Algebras and Representation Theory
https://doi.org/10.1007/s10468-025-10327-8

RESEARCH

A Partial Classification of Simple Regular Representations of Bimodules Type (2, 2) Over the Field of Laurent Series

Hernán Giraldo[1] · David Reynoso-Mercado[1] · Pedro Rizzo[1]

Received: 20 November 2023 / Accepted: 2 March 2025

Abstract
In this paper, we use Galois descent techniques to find suitable representatives of the regular simple representations of the species of type $(2, 2)$ over $k_n := k[\varepsilon^{1/n}]$, where n is a positive integer and $k := \mathbb{C}(\!(\varepsilon)\!)$ is the field of Laurent series over the complexes. These regular representations are essential for the definition of canonical algebras. Our work is inspired by the work done for species of type $(1, 4)$ on k in Geiss and Reynoso-Mercado (Bol. Soc. Mat. Mex. **30**(3):87, 2024). We presents all the regular simple representations on the n-crown quiver, and from these, we establish a partial classification of regular simple representations of bimodules type $(2, 2)$.

Keywords Galois descent · Species · Regular simple representations · Canonical algebras · n-crown quiver

Mathematics Subject Classification (2010) 16G20 · 16S35

1 Introduction

In [5], Geiss et al. began investigating the representation theory of a new class of quiver algebras associated with symmetrizable generalized Cartan matrices. Their goal was to lay the foundation for generalizing many of the connections between path algebras, preprojective algebras, Lie algebras, and cluster algebras from the symmetric to the symmetrizable case. One of their specific goals was to obtain new geometric constructions of the positive part of a symmetrizable Kac-Moody algebra in terms of varieties of representations of these quiver algebras. For example, they introduced a new class of $\mathbb{F}$-algebras, denoted by $H_{\mathbb{F}}(C,\ D,\ \Omega)$,

Presented by: Karin Baur

✉ David Reynoso-Mercado
david.reynoso@udea.edu.co

Hernán Giraldo
hernan.giraldo@udea.edu.co

Pedro Rizzo
pedro.hernandez@udea.edu.co

[1] Álgebra, Teoría de Números y Aplicaciones: ERM (ALTENUA) y Álgebra UdeA, Instituto de Matemáticas, Facultad de Ciencias Exactas y Naturales, Universidad de Antioquia UdeA, Calle 70 No. 52-21, Medellín, Colombia

Published online: 26 March 2025

on the field $\mathbb{F}$. These algebras allow one to model species representations as locally free modules over them, especially when $\mathbb{F}$ is algebraically closed. Since the algebra $H_{\mathbb{F}}(C, D, \Omega)$ is defined via quivers with relations, one can study its module varieties over any field $\mathbb{F}$. In the particular case that $\mathbb{F} = \mathbb{C}$, they generalize Lusztig's nilpotent varieties from the symmetric to the symmetrizable case.

In this article, we study normal forms for the regular simple representations of species of type $(2, 2)$, also denoted as $\tilde{A}_{11}$ in the notation introduced in [4]. Our goal is to obtain a partial classification of these representations. From a broader perspective, our partial classification can be seen as a new step forward in the study of species representations of the $\mathbb{F}$-algebra $H_{\mathbb{F}}(C, D, \Omega)$.

Our classification technique, following the ideas in [7], consists of constructing a canonical algebra, as outlined by Ringel in [12], associated to a bimodule of type $(2, 2)$. Specifically, we focus on bimodules with product dimension equal to 4 over division algebras with center $\mathbb{C}(\!(\varepsilon)\!)$. There are two distinct families of such bimodules, of type $(1, 4)$ and of type $(2, 2)$, as discussed in Section 2.1. The former case has been extensively examined in [7]. In this paper, we shift our focus to the latter type and use a special case of representations of the $n-$crown quiver (see Section 2.4) to obtain a classification that differs substantially from that in [7].

Inspired by [6], we first need to find explicit normal forms for canonical algebras of type $(2, 2)$ over the field of complex Laurent series, $\mathbb{C}(\!(\varepsilon)\!)$. To this end, we must first find normal forms of the regular simple representations of species of type $(2, 2)$. Since $\mathbb{C}(\!(\varepsilon)\!)$ is a quasi-finite field, the technique of Galois descent is an efficient tool for achieving our objectives in this context.

This paper is organized as follows. In Section 2, we recall several concepts and fix some notations that we will use throughout the paper. More precisely, we introduce a brief study of root systems for quivers in Section 2.2. We present a quick tour of string and band modules on string algebras in Subsection 2.3. In Section 2.4, we focus on the classification of indecomposable modules for a path gentle (and therefore string) algebra constructed on the *n-crown quiver* $\mathcal{Q}_n$.

In Section 3, we develop the main results of this paper. In fact, in Section 3.1 we will establish a decisive isomorphism as k_d-algebras between the tensor algebra $k_d \otimes_k \Lambda_n$ and the path algebra $k_d\mathcal{Q}_n$, where Λ_n is the *Euclidean species* $\Lambda_n := \begin{bmatrix} k_n & k_n \oplus {}^{\sigma_n}k_n \\ 0 & k_n \end{bmatrix}$. The purpose of this paper is to classify the Λ_n-modules of type $(2, 2)$ over k_n. We partially resolve this problem from of the next subsections. Our strategy is as follows. We show that, in Section 3.1, the path algebra $k_d\mathcal{Q}_n$ admits a k_n-linear automorphism of order d, which is compatible with the action of σ_d on $k_d \otimes \Lambda_n$. We denote this automorphism by $\gamma_{d,n}$. In Section 3.2, we describe how $\gamma_{d,n}$ acts on the representations of the n-crown quiver $\mathcal{Q}_n$. Using Theorem 3.2, we show how a regular simple representation of Λ_n can be regarded as a $\mathcal{Q}_n$-representation. In Section 3.3, we prove that there is a bijection between certain isoclasses of regular simple Λ_n-modules and certain special $\langle\sigma_d\rangle$-orbits of regular simple $k_d\mathcal{Q}_n$-modules (see Theorem 3.9). In Section 3.4, we prove that any regular simple representation is isomorphic to a direct sum of irreducible, invariant modules of the form M_a (for generic $a \in k_m$), $M_{(1)}$ or $N_{(1)}$ (see Theorem 3.18). Finally, we present a total classification of regular simple non-homogeneous representations on Λ_n and a partial classification of regular simple homogeneous representations on Λ_n, for any $n \geq 2$. The classes of regular simple non-homogeneous and homogeneous representations correspond to string and band (indecomposable) modules of the path algebra $k_d\mathcal{Q}_n$, respectively. In Section 3.5 we conclude our classification with the study of the case $n = 1$.

2 Preliminaries

Throughout our work, $k := \mathbb{C}(\!(\varepsilon)\!)$ denotes the field of Laurent series over the complex numbers, which is a quasi-finite because $\mathbb{C}$ is an algebraically closed field of characteristic 0 (see [14, Chapter XIII, §2]). Thus, each division algebra over k is isomorphic to $k_n := \mathbb{C}(\!(\varepsilon^{\frac{1}{n}})\!)$, for some $n \in \{1, 2, 3, \cdots\} = \mathbb{Z}_{>0}$. In particular, every finite-dimensional division algebra is commutative, and the Galois group of the extension $k_n|k$ is the cyclic group $C_n = \langle \sigma_n \rangle$ of order n. Here, σ_n acts k-linearly on k_n via $\sigma_n(\varepsilon^{\frac{j}{n}}) = \zeta_n^j \varepsilon^{\frac{j}{n}}$, where $\zeta_n := e^{\frac{2\pi i}{n}}$.

For m a divisor of n, we identify k_m with the subfield of k_n generated by $(\varepsilon^{\frac{1}{n}})^{\frac{n}{m}}$, thus $\sigma_n|_{k_m} = \sigma_m$. We say that $x \in k_n$ is *generic* if $|C_n \cdot x| = n$. This is equivalent to prove that $\Pi_{j=0}^{n-1}(y - \sigma_n^j(x)) \in k[y]$, which is an irreducible polynomial. For $0 \neq x = \sum_{j \in \mathbb{Z}} x_j \varepsilon^{\frac{j}{n}} \in k_n$ we define $\operatorname{codeg}_n(x) := \min\{j \in \mathbb{Z} | x_j \neq 0\}$ and $\operatorname{codeg}(0) = \infty$.

