construct patterns with ordered space structures, mainly due to the sensitivity of chaotic trajectories to their initial conditions and the unpredictability of long term evolution of the orbits. However the possibility to study a simpler asymmetric image system should simplify, at least, the computational effort. In this work, we show how lattices of locally similar dynamical systems, although topologically inequivalent, can exhibit different pattern formations, and that is possible to perform global bifurcations by means of a parametric change in the coupling strength. This bifurcations are not the same for different topologies of the unit cells. We also investigate how this discrete equivariant transformation affect the dynamical behavior of 1-dimensional arrays of chaotic oscillators.

## 2. Model System

In order to study global differences between a cover and its image, we choose two different configurations, one and two dimensional lattices, and explore their spatiotemporal behavior with fixed control parameters.

### 2.1. Two-dimensional lattices

We take a two-dimensional lattice of coupled Lorenz oscillators, with fixed boundary conditions  $x_{0,j} = x_{N,j}$  and  $x_{i,0} = x_{i,N}$  for each variable:

$$\begin{aligned} \dot{x}_{ij} &= -\sigma(x_{ij} - y_{ij}) + \epsilon \nabla^2 x; \\ \dot{y}_{ij} &= -x_{ij} z_{ij} + r x_{ij} - y_{ij} + \epsilon \nabla^2 y; \\ \dot{z}_{ij} &= x_{ij} y_{ij} - \beta z_{ij} + \epsilon \nabla^2 z, \end{aligned} \quad (1)$$

where

$$\nabla^2 w = w_{i+1,j} + w_{i-1,j} + w_{i,j+1} + w_{i,j-1} - 4w_{i,j}, \quad (2)$$

is a two-dimensional diffusive coupling, and  $i, j = 1, 2, \dots, N$ , with  $N$  the dimension of the  $N \times N$  lattice. The parameter settings are:  $\sigma = 10$ ,  $\beta = 8/3$  and  $r = 27$ , so that each single independent Lorenz oscillator is chaotic, and are left unchanged for the following. The spatial orderings of the system are obtained varying homogeneously the control parameter  $\epsilon$ , and in order to study this transition, we use the following error functions [7]:

$$\begin{aligned} E_{min} &= \min(E_1(m), E_2(m)); \\ E_{max} &= \max(E_1(m), E_2(m)), \end{aligned} \quad (3)$$

where

$$\begin{aligned} E_1(m) &= \frac{1}{N} \sum_i^N \left\| x\left(i, \frac{N}{2} + m\right) - x\left(i, \frac{N}{2} - m\right) \right\|; \\ E_2(m) &= \frac{1}{N} \sum_j^N \left\| x\left(\frac{N}{2} + m, j\right) - x\left(\frac{N}{2} - m, j\right) \right\|, \end{aligned} \quad (4)$$

and  $m$  is a spatial delay, and its value is chosen in the range  $0 < m < N/2$ . We use the notation  $x(i, j) = x_{ij}$  writing eq. (3) for the sake of simplicity. With this definitions, we can identify three types of behavior: *i*)  $E_{min} > 0$ ,  $E_{max} > 0$  there is no pattern in the array; *ii*)  $E_{min} = 0$ ,  $E_{max} > 0$  there is a mirror symmetry respect to one of the axis:  $i = N/2$ ,  $j = N/2$  or the diagonal axis; *iii*)  $E_{min} = 0$ ,  $E_{max} = 0$ , the symmetry is central, *i.e.*, there is a mirror symmetry for each axis.

Using the fact that eq. (1) is equivariant under  $z$ -axis rotations, we have the diffeomorphism

$$\begin{aligned} u &= x^2 - y^2; \\ v &= 2xy; \\ w &= z, \end{aligned} \quad (5)$$

and the mapping of the system eq. (1) into its locally-equivariant *image* system [8] is

$$\begin{aligned} \dot{u}_{ij} &= (-\sigma - 1)u_{ij} + (\sigma - r)v_{ij} + v_{ij}w_{ij} + (1 - \sigma)\rho + \epsilon \nabla^2 u; \\ \dot{v}_{ij} &= (r - \sigma)u - (\sigma + 1)v_{ij} - u_{ij}w_{ij} + (r + \sigma)\rho - w_{ij}\rho + \epsilon \nabla^2 v; \\ \dot{w}_{ij} &= -\beta w_{ij} + 0.5v_{ij} + \epsilon \nabla^2 w, \end{aligned} \quad (6)$$

where  $\rho = \sqrt{u^2 + v^2}$ . The system eq. (6) is topologically similar to a Rössler oscillator. We have not made the transformation on the coupling, in order to assure that the two systems are coupled in the same diffusive way.