2.1 Representations of the Species of Type (2, 2) Over Finite-Dimensional Extensions of $\mathbb{C}(\!(\varepsilon)\!)$

In this subsection, we consider bimodules for which the field k, acts centrally. The category of these k_n-k_m-bimodules is equivalent to the category of left $k_n \otimes_k k_m$-modules, where n and m are positive integers.

Let X be a k_n-k_m-bimodule, and let σ_n be an automorphism of k_n. We define the bimodule ${}^{\sigma_n}X$, where the left multiplication is defined as $y * x := \sigma_n(y)x$ for any $x \in X$, $y \in k_n$, and the right multiplication is the usual multiplication as a k_m-module. Similarly, we can define X^{σ_m} as a k_n-k_m-bimodule.

As we will show in Proposition 3.1, $k_n \otimes_k k_m$ is isomorphic to $k_{\mathrm{lcm}(n,m)} \times \cdots \times k_{\mathrm{lcm}(n,m)}$, $\gcd(n, m)$−times, as k-algebras. Thus, the category of k_n-k_m-bimodules is semisimple with $\gcd(n, m)$-isoclasses of simple objects.

Let $n, m \in \mathbb{Z}_{>0}$ and X be a k_n-k_m-bimodule such that $\dim({}_{k_n}X)\dim(X_{k_m}) = 4$. Then it is easy to see that only two cases are possible:

a) $k_n = k_4$, $k_m = k$ an $X = k_4$.
b) $k_n = k_m$ and $X = k_n \oplus {}^{\sigma_n}k_n$, for some $n \in \mathbb{Z}_{>0}$.

The first case has already been addressed in [7], so we will focus on the second case.

Recall that a representation M of $k_n \oplus {}^{\sigma_n}k_n$ as a k_n-k_n-bimodule takes the form $(k_n^m, k_n^{m'}, \varphi_M)$, where $\varphi_M : (k_n \oplus {}^{\sigma_n}k_n) \otimes_k k_n^{m'} \to k_n^m$ is a k_n-linear morphism, and $m, m' \in \mathbb{Z}_{>0}$, as defined in [12]. Let N be the matrix of the mapping φ_M.

By definition, the objects in the category $\operatorname{Rep}_{k_n}(2, 2)$ are $m \times 2m'$ matrices with entries in k_n. For $N \in \operatorname{Mat}_{m \times 2m'}(k_n)$ and $N' \in \operatorname{Mat}_{l \times 2l'}(k_n)$, the morphisms from N to N' are given by the k_n-vector space:

$$\operatorname{Hom}_{(2,2)}(N, N') := \{(f_2, f_1) \in \operatorname{Mat}_{m \times l}(k_n) \times \operatorname{Mat}_{2l' \times 2m'}(k) | f_2 N = N' f_1\}.$$

It is straightforward to see that $\operatorname{Rep}_{k_n}(2, 2)$ is an abelian k-linear category. It is naturally equivalent to Λ_n-mod for the Euclidean species $\Lambda_n := \begin{bmatrix} k_n & k_n \oplus {}^{\sigma_n}k_n \\ 0 & k_n \end{bmatrix}$ of type (2, 2) over k_n.

Let M be a representation and $N \in \operatorname{Mat}_{m \times 2m'}(k_n)$ be the matrix of the morphism

$$\varphi_M : (k_n \oplus {}^{\sigma_n}k_n) \otimes_k k_n^{m'} \to k_n^m.$$

We denote by $\underline{\dim} N = (m', m) \in \mathbb{N}_0^2$. Following Dlab-Ringel [4, §3], $N \in \mathrm{Rep}_k(2, 2)$ is called *regular simple* if $\underline{\dim} N = (m, m)$ for some $m \in \mathbb{Z}_{>0}$ and $\mathrm{End}_{(2,2)}(N)$ is a division algebra. Since k_n is quasi-finite, the previous condition is equivalent to $\mathrm{End}_{(2,2)}(N) \cong k_l$ for some l.

2.2 Quivers and Roots

Let Q be a finite and connected quiver. As usual, we denote by Q_0 (resp. by Q_1) the set of vertices (resp. the set of arrows) of Q. Also, $h(a)$ (resp. $t(a)$) denotes the vertex of Q_0 where the arrow a starts (resp. ends).

A *subquiver* of a quiver $Q = (Q_0, Q_1, h, t)$ is a quiver $Q' = (Q'_0, Q'_1, h', t')$ where $Q'_0 \subset Q_0$, $Q'_1 \subset Q_1$, $h' = h|_{Q'_1}$ and $s' = s|_{Q'_1}$. A subquiver Q' of Q is *full* if $Q'_1 = \{a \in Q_1 | h(a), t(a) \in Q'_0\}$.

Associated to every quiver Q we have a *symmetric generalized Cartan matrix*, $A_Q = (a_{ij})$, whose entries are defined as follows:

$$a_{ij} = \begin{cases} 2 & \textit{if } i = j; \\ -\#\{\textit{edges between } i \textit{ and } j\} & \textit{if } i \neq j. \end{cases} \tag{1}$$

Note that, A_Q not depend of the orientation on Q.

The *root lattice* of Q is defined by the free abelian group $\mathbb{Z}^{Q_0}$. We identify each trivial path e_i with the $i-$th standard basis vector of $\mathbb{Z}^{Q_0}$, for each $i \in Q_0$. This group admits a partially order "$\geq$" induced, for $a = \sum_{i \in Q_0} a_i e_i \in \mathbb{Z}^{Q_0}$, by the relation:

$$a = \sum_{i \in Q_0} a_i e_i \geq \mathbf{0} \text{ if and only if } a_i \geq 0 \text{ for all } i \in Q_0,$$

where $\mathbf{0}$ is the zero object in $\mathbb{Z}^{Q_0}$. In consequence, $a = \sum_{i \in Q_0} a_i e_i \geq b = \sum_{i \in Q_0} b_i e_i$ if and only if $a - b \geq \mathbf{0}$.

From A_Q we can define a *symmetric bilinear form* $(-, -) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$, which satisfies $(e_i, e_j) := a_{ij}$, for all i, j. This yields a *reflection* $r_i : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q_0}$ *on* $\mathbb{Z}^{Q_0}$, which is defined by $a \mapsto a - (a, e_i)e_i$, for each $i \in Q_0$. We define the *Weyl Group* of Q, denoted by $\mathcal{W}$, as the subgroup of $\mathrm{Aut}(\mathbb{Z}^{Q_0})$ generated by the all reflections r_i. We now introduce the *set of (positive) imaginary roots for* Q, denoted by $\Delta^+_{\mathrm{im}} := \bigcup_{w \in \mathcal{W}} w(F)$, as the union of the sets of images $w(F)$, where w ranges over all elements of the Weyl group $\mathcal{W}$. Here the set F, which is called *the fundamental region*, is defined by

$$F := \{a \in \mathbb{Z}^{Q_0} \mid a \neq 0,\ (a, e_i) \leq 0 \text{ for all } i \in Q_0, \text{ and } \operatorname{supp} a \text{ is connected}\},$$

where $\operatorname{supp} a$ represents the full subquiver of Q with vertex set $\{i \in Q_0 \mid a_i \neq 0\}$. The quiver $\operatorname{supp} a$ is called *the support of* $a \in \mathbb{Z}^{Q_0}$.

Due to the above comments, we denote *the minimal positive imaginary root* by δ. If $M = (V_i, f_\rho)$ is an indecomposable representation of the quiver Q over field $\mathbb{F}$, then the *defect* of M is defined by the integer

$$\operatorname{defect}(M) := \langle \delta, \dim M \rangle,$$

where "$\langle \cdot, \cdot \rangle$" is *the Euler form of* Q, which is defined by

$$\langle a, b \rangle := \sum_{i \in Q_0} a_i b_i - \sum_{\rho : i \to j} a_i b_j,$$

and $\dim M := \sum_{i \in Q_0} (\dim V_i) e_i$.

Let Q be a connected quiver without oriented cycles. According to the theory above, we have the following important criteria for the isomorphism classes of indecomposable representations M of Q:

a) M is preprojective if and only if $\tau^r M = 0$ for some sufficiently large integer r if and only if $\mathrm{defect}(M) < 0$.
b) M is preinjective if and only if $\tau^{-r} M = 0$ for some sufficiently large integer r if and only if $\mathrm{defect}(M) > 0$.
c) M is regular if and only if $\tau^{-r}\tau^r M = M$ for all integers r if and only if $\mathrm{defect}(M) = 0$.

Here, τ represents the *Auslander-Reiten translate* (see [1]).

2.3 String Algebras

Let $\mathbb{F}Q/I$ be a bound quiver algebra. We say that $\mathbb{F}Q/I$ is a *string algebra* (see [3]) if it satisfies the following three conditions:

(S1) Any vertex of Q is starting or ending point of at most two arrows.
(S2) Given an arrow $\beta \in Q_1$, there exists at most one arrow $\rho \in Q_1$ such that either $\beta\rho \notin I$ or $\rho\beta \notin I$.
(S3) The ideal I is generated by null relations.

Let $\mathbb{F}Q/I$ be a string algebra and $a \in Q_1$ be an arrow. The *formal inverse of a* is the arrow a^{-1} such that $t(a^{-1}) = h(a)$ and $h(a^{-1}) = t(a)$. A *string* S is a sequence $S = s_1 \cdots s_n$, where $s_i \in Q_1$ or $s_i^{-1} \in Q_1$, $t(s_i) = h(s_{i-1})$ and $s_i \neq s_{i-1}^{-1}$ for all $1 < i \leq n$. Let $\mathcal{S}$ be the set of all strings. We define an equivalence relation $S \sim_s S'$ if and only if $S = S'$ or $S' = S^{-1}$. We denote the set of representatives of the equivalence classes of $\mathcal{S}/\sim_s$ as $\underline{\mathcal{S}}$. Here S^{-1} corresponds to the string $S^{-1} = s_n^{-1} \cdots s_1^{-1}$.

Let $\mathbb{F}Q/I$ be a string algebra and $S = s_1 s_2 \cdots s_n \in \underline{\mathcal{S}}$ be a string. We define a map $u : \{0, 1, \cdots, n\} \to Q_0$ by $u(0) = t(s_1)$ and $u(i) = h(s_i)$, for all $1 \leq i \leq n$. We will define a representation $M(S)$ of the quiver Q bounded by the relations I associated to the string S. Indeed, for each vertex $v \in Q_0$, let $I_v = u^{-1}(v) \subset \{0, 1, \cdots, n\}$ be the set of indexes. To each vertex v we associate the vector space $M(S)_v = \mathbb{F}^{\#I_v}$, where $\#I_v$ denotes the cardinality of the set I_v. Now, from the choice of a basis $\{z_0, z_1, \cdots, z_n\}$ of $\mathbb{F}^{n+1}$, we take the corresponding elements indexed by I_v to obtain a basis of the vector space $\mathbb{F}^{\#I_v}$, for each $v \in Q_0$. We define, for each arrow $a \in Q_1$ such that $M(S)_{t(a)} \neq 0$ and $M(S)_{h(a)} \neq 0$, a morphism $f_a : M(S)_{t(a)} \to M(S)_{h(a)}$ by

$$f_a(z_i) = \begin{cases} z_{i+1} & \text{if } s_{i+1} = a, \\ z_{i-1} & \text{if } s_i = a^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $M(S) = (M(S)_v, f_a)_{v \in Q_0, a \in Q_1}$ is a representation of the quiver Q bounded by the relations I and $M(S) \cong M(S')$ if and only if $S \sim_s S'$. The class by isomorphism of this modules are called *string modules*.

Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ consisting of all non-trivial strings S such that S^n is defined for all $n \in \mathbb{N}$, and S is not a power of a substring. We define an equivalence relation on $\mathcal{S}'$ as follows: $S \sim_r S'$ if and only if S' is a cyclic permutation of S. We denote the set of representatives of the equivalence classes of $\mathcal{S}'/\sim_r$ as $\underline{\mathcal{S}'}$.

Let Z be a vector space and $\varphi : Z \to Z$ be an automorphism. It is possible to prove that (see [3]) the pair (Z, φ) induce a $\mathbb{F}[T, T^{-1}]$−module structure over Z. We want to define a module $M(S,\ \varphi)$ associated to each string S in this new class. As before, like for string modules, for each vertex $v \in Q_0$ we consider $I_v = u^{-1}(v)$, $I'_v = I_v \cap \{0,\ 1, \cdots,\ n-1\}$ and $s_n : u(n-1) \to u(0)$.

Let $\bigoplus_{i \in I'_v} Z_i$ be the $\mathbb{F}$-vector space, where $Z_i = Z$ for all $i \in I'_v$. Now, for each arrow $a \in Q_1$, we define $f_a : (\bigoplus Z_i)_{i \in I'_{t(a)}} \to (\bigoplus Z_j)_{j \in I'_{h(a)}}$, whose domain and codomain are not null spaces. Thus, we denote by f_i, for all $1 \leq i \leq n$, the following morphisms:

$$f_i = \begin{cases} \mathrm{id} : Z_{i-1} \to Z_i & \text{if } s_i \in Q_1 \text{ and } i < n, \\ \mathrm{id} : Z_i \to Z_{i-1} & \text{if } s_i^{-1} \in Q_1 \text{ and } i < n, \\ \varphi : Z_{n-1} \to Z_0 & \text{if } s_i \in Q_1 \text{ and } i = n, \\ \varphi^{-1} : Z_i \to Z_{i-1} & \text{if } s_i^{-1} \in Q_1 \text{ and } i = n. \end{cases}$$

Once we have chosen an order for the direct sums $(\bigoplus Z_i)_{i \in I'_{t(a)}}$ and $(\bigoplus Z_j)_{j \in I'_{h(a)}}$ we can define a matrix

$$g_{rs} = \begin{cases} f_i \ \text{if } t(s_i) \text{ is the } r\text{-th component of } (\bigoplus Z_i)_{i \in I'_{t(a)}} \\ \quad \text{and } h(s_i) \text{ is the } s\text{-th component of } (\bigoplus Z_j)_{j \in I'_{h(a)}}, \\ 0 \ \ \text{otherwise.} \end{cases}$$

It is not difficult to verifies that $M(S, \varphi)$ is a representation of the quiver Q bounded by the relations I and $M(S, \varphi) \cong M(S', \varphi')$ if and only if (Z, φ) and (Z, φ') are isomorphic as $\mathbb{F}[T, T^{-1}]-$modules and $S \sim_r S'$. These modules are called *band modules*.

The main result related to the classification of indecomposable modules for string algebras is the following theorem of Butler and Ringel in [3].

Theorem 2.1 *Let $\mathbb{F}Q/I$ be a string algebra of finite dimension. The string modules $M(S)$ with $S \in \underline{S}$ and band modules $M(S', \varphi)$ with $S' \in \underline{S'}$ and $\varphi : Z \to Z$ be an automorphism on the vector space Z, provide a complete list of indecomposable $\mathbb{F}Q/I$-modules, up to isomorphism.*

2.4 About the *n*-Crown Quiver

If $\mathcal{Q} = (Q_0, Q_1, h, t)$ is a quiver, then we call a vertex a *source* if it is not head of any arrow and it is a *sink* if it is not the tail of any arrow. Let $\mathcal{Q}_n$ be a quiver whose set of vertices $Q_0 = U \cup W$ is such that $U = \{1,\ 2, \cdots,\ n\}$ corresponds to the set of sources and $W = \{1',\ 2', \cdots,\ n'\}$ corresponds to the set of sinks. The quiver $\mathcal{Q}_n$ is called a $n-$*crown* if $n \geq 2$ and:

(C1) There exists an arrow from i to j' if: $i > 1$ and $i \leq j' \leq i+1$; or $i = 1$ and $j' = 1'$; or $i = n$ and $j' = 1'$.

(C2) If $n = 2$, no path from 1 to $2'$ shares a vertex with any path starting at 2 and ending at $1'$.

For instance, the $2-$crown quiver $\mathcal{Q}_2$ is represented by

$$\begin{tikzcd} 1 \arrow[d] \arrow[dr] & 2 \arrow[dl] \arrow[d] \\ 1' & 2' \end{tikzcd}$$

For $n > 2$, the n-crown quiver $\mathcal{Q}_n$ corresponds to the Fig. 1.