### 2.2. One-dimensional lattices

Using both eq. (1) and eq. (6), we built one-dimensional lattices, with fixed and free borders, i.e., chain and ring configurations. The diffusive terms used are

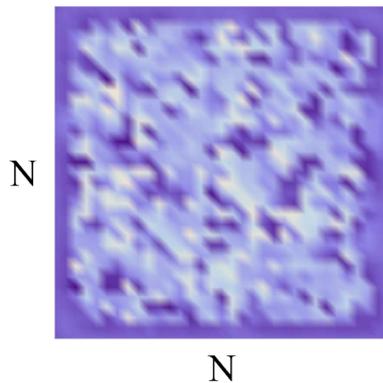
$$\nabla w = w_{i+1} + w_{i-1} - 2w_i \quad \text{chain and in the bidirectional ring;} \quad (7)$$

$$\nabla w = w_{i-1} \quad \text{unidirectional ring,} \quad (8)$$

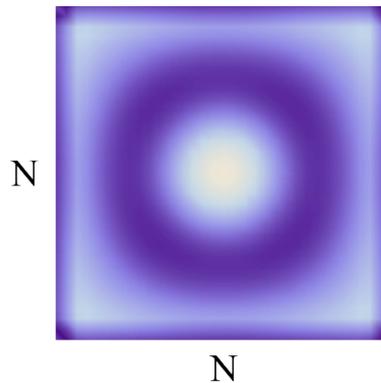
with  $i = 1, 2, \dots, N$ , where  $N$  is the number of cells in the lattice, and  $w$  stands for  $x, y$  or  $z$ , and  $u, v$  or  $w$ . The parameter settings are the same as in the two-dimensional case, so each single independent Lorenz oscillator and his *image* are chaotic. The coupling parameter was set as  $\epsilon = 2.5$ .

### 3. Results

In Figures (1) and (2) the two regimes of the system eq. (1) are plotted for times greater than the transient. The only dynamical transition present when  $\epsilon$  is changed, is from disorder to a central symmetry. By means of the error functions eq. (3), we observe that the dependency



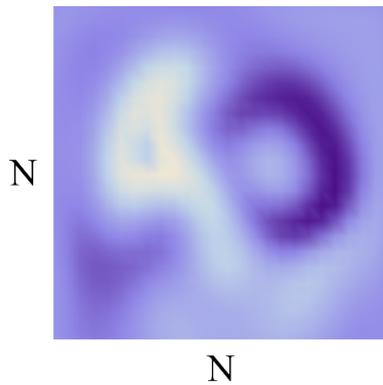
**Figure 1.** Space pattern of a Lorenz oscillator lattice with  $\epsilon = 0.5$  and  $N = 31$ . There is an evident lack of synchronization.



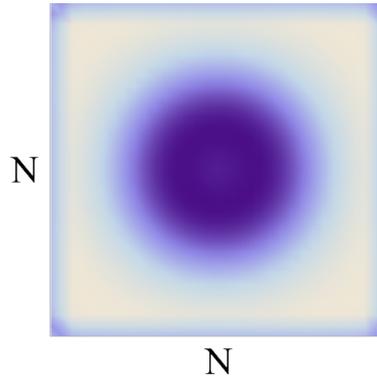
**Figure 2.** Space pattern of a Lorenz oscillator lattice with  $\epsilon = 0.9$  and  $N = 31$ . The lattice behaves like a breathing system.

of the parameter  $\epsilon_b$  where the bifurcation occur with the size of the array is roughly parabolic, different from the exponential dependence as was proposed in [7] for a configuration of two-lobe attractors. Although each central configuration at which the system arrives after the bifurcation depends on the initial conditions, the same qualitative behavior is observed for long enough times.

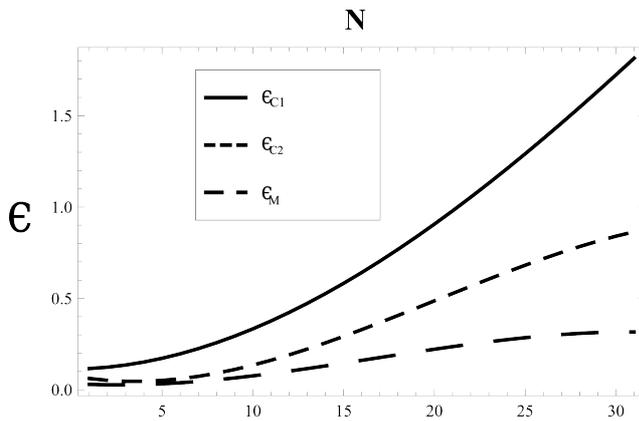
The results for the image system are plotted in Figures (3) and (4). We can see a very different behavior: in this system, made up of locally equivalent units of eq. (1), the array present two bifurcations: one that goes from disorder to a mirror symmetry, and from there to a central symmetry. We found that the most common mirror symmetry present in this array is about the diagonal axis, so a redefinition of the error functions was done in order to quantify it. This behavior is observed for systems without discrete symmetries, like Rössler oscillator, and