If we consider the n-crown quiver $\mathcal{Q}_n$ and we denote by $\alpha_i : i \to i'$ and $\alpha_{i'} : i-1 \to i'$ the arrows connecting source and sinks, then, it is not difficult to verifies that, the strings with length j, where $1 \le j \le n$, whose starting vertex is $1'$ are defined by

$$s_{1',j} := \begin{cases} \alpha_1^{-1}\alpha_2'\alpha_2^{-1}\cdots\alpha_{\frac{j+1}{2}}^{-1} & \text{if } j \text{ is odd,} \\ \\ \alpha_1^{-1}\alpha_2'\alpha_2^{-1}\cdots\alpha'_{\frac{j+2}{2}} & \text{if } j \text{ is even.} \end{cases}$$

Similarly, it is not difficult to verifies that, the strings with length j, where $1 \le j \le n$, whose starting vertex is 1 are defined by

$$s_{1,j} := \begin{cases} \alpha_2'\alpha_2^{-1}\cdots\alpha'_{\frac{j+3}{2}} & \text{if } j \text{ is odd,} \\ \\ \alpha_2'\alpha_2^{-1}\cdots\alpha_{\frac{j+2}{2}}^{-1} & \text{if } j \text{ is even.} \end{cases}$$

Now, if we consider $\gamma_n : \{1, \cdots, n\} \to \{1, \cdots, n\}$ a permutation defined by $\gamma_n(1) = n$ and $\gamma_n(i) = i - 1$ for all $1 < i \le n$, then, for $1 \le i \le n$, we have that

$$\gamma_n^{n+1-i}(1) = \gamma_n^{n-i}(n) = n - (n-i) = i.$$

In consequence, the strings with length j, where $1 \le j \le n$, whose starting vertices are i' and i, respectively, are denoted by

$$s_{i',j} := \begin{cases} \alpha^{-1}_{\gamma_n^{n+1-i}(1)}\alpha'_{\gamma_n^{n+1-i}(2)}\cdots\alpha^{-1}_{\gamma_n^{n+1-i}(\frac{j+1}{2})} & \text{if } j \text{ is odd,} \\ \\ \alpha^{-1}_{\gamma_n^{n+1-i}(1)}\alpha'_{\gamma_n^{n+1-i}(2)}\cdots\alpha'_{\gamma_n^{n+1-i}(\frac{j+2}{2})} & \text{if } j \text{ is even,} \end{cases}$$

$$s_{i,j} := \begin{cases} \alpha'_{\gamma_n^{n+1-i}(2)}\alpha^{-1}_{\gamma_n^{n+1-i}(2)}\cdots\alpha'_{\gamma_n^{n+1-i}(\frac{j+3}{2})} & \text{if } j \text{ is odd,} \\ \\ \alpha'_{\gamma_n^{n+1-i}(2)}\alpha^{-1}_{\gamma_n^{n+1-i}(2)}\cdots\alpha^{-1}_{\gamma_n^{n+1-i}(\frac{j+2}{2})} & \text{if } j \text{ is even.} \end{cases}$$

We claim that each vertex appears at most once in a string of length j. Indeed, we have:

a) For j odd, $v = h(\alpha)$ or $v = t(\alpha)$ for some $\alpha \in s_{i',j}$ if and only if $v \in U_{i,\frac{j+1}{2}} \cup W_{i,\frac{j+1}{2}}$,
b) for j even, $v = h(\alpha)$ or $v = t(\alpha)$ for some $\alpha \in s_{i',j}$ if and only if $v \in U_{i,\frac{j}{2}} \cup W_{i,\frac{j+2}{2}}$,
c) for j odd, $v = h(\alpha)$ or $v = t(\alpha)$ for some $\alpha \in s_{i,j}$ if and only if $v \in U_{i,\frac{j+1}{2}} \cup W_{i,\frac{j+3}{2}} - \{i'\}$,
d) for j even, $v = h(\alpha)$ or $v = t(\alpha)$ for some $\alpha \in s_{i',j}$ if and only if $v \in U_{i,\frac{j+2}{2}} \cup W_{i,\frac{j+2}{2}} - \{i'\}$,

where the sets of sources and sinks are, respectively, $U_{i,l} := \left\{i, \gamma_n^{n+1-i}(2), \cdots, \gamma_n^{n+1-i}(l)\right\} \subseteq U$ and $W_{i,l} := \left\{i', \gamma_n^{n+1-i}(2)', \cdots, \gamma_n^{n+1-i}(l)'\right\} \subseteq W$, with $1 \le i, l \le n$.

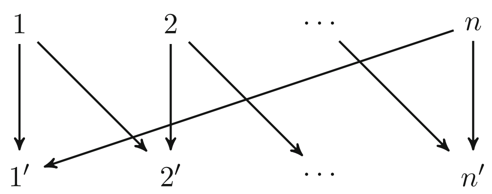

Fig. 1 The n-crown quiver $\mathcal{Q}_n$, for $n \ge 3$

Therefore, for each string $s_{i',j}$, the corresponding string module $M(s_{i',j}) = (M_v, f_a)_{v\in Q_0, a\in Q_1}$ is defined by:

$$M_v = \begin{cases} k_n \text{ if } j \text{ is odd and } v \in U_{i,\frac{j+1}{2}} \cup W_{i,\frac{j+1}{2}}; \\ k_n \text{ if } j \text{ is even and } v \in U_{i,\frac{j}{2}} \cup W_{i,\frac{j+2}{2}}; \\ 0 \ \text{ otherwise}; \end{cases}$$

$$f_a = \begin{cases} 1 \text{ if } a \in s_{i',j} \text{ or } a^{-1} \in s_{i',j}; \\ 0 \text{ otherwise}; \end{cases}$$

Analogously, for each string $s_{i,j}$, the corresponding string module $M(s_{i,j}) = (M_v, f_a)_{v\in Q_0, a\in Q_1}$ is defined by:

$$M_v = \begin{cases} k_n \text{ if } j \text{ is odd and } v \in U_{i,\frac{j+1}{2}} \cup W_{i,\frac{j+3}{2}} - \{i'\}; \\ k_n \text{ if } j \text{ is even and } v \in U_{i,\frac{j+2}{2}} \cup W_{i,\frac{j+2}{2}} - \{i'\}; \\ 0 \ \text{ otherwise}; \end{cases}$$

$$f_a = \begin{cases} 1 \text{ if } a \in s_{i,j} \text{ or } a^{-1} \in s_{i,j}; \\ 0 \text{ otherwise}. \end{cases}$$

Thus, under the notations in Section 2.2, we obtain that

$$\dim(M(s_{i',j})) = \begin{cases} \sum_{v\in U_{i,\frac{j+1}{2}}} e_v + \sum_{v\in W_{i,\frac{j+1}{2}}} e_v & \text{if } j \text{ is odd;} \\ \sum_{v\in U_{i,\frac{j}{2}}} e_v + \sum_{v\in W_{i,\frac{j+2}{2}}} e_v & \text{if } j \text{ is even;} \end{cases}$$

$$\dim(M(s_{i,j})) = \begin{cases} \sum_{v\in U_{i,\frac{j+1}{2}}} e_v + \sum_{v\in W_{i,\frac{j+3}{2}}-\{i'\}} e_v & \text{if } j \text{ is odd;} \\ \sum_{v\in U_{i,\frac{j+2}{2}}} e_v + \sum_{v\in W_{i,\frac{j+2}{2}}-\{i'\}} e_v & \text{if } j \text{is even.} \end{cases}$$

Hence, the Cartan matrix of $\mathcal{Q}_n$, indexed by $U \cup W$, is defined by:

$$a_{ij} = \begin{cases} 2 \ \text{ if } i = j, \\ -1 \ \text{ if } i \in U \text{ and j=i' or j=(i+1)'}, \\ -1 \ \text{ if } j \in U \text{ and i=j' or i=(j+1)'}, \\ 0 \ \text{ otherwise.} \end{cases}$$

In consequence, it is not difficult to prove that $\delta := \sum_{i\in U} e_i + \sum_{i'\in W} e_{i'}$ is a minimal (positive) imaginary root and, for $a := \sum_{i\in U} a_i e_i + \sum_{i\in W} a'_i e_i \in \mathbb{Z}^{Q_0}$, we have that:

$$\langle \delta, a\rangle = \sum_{i\in U} a_i + \sum_{i\in W} a'_i - \sum_{i\in U} (a'_i + a'_{i+1}). \tag{2}$$

Lemma 2.2 *Let S be a string of length $1 \le j \le n$ in the quiver $\mathcal{Q}_n$ and let $M(S)$ be the corresponding string module. Then,*

a) $\text{defect}(M(S)) = 0$ *if j is odd,*
b) $\text{defect}(M(S)) \neq 0$ *if j is even.*

Proof Let $S = s_1 \cdots s_j$ be a string of $\mathcal{Q}_n$ of length j, with $1 \le j \le n$. We denote by

$$U_S = \{v \in U \mid v = t(s_1) \text{ or } v = h(s_i) \ \text{ for some } \ 1 \le i \le j\}$$

the corresponding set of sources of S and by

$$W_S = \{v \in W \mid v = t(s_1) \text{ or } v = h(s_i) \ \text{ for some } 1 \le i \le j\}$$

the corresponding set of sinks of the string S. Thus,

$$\begin{aligned}\operatorname{defect}(M(S)) = \langle \delta, \dim(M(S)) \rangle &= \sum_{i \in U_S} 1 + \sum_{i \in W_S} 1 - \sum_{i \in U_S} 2 \\ &= |U_S| + |W_S| - 2|U_S| = |W_S| - |U_S|.\end{aligned}$$

In consequence, if either $S = s_{i',j}$ or $S = s_{i,j}$, then it is easy to check that, $|W_S| = |U_S|$, if j is odd, and $|W_S| \neq |U_S|$, if j is even. Therefore, $\operatorname{defect}(M(S)) = 0$ if j is odd, and $\operatorname{defect}(M(S)) \neq 0$ if j is even, which completes the proof of lemma. □

For the case of bands, it is easy to check that there exists a unique band representant in $\underline{S'}$ in the quiver $\mathcal{Q}_n$. In fact, it is sufficient consider $S' = (\alpha_1')^{-1}\alpha_n(\alpha_n')^{-1} \cdots (\alpha_2')^{-1}\alpha_1'$. In this case, if V is a k_n vector space and φ is an automorphism of V, the corresponding band module is defined by $M(S', \varphi) = (M_v, f_a)_{v \in Q_0, a \in Q_1}$, where $M_v = V$ for each $v \in Q_0$, and the maps

$$f_a = \begin{cases} \varphi & \text{if } a = \alpha_1, \\ \mathrm{id}_V & \text{otherwise,} \end{cases}$$

$M(S', \varphi)$ is represented in the Fig. 2.

Lemma 2.3 *Let S' be the unique band, up to isomorphism, in the quiver $\mathcal{Q}_n$ and let $M(S', \varphi)$ be the corresponding band module. Then,* $\operatorname{defect}(M(S', \varphi)) = 0$.

Proof In this case, we have that $\dim(M(S', \varphi)) = \sum_{v \in Q_0} \dim(V) e_v = \dim(V)\delta$. Thus,

$$\operatorname{defect}(M(S', \varphi)) = \langle \delta, \dim(M(S', \varphi)) \rangle = \langle \delta, \dim(V)\delta \rangle = \dim(V)\langle \delta, \delta \rangle = 0.$$

□

We will see in 3.1 that $k_d \otimes_{k_n} \Lambda_n \cong k_d \mathcal{Q}_n$ as k_d-algebras, for $n \in \mathbb{Z}_{>0}$ and d be a multiple of n. The path algebra $k_d \mathcal{Q}_n$ admits a k_n-linear automorphism $\gamma_{d,n}$ of order d, which is compatible with the action of σ_d on $k_d \otimes \Lambda_n$. Indeed, $\gamma_{d,n} : k_d \mathcal{Q}_n \to k_d \mathcal{Q}_n$ is the authomorphism of k_d-algebras defined by $k\rho \mapsto \sigma_d(k)\gamma_n(\rho)$, where γ_n is the automorphism given by the "reverse shift" shown in Fig. 3.

3 Representations of Twisted Quivers and Galois Descent

3.1 Quivers with Automorphisms

Our primary objective in this subsection is to establish the isomorphism of the k_d-algebras $k_d \otimes_k \Lambda_n$ and the path algebra $k_d \mathcal{Q}_n$. For this purpose, let $k_n = \mathbb{C}((\varepsilon^{\frac{1}{n}}))$ denote the degree n extension of the field k of Laurent series over the complex numbers. If we consider k_n and

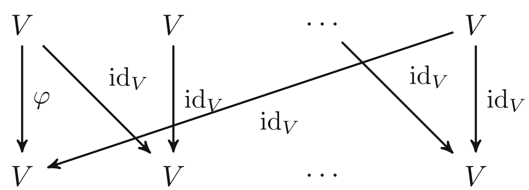

Fig. 2 Let V be a k_n-vector space and φ is an automorphism of V, the band module $M(S', \varphi)$

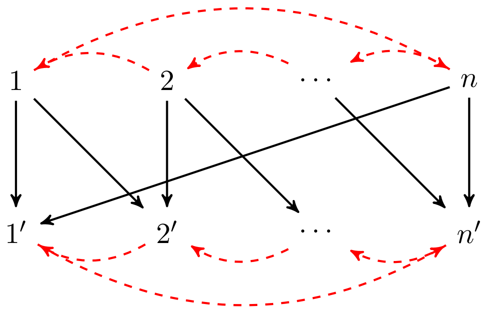

Fig. 3 γ_n is the "reverse shift" automorphism of order n on the quiver $\mathcal{Q}_n$

k_m as two extensions of k, the following proposition show that $k_n \otimes_k k_m$ is an extension of the field k isomorphic to $\gcd(n, m)$ copies of $k_{\mathrm{lcm}(n,m)}$.

Proposition 3.1 *If* $k_n = \mathbb{C}((\varepsilon^{\frac{1}{n}}))$, *for any integer* $n > 0$, *then* $k_n \otimes_k k_m \cong \underbrace{k_l \times \cdots \times k_l}_{g\text{-times}}$, *where* $l = \mathrm{lcm}(m, n)$ *and* $g = \gcd(m, n)$.

Proof See [7, Section 3.1]. □

In particular, if d is a multiple of n, then we obtain the isomorphism $k_d \otimes_k k_n \cong \underbrace{k_d \times \cdots \times k_d}_{n-\text{times}}$ as k_d-algebras.

For every positive integer n, it is straightforward to verify that the set

$$\left\{ \beta_j = \frac{1}{n\varepsilon} \sum_{l=0}^{n-1} \varepsilon^{\frac{n-l}{n}} \otimes \sigma^{j-1}(\varepsilon^{\frac{l}{n}}) \;\middle|\; 1 \le j \le n \right\}, \tag{3}$$

forms a basis of $k_n \otimes k_n$ as k_n-vector space. Additionally, the β_j's satisfy the equations $\beta_j \beta_l = \delta_{jl}\beta_j$, for all $1 \le j, l \le n$. From of the definition $\Lambda_n = \begin{bmatrix} k_n & k_n \oplus {}^{\sigma_n}k_n \\ 0 & k_n \end{bmatrix}$, we obtain the isomorphism

$$k_d \otimes_k \Lambda_n \cong \begin{bmatrix} k_d \otimes k_n & k_d \otimes (k_n \oplus {}^{\sigma_n}k_n) \\ 0 & k_d \otimes k_n \end{bmatrix}$$

as k_d-algebras. Consequently, we can construct the sets of linearly independent matrices for this tensor product from the basis in Eq. 3 in the following straightforward manner:

$$\mathcal{B}_1 = \left\{ \beta_{j1} = \frac{1}{n\varepsilon} \sum_{l=0}^{n-1} \varepsilon^{\frac{n-l}{n}} \otimes \sigma^{j-1}(\varepsilon^{\frac{l}{n}}) E_{11} \;\middle|\; 1 \le j \le n \right\},$$

$$\mathcal{B}_2 = \left\{ \beta_{j2} = \frac{1}{n\varepsilon} \sum_{l=0}^{n-1} \varepsilon^{\frac{n-l}{n}} \otimes \begin{bmatrix} 0 & (\sigma^{j-1}(\varepsilon^{\frac{l}{n}}), 0) \\ 0 & 0 \end{bmatrix} \;\middle|\; 1 \le j \le n \right\},$$

$$\mathcal{B}_3 = \left\{ \beta'_{j1} = \frac{1}{n\varepsilon} \sum_{l=0}^{n-1} \varepsilon^{\frac{n-l}{n}} \otimes \begin{bmatrix} 0 & (0, \sigma^{j-1}(\varepsilon^{\frac{l}{n}})) \\ 0 & 0 \end{bmatrix} \;\middle|\; 1 \le j \le n \right\},$$

$$\mathcal{B}_4 = \left\{ \beta'_{j2} = \frac{1}{n\varepsilon} \sum_{l=0}^{n-1} \varepsilon^{\frac{n-l}{n}} \otimes \sigma^{j-1}(\varepsilon^{\frac{l}{n}}) E_{22} \;\middle|\; 1 \le j \le n \right\},$$

where E_{11} and E_{22} are elements of the standard basis of $\mathrm{Mat}_{2\times 2}(k_d)$. Thus, the union of these four sets

$$\mathcal{B} := \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4, \tag{4}$$

forms naturally a basis of $k_d \otimes_k \Lambda_n$ as a k_d-algebra.