**Figure 3.** Space pattern of a Lorenz oscillator lattice with  $\epsilon = 0.4$  and  $N = 31$ . The lattice behaves like a breathing system with two lobes



**Figure 4.** Space pattern of a Lorenz oscillator lattice with  $\epsilon = 2.3$  and  $N = 31$ . The lattice behaves like a breathing system.



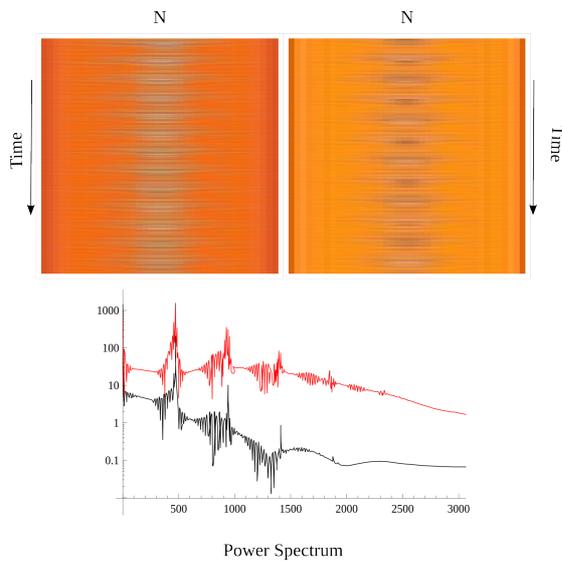
**Figure 5.** Variation of the control parameter  $\epsilon$  with the lattice dimension. Solid line shows the random-to-central symmetry bifurcation in eq. (1); the other two show the bifurcation points in the transformed system eq. (6): dashed line shows the mirror to central bifurcation parameter  $\epsilon_M$  and the long dashed line shows the *none to mirror* bifurcation parameter  $\epsilon_{C2}$ .

the change of the control parameter with the dimension of the lattice is almost linear for both bifurcation points; the change of  $\epsilon$  for the two systems considered are shown in the Figure (5).

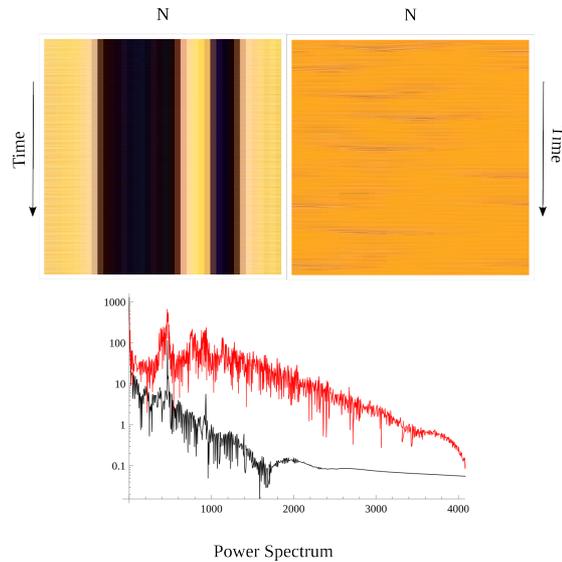
Is important to note from this results that with the tools available to study spatiotemporal chaos, is impossible to claim for the origin of the double-bifurcated system: an observer can not say if the array is made with Rössler oscillators, or with Lorenz *images* oscillators. This means that there are unrecoverable information lost in the transformation, and that in coupled 2D lattices of oscillators is more important the topology (one or two lobes) than the local equivariance.

In Figures (6),(7) and (8), we plot the spatiotemporal dynamics of one-dimensional arrays of diffusively coupled Lorenz attractors and their *images*. In order to differentiate global behaviors we used their power spectrum, because patterns are most difficult to identify in this case. In the three different configurations we can see similar global dynamics: although in each case the way in which the unit cells synchronize are different, between a system and its image there are more complex correlations. This correlations are well described because the main frequencies in both (cover and image) configurations are the same in the first two lattices couplings (fixed borders, using eq. (8) and free borders, using eq. (7)). We saw cluster synchronization, correlated states and total synchronization in Figures (6),(7) and (8) respectively, but no spatio-temporal chaos.