Theorem 3.2 *For a positive integer n and a multiple d of n, the k_d-algebras $k_d \otimes_k \Lambda_n$ and $k_d \mathcal{Q}_n$ are isomophic (as k_d-algebras), where $\Lambda_n = \begin{bmatrix} k_n & k_n \oplus {}^{\sigma_n}k_n \\ 0 & k_n \end{bmatrix}$ and $\mathcal{Q}_n$ is the n-crown quiver (see* Fig. 1*).*

Proof Consider $\mathcal{B}$, the basis of $k_d \otimes_k \Lambda_n$, in Eq. 4. It is straightforward to see that, for every $1 \le i, j \le n$:

$$\beta_{j1}\beta_{i2} = \delta_{j,i}\,\beta_{i2}; \quad \beta_{j1}\beta'_{i1} = \delta_{j-1,i}\,\beta'_{i1}; \quad \beta_{j2}\beta'_{i2} = \delta_{j,i}\,\beta_{j2}; \quad \beta'_{j1}\beta'_{i2} = \delta_{j,i}\,\beta'_{j1}.$$

Consequently, the morphism $k_d \otimes_k \Lambda \longrightarrow k_d \mathcal{Q}_n$, defined by the assignment

$$\beta_{j1} \mapsto j,\ \beta_{j2} \mapsto \alpha_j,\ \beta'_{j1} \mapsto \alpha_{j'},\ \beta'_{j2} \mapsto j'$$

is an isomorphism of k_d-algebras, as it is easy to verify. Here, j (resp. j') is the j-th vertex (resp. j'-th vertex) and $\alpha_i : i \to i'$ and $\alpha_{i'} : (i-1) \to i'$ are arrows in the path algebra $k_d \mathcal{Q}_n$ of the n-crown $\mathcal{Q}_n$. □

Our second objective in this subsection is to show that Galois group $\langle \sigma_d \rangle$ acts on $k_d \otimes \Lambda_n$. Indeed, this action is induced by the k-automorphism defined on $\mathcal{B}$, $\sigma_d \otimes \mathrm{id} : k_d \otimes_k \Lambda_n \to k_d \otimes_k \Lambda_n$, as follows:

$$(\sigma_d \otimes \mathrm{id})(\beta_{jk}) = \beta_{j-1,k}, \text{ and } (\sigma_d \otimes \mathrm{id})(\beta'_{jk}) = \beta'_{j-1,k},\ 1 \le k \le 2.$$

Thus, allow us define an action of the group $\langle \sigma_d \rangle$ on $k_d \mathcal{Q}_n$ through the automorphism:

$$\begin{aligned} \gamma_{d,n} : k_d \mathcal{Q}_n &\longrightarrow k_d \mathcal{Q}_n \\ k\rho &\longmapsto \sigma_d(k)\gamma_n(\rho), \end{aligned}$$

where γ_n is the of order$-n$ automorphism of the n-crown $\mathcal{Q}_n$, see Fig. 3. In this case the automorphism $\gamma_{d,n}$ involves γ_n, an automorphism of $\mathcal{Q}_n$, and $\sigma_d : k_d \to k_d$, so we will said $\gamma_{d,n}$ is a "reverse shift" automorphism.

The action of the group $\langle \sigma_d \rangle$ on $k_d \mathcal{Q}_n$, is crucial for defining the skew group algebra $(k_d \mathcal{Q}_n)\langle \sigma_d \rangle$ in Subsection 3.3.

3.2 Invariant Representantions

In this subsection, we investigate the behavior of the function resulting from applying $\mathrm{id} \otimes -$ to the morphism $\varphi_M : k_n^m \oplus {}^{\sigma_n}k_n^m \to k_n^m$. To begin, consider an arbitrary element $a \in k_n$. Following this, we can define the following k_n-morphisms $f_a : k_n \to k_n$ and $g_a : k_n \to {}^{\sigma_n}k_n$ as follows $1 \mapsto a$. The following lemma illustrates how the morphisms $\mathrm{id} \otimes f_a$ and $\mathrm{id} \otimes g_a$ act on the basis elements of $k_n \otimes k_n$.

Lemma 3.3 *Let $a \in k_n$, $f_a : k_n \to k_n$ and $g_a : k_n \to {}^{\sigma_n}k_n$ be k_n-morphisms defined by $1 \mapsto a$. We have* $\mathrm{id} \otimes f_a : k_n \otimes k_n \to k_n \otimes k_n$ *and* $\mathrm{id} \otimes g_a : k_n \otimes k_n \to k_n \otimes {}^{\sigma_n}k_n$*, which are k_n-morphisms and act on the basis elements as follows:*

$$(\mathrm{id} \otimes f_a)(\beta_j) = \sigma_n^{n+1-j}(a)\beta_j$$
$$(\mathrm{id} \otimes g_a)(\beta_j) = \sigma_n^{n-j}(a)\beta_{j+1}.$$

Proof Let $\beta_j = \frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\sigma_n^{j-1}(\varepsilon^{\frac{l}{n}})$ be an element of the basis $\mathcal{B}$ in Eq. 3. It follows that:

$$
\begin{aligned}
(\mathrm{id}\otimes f_a)(\beta_j) &= \frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\mathrm{id}(\varepsilon^{\frac{n-l}{n}})\otimes f_a(\sigma_n^{j-1}(\varepsilon^{\frac{l}{n}}))\\
&= \frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes a\sigma_n^{j-1}(\varepsilon^{\frac{l}{n}}),
\end{aligned}
$$

Since $a\in k_n$ can be expressed as a linear combination of elements on k, we say $a=\sum_{i=0}^{n-1}a_i\varepsilon^{\frac{i}{n}}$, and since $\sigma_n(\varepsilon^{\frac{l}{n}})=\zeta_n^l\varepsilon^{\frac{l}{n}}$, we obtain that

$$
\begin{aligned}
\frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes a\sigma_n^{j-1}(\varepsilon^{\frac{l}{n}}) &= \frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\left(\sum_{i=0}^{n-1}a_i\varepsilon^{\frac{i}{n}}\right)\zeta_n^{l(j-1)}\varepsilon^{\frac{l}{n}}\\
&= \frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\left(\sum_{i=0}^{n-1}a_i\varepsilon^{\frac{i+l}{n}}\right)\zeta_n^{l(j-1)}.
\end{aligned}
$$

Now, due to the fact that $a_i\in k$ and summations properties, we have

$$
\begin{aligned}
\frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\left(\sum_{i=0}^{n-1}a_i\varepsilon^{\frac{i+l}{n}}\right)\zeta_n^{l(j-1)} &= \frac{1}{n\varepsilon}\sum_{l=0}^{n-1}\sum_{i=0}^{n-1}a_i\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)}\\
&= \frac{1}{n\varepsilon}\sum_{i=0}^{n-1}\sum_{l=0}^{n-1}a_i\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)}\\
&= \sum_{i=0}^{n-1}a_i\frac{1}{n\varepsilon}\left(\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)}\right).
\end{aligned}
\tag{5}
$$

If we fix an index i, we have $\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)}$ and using summations properties we obtain that

$$
\begin{aligned}
\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)} &= \sum_{l=0}^{n-(i+1)}\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)}+\sum_{l=n-i}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)}\\
&= \sum_{l'=i}^{n-1}\varepsilon^{\frac{n+i-l'}{n}}\otimes\varepsilon^{\frac{l'}{n}}\zeta_n^{(l'-i)(j-1)}+\sum_{l'=0}^{i-1}\varepsilon^{\frac{i-l'}{n}}\otimes\varepsilon^{\frac{n+l'}{n}}\zeta_n^{(l'-i+n)(j-1)}.
\end{aligned}
$$

Nevertheless, $\varepsilon^{\frac{i-l'}{n}}\otimes\varepsilon^{\frac{n+l'}{n}}\zeta_n^{(l'-i+n)(j-1)}=\varepsilon^{\frac{n+i-l'}{n}}\otimes\varepsilon^{\frac{l'}{n}}\zeta_n^{(l'-i)(j-1)}=(\zeta_n^{-i(j-1)}\varepsilon^{\frac{i}{n}})\varepsilon^{\frac{n-l'}{n}}\otimes\varepsilon^{\frac{l'}{n}}\zeta_n^{(l')(j-1)}$, since $\varepsilon^{\frac{n}{n}}=\varepsilon$, $\zeta_n^{-i(j-1)}\in k$ and $\zeta_n^n=1$. Therefore

$$
\begin{aligned}
\sum_{l=0}^{n-1}\varepsilon^{\frac{n-l}{n}}\otimes\varepsilon^{\frac{i+l}{n}}\zeta_n^{l(j-1)} &= (\zeta_n^{-i(j-1)}\varepsilon^{\frac{i}{n}})\sum_{l'=i}^{n-1}\varepsilon^{\frac{n-l'}{n}}\otimes\varepsilon^{\frac{l'}{n}}\zeta_n^{l'(j-1)}+\sum_{l'=0}^{i-1}\varepsilon^{\frac{n-l'}{n}}\otimes\varepsilon^{\frac{l'}{n}}\zeta_n^{(l')(j-1)}\\
&= (\zeta_n^{-i(j-1)}\varepsilon^{\frac{i}{n}})\sum_{l'=0}^{n-1}\varepsilon^{\frac{n-l'}{n}}\otimes\varepsilon^{\frac{l'}{n}}\zeta_n^{(l'-i)(j-1)}.
\end{aligned}
\tag{6}
$$