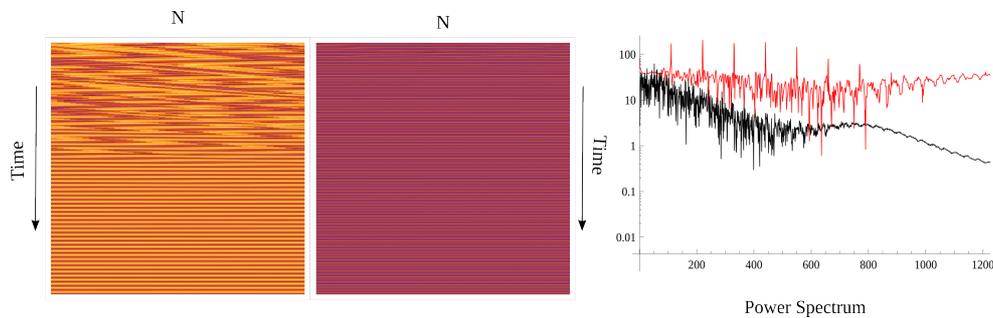
The way in which the borders of the arrays are set, fixed the spatiotemporal dynamic, in both the cover and the image system. The fact that the phase space state of each cell is not chaotic, tell



**Figure 6.** Evolution of  $N = 40$  attractors coupled in a diffusive way, with fixed borders, using eq. (8). Upper left: Lorenz lattice; upper right: Lorenz Image lattice; bottom: Power Spectrum of (red) Lorenz and (black) its image.



**Figure 7.** Evolution of  $N = 40$  attractors coupled in a diffusive way, with variable borders, using eq. (7). Upper left: Lorenz lattice; upper right: Lorenz Image lattice; bottom: Power Spectrum of (red) Lorenz and (black) its image.



**Figure 8.** Evolution of  $N = 40$  attractors coupled in a diffusive way, with variable borders, using eq. (8). Left: Lorenz lattice; center: Lorenz Image lattice; right: Power Spectrum of (red) Lorenz and (black) its image.

us that is the local nature which determine the global dynamic, and hence is the equivalent local nature between the cover and its image which makes the spatiotemporal dynamic to behave in a similar fashion, i.e., to oscillate around the same fundamental frequencies. However is important to note that when both system shows total synchronization, as in Figure (8), their frequencies are not correlated.

#### 4. Conclusion

Although identical dynamic information can be gathered from a cover system and his image (Lyapunov exponent, fractal dimensions), their behavior in phase space is topologically different. The system without symmetry is more easy to study, since there is a reduction of the manifolds in the *asymmetry* space. However we observed that the information that can be recovered from

the mapped lattice is different in an essential way from the former. More important is the fact that although the phase space dynamics of a individual cell image is *simpler*, because in the case of a Lorenz oscillator it is a one-lobe attractor, the dynamic of *images* lattice is more complicated: it seems to show globally a pattern that recalls the untransformed individual cell phase space dynamics, a *two-lobe* global attractor. We found that for large enough values of  $\epsilon$ , the systems reach a stable state with no chaotic unit-cell behavior as expected. We also show numerically that the dependence of the control parameter with the dimension of an array of one-lobe coupled systems is almost linear and that of two-lobe coupled systems is parabolic, in clear difference with the results found in [7], making possible to study finite but high dimensional lattices without high computational efforts.

In one-dimensional lattices the way in which the borders of the arrays are set, fixed the spatiotemporal dynamics: in the cases shown by Figures (6) and (7) there is the same fundamental frequency, as it is shown by the Power Spectrum. When the chaotic nature of each cell is lost, the shape (topology) of the attractors is also lost. This suggest that is the local nature which determine the global dynamic, and hence is the equivariance between the cover and its image which makes the spatiotemporal dynamic to behave in a similar fashion, i.e., to oscillate around the same fundamental frequencies.

This imply that although in a 2-D array is not possible to relate an image to its cover based on the global lattice behavior, in a 1D array it is possible, even when there are not chaotic information (Lyapunov exponent, fractal dimensions, etc.), given that they will oscillate with the same frequencies. The fact that when both systems (image and cover) shows total synchronization their frequencies are not related, suggests that there is a global mixing destroying all the information given to the system by the way in which the cells are coupled, or by the way in which the borders are fixed.

Although more work on theoretical differentiation of lattice dynamics is needed, a small insight has been gained through this work.

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