We can substitute Eq. 6 into Eq. 5 and utilize the properties $\zeta_n^n = 1$ and the definition of σ_n to obtain that

$$\begin{aligned}
\sum_{i=0}^{n-1} a_i \frac{1}{n\varepsilon}\left(\sum_{l=0}^{n-1} \varepsilon^{\frac{n-l}{n}} \otimes \varepsilon^{\frac{i+l}{n}} \zeta_n^{l(j-1)}\right) &= \sum_{i=0}^{n-1} a_i \zeta_n^{(-i)(j-1)} \varepsilon^{\frac{i}{n}} \left(\frac{1}{n\varepsilon}\sum_{l'=0}^{n-1} \varepsilon^{\frac{n-l'}{n}} \otimes \varepsilon^{\frac{l'}{n}} \zeta_n^{l'(j-1)}\right) \\
&= \sum_{i=0}^{n-1} a_i \zeta_n^{(i)(1-j)} \varepsilon^{\frac{i}{n}} \left(\beta_j\right) \\
&= \sum_{i=0}^{n-1} a_i \zeta_n^{(i)(n+1-j)} \varepsilon^{\frac{i}{n}} \left(\beta_j\right) \\
&= \sum_{i=0}^{n-1} \sigma_n^{n+1-j}(a_i \varepsilon^{\frac{i}{n}}) \left(\beta_j\right) \\
&= \sigma_n^{n+1-j}(a) \left(\beta_j\right).
\end{aligned}$$

Hence, $(\mathrm{id} \otimes f_a)(\beta_j) = \sigma_n^{n+1-j}(a)\left(\beta_j\right)$. A similar argument, utilizing the fact that $g_a(x) = \sigma_n(x)a$, can be employed to compute $\mathrm{id} \otimes g_a$. This concludes the proof of the lemma. $\square$

Let $a, b \in k_n$ be elements such that $(f_a, g_b) : k_n \oplus {}^{\sigma_n}k_n \to k_n$ is a regular simple representation of Λ_n, and d a multiple of n. Now, by application of $\mathrm{id} \otimes -$ to (f_a, g_b), the induced morphism

$$(\mathrm{id} \otimes f_a,\ \mathrm{id} \otimes g_b) : (k_d \otimes k_n) \oplus (k_d \otimes {}^{\sigma_n}k_n) \to (k_d \otimes k_n),$$

is, by Lemma 3.3, such that

$$\beta_j \oplus 0 \mapsto \sigma_n^{n+1-j}(a)(\beta_j) \quad \text{and} \quad 0 \oplus \beta_j \mapsto \sigma_n^{n-j}(a)\beta_{j+1}.$$

Since, by Proposition 3.1, $k_d \otimes k_n \cong \underbrace{k_d \times \cdots \times k_d}_{n\text{-times}}$, we have that $(\mathrm{id} \otimes f_a,\ \mathrm{id} \otimes g_b)$ can be identified with the following representation:

$$\begin{tikzcd}
k_d \arrow[d, "a"] \arrow[dr, "\sigma_d^{n-1}(b)"] & k_d \arrow[d, "\sigma_d^{n-1}(a)"] & \cdots \arrow[dr, "\sigma_d(b)"] & k_d \arrow[d, "\sigma_d(a)"] \arrow[dlll, "b"] \\
k_d & k_d & \cdots & k_d
\end{tikzcd}$$

Similarly, let $\varphi_M : k_n^m \oplus {}^{\sigma_n}k_n^m \to k_n^m$ be a regular simple representation of Λ_n. Let $[AB]$ be the matrix associated with φ_M, where $A, B \in \mathrm{Mat}_{m\times m}(k_n)$, and $d := \mathrm{lcm}(n, m)$. We have that $k_d \otimes_k \varphi_M$ can be identified with the following representation:

$$\begin{tikzcd}
k_d^m \arrow[d, "A"] \arrow[dr, "\sigma_d^{n-1}(B)"] & k_d^m \arrow[d, "\sigma_d^{n-1}(A)"] & \cdots \arrow[dr, "\sigma_d(B)"] & k_d^m \arrow[d, "\sigma_d(A)"] \arrow[dlll, "B"] \\
k_d^m & k_d^m & \cdots & k_d^m
\end{tikzcd}$$

Here, σ_d is the automorphism defined by $\varepsilon^{\frac{1}{d}} \mapsto \zeta_d \varepsilon^{\frac{1}{d}}$, and $\sigma_d^j(A)$ (resp. $\sigma_d^j(B)$) denotes the matrix obtained by application of the morphism σ_d^j to each entry of the matrix A (resp. B). This is a representation of our n-crown quiver $\mathcal{Q}_n$, see Fig. 1.

From the ideas of Hubery's work in [8, Section 3], with appropriate modifications to our context, we define a module ${}^{\gamma_{d,n}}\mathcal{M}$ using the underlying vector space structure of $\mathcal{M}$, where $\mathcal{M}$ is a $k_d\mathcal{Q}_n$-module. To this end, we introduce a new product $*$ as follows:

$$p * m = \gamma_{d,n}^{-1}(p)m, \text{ with } p \in k_d\mathcal{Q}_n.$$

For each $f : \mathcal{M} \to \mathcal{N}$ a module homomorphism, we obtain a homomorphism ${}^{\gamma_{d,n}}f : {}^{\gamma_{d,n}}\mathcal{M} \to {}^{\gamma_{d,n}}\mathcal{N}$ as follows. Since f is, in particular, a homomorphism of vector spaces, we have ${}^{\gamma_{d,n}}f = f$. Therefore

$$f(p*m) = f(\gamma_{d,n}^{-1}(p)m) = \gamma_{d,n}^{-1}(p)f(m) = p*f(m).$$

This defines an autoequivalence $F(\gamma_{d,n})$ on the category of (right) finitely generated modules on $k_d\mathcal{Q}_n$ and satisfies $F(\gamma_{d,n}^r) = F(\gamma_{d,n})^r$, for any $r \in \mathbb{Z}$. Since $F(\gamma_{d,n})$ is an additive functor, we conclude that $\mathcal{M}$ is indecomposable if and only if ${}^{\gamma_{d,n}}\mathcal{M}$ is indecomposable.

The categories mod$-k_d\mathcal{Q}_n$ of right finitely generated k_d-modules and $\mathrm{Rep}(\mathcal{Q}_n, k_d)$ of k_d-linear representations of $\mathcal{Q}_n$ are known to be equivalent (see [1, Sec. III]). Consequently, the functor $F(\gamma_{d,n})$ must also act on $\mathrm{Rep}(\mathcal{Q}_n, k_d)$.

Let $M = (V_i, V_i', f_i, g_i)$ be a k_d-representation of $\mathcal{Q}_n$, where $f_i : V_i \to V_i'$ and $g_i : V_{i-1} \to V_i'$. Let $\mathcal{M}$ be the corresponding $k_d\mathcal{Q}_n$-module, so $\mathcal{M}$ has underlying k_d-vector space $V = \bigoplus_{j=1}^n (V_i \oplus V_i')$. Let ${}^{\gamma_{d,n}}\mathcal{M}$ be the corresponding module and ${}^{\gamma_{d,n}}M = (W_i, W_i', \hat{f}_i, \hat{g}_i)$ its representation. We wish to describe ${}^{\gamma_{d,n}}M$ in terms of the original M. To this end, note that

$$W_i = \epsilon_i * V = \gamma_{d,n}^{-1}(\epsilon_i)V = \epsilon_{\gamma_n^{-1}(i)}V = V_{\gamma_n^{-1}(i)},$$

similarly $W_i' = V'_{\gamma_n^{-1}(i)}$. On one hand, the product on W_i is defined by $a * w_j = \gamma_{d,n}^{-1}(a)w_i = \sigma_d^{-1}(a)w_i$, for all $a \in k_d$ and $w_i \in W_i$. On the other hand, for each $f_i : V_i \to V_i'$ (resp. $g_i : V_{i-1} \to V_i'$) we can associate a matrix $A_i \in \mathrm{Mat}_{\dim V_i' \times \dim V_i}(k_d)$ (resp. $B_i \in \mathrm{Mat}_{\dim V_i' \times \dim V_{i-1}}(k_d)$). Thus, $\hat{f}_i : W_i \to W_i'$ is defining by

$$\begin{aligned} \hat{f}_i(w) = \alpha_i * w = \gamma_{d,n}^{-1}(\alpha_i)w = \alpha_{\gamma_n^{-1}(i)}w = A_{\gamma_n^{-1}(i)} \\ w = \sigma_d^{-1}(\sigma_d(A_{\gamma_n^{-1}(i)}))w = \sigma_d(A_{\gamma_n^{-1}(i)}) * w, \end{aligned}$$

where $\sigma_d(A_i)$ means that σ_d is applied to each entry of the matrix A_i and α_i is the arrow between i and i'. Similarly, $\hat{g}_i(w) = \sigma_d(B_{\gamma_n^{-1}(i)}) * w$. Hence, if M is a k_d-representation of $\mathcal{Q}_n$ as below

$$\begin{tikzcd}
V_1 \arrow[d, "A_1"] \arrow[dr, "B_2"] & V_2 \arrow[d, "A_2"] & \cdots \arrow[dr, "B_n"] & V_n \arrow[d, "A_n"] \arrow[dlll, "B_1"] \\
V_1' & V_2' & \cdots & V_n'
\end{tikzcd}$$

then the corresponding representation ${}^{\gamma_{d,n}}M$ is:

$$\begin{tikzcd}
V_2 \arrow[d, "\sigma_d(A_2)"] \arrow[dr, "\sigma_d(B_3)"] & V_3 \arrow[d, "\sigma_d(A_3)"] & \cdots \arrow[dr, "\sigma_d(B_1)"] & V_1 \arrow[d, "\sigma_d(A_1)"] \arrow[dlll, "\sigma_d(B_2)"] \\
V_2' & V_3' & \cdots & V_1'
\end{tikzcd}$$

Similar to the approach in [8], we will refer to M as an *isomorphically invariant representation*, denoted by ii-representation, if ${}^{\gamma_{d,n}}M \cong M$. We will say that M is *isomorphically invariant indecomposable* if it is not isomorphic to the proper direct sum of two such isomorphically invariant representations.The general form of these isomorphically invariant indecomposables is demonstrated by the following lemma:

Lemma 3.4 *Let M be a representation of $\mathcal{Q}_n$. The representation M is an isomorphically invariant indecomposable if and only if there exists N be an indecomposable representation such that*

$$M \cong N \oplus {}^{\gamma_{d,n}}N \oplus {}^{\gamma_{d,n}^2}N \oplus \cdots \oplus {}^{\gamma_{d,n}^{m-1}}N,$$

and $m \geq 1$ is the smallest integer such that ${}^{\gamma_{d,n}^m}N \cong N$.

Proof Suppose that M is an ii-representation of $\mathcal{Q}_n$. By Krull-Remak-Schmidt theorem for $\mathcal{Q}_n$-representations, we can write M as a direct sum of indecomposable representations as follows

$$M \cong N_1 \oplus N_2 \oplus \cdots \oplus N_{m-1}.$$

Given that ${}^{\gamma_{d,n}}M \cong M$ and $F(\gamma_{d,n})$ is an additive functor, it must act (up to isomorphism) as a permutation of these indecomposable summands. Without loss of generality, we can assume that $N_i = {}^{\gamma_{d,n}^{i-1}}N_1$. If $N_j \cong N_1$, for some $1 \leq j \leq m-1$, with j minimum with this property. Then $\bigoplus_{i=1}^{j-1} N_i = \bigoplus_{i=1}^{j-1} {}^{\gamma_{d,n}^{i-1}}N_1$ is an ii-representation, since $F(\gamma_{d,n})$ is additive.

Since M and $\bigoplus_{i=1}^{j-1} N_i$ are ii-representations, it follows that $\bigoplus_{i=j}^{m} N_i$ is also an ii-representation, which contradicts the fact that M is ii-indecomposable. Therefore $N_i \cong N_j$ if and only if $i = j$. Besides, ${}^{\gamma_{d,n}}N_i \cong N_{i+1}$ for $1 \leq i \leq m-2$ and ${}^{\gamma_{d,n}}N_{m-1} \cong N_1$, *i.e.*, m is the smallest integer such that ${}^{\gamma_{d,n}^m}N_1 \cong N_1$.

Conversely, let N be an indecomposable representation, where $m \geq 1$ is the smallest integer such that ${}^{\gamma_{d,n}^m}N \cong N$. Since $F(\gamma_{d,n})$ is an additive functor, we obtain that $\bigoplus_{i=1}^{m-1} {}^{\gamma_{d,n}^{i-1}}N$

is an ii-representation. The claim that $\bigoplus_{i=1}^{m-1} \gamma_{d,n}^{i-1} N$ is ii-indecomposable follows from the fact that $m \geq 1$ is the smallest integer such that $\gamma_{d,n}^{m} N \cong N$. $\square$

3.3 Skew Group Algebras

Consider a field $\mathbb{F}$, an $\mathbb{F}$-algebra Λ and a finite group G acting on Λ. The *skew group algebra* ΛG can be defined (see [11]). It has an $\mathbb{F}$-basis consisting of elements of the form λg, where $\lambda \in \Lambda$ and $g \in G$. The product in ΛG is defined as follows:

$$\lambda g \cdot \mu g' := \lambda g(\mu) g \circ g'.$$

An illustrative example of a skew group algebra is $\mathbb{F}_n G$, where $\mathbb{F}_n$ is a Galois extension of $\mathbb{F}$ and $G = \langle \sigma_n \rangle$ is its Galois group. A significant characterization of this algebra is provided by the following proposition.

Proposition 3.5 *If* $\mathbb{F}_n \mid \mathbb{F}$ *is a Galois extension with Galois group* $G = \langle \sigma_n \rangle$*, then the skew group algebra* $\mathbb{F}_n G$ *is isomorphic to* $\mathrm{Mat}_{n\times n}(\mathbb{F})$.

Proof See [2, Appendix] or [10, Appendix A64]. $\square$

In a broader context, when a finite cyclic group G acts on Λ the category of ΛG-modules can be viewed as a specific type of $\Lambda-$modules, as demonstrated by the following proposition.

Proposition 3.6 *Let* $G = \langle \sigma \rangle$ *be a finite cyclic group acting on an* $\mathbb{F}$*-algebra* Λ*, where* $\mathbb{F}$ *is a field. If* ΛG *denotes the corresponding skew group algebra, then the category of* ΛG*-modules is equivalent to the category of pairs* (M, f)*, where* M *is a* Λ*-module and* $f : {}^{\sigma}M \to M$ *is an isomorphism satisfying the following condition:*

$$f \circ {}^{\sigma} f \circ \cdots \circ {}^{\sigma^{n-1}} f = \mathrm{id}_M .$$

Proof A sketch of the proof will be provided here. For further details, refer to [11]. Consider an $\mathbb{F}$-algebra Λ and a group $G = \langle \sigma \rangle$ of order n. Recall that the skew group algebra ΛG possesses an $\mathbb{F}$-basis consisting of elements of the form $\lambda \sigma^r$, and the product is defined as follows: $\lambda \sigma^r \mu \sigma^s := \lambda \sigma^r(\mu) \sigma^{r+s}$.

If M is a ΛG-module, it can also be considered as a Λ-module under the product $\lambda m = \lambda \sigma^0 m$. Additionally, the action of σ on M is given by $\sigma(m+n) = \sigma m + \sigma n$, and $\sigma(\lambda m) = (\sigma \lambda) m = (\sigma(\lambda)\sigma) m = \sigma(\lambda)(\sigma m)$. Now, define an isomorphism $f : {}^{\sigma}M \to M$ by mapping $m \mapsto \sigma m$. This isomorphism satisfies the condition $f \circ {}^{\sigma} f \circ \cdots \circ^{\sigma^{n-1}} f = \mathrm{id}_M$. Conversely, given a pair (M, f), where M is a Λ-module and $f : {}^{\sigma}M \to M$ is an isomorphism satisfying: $f \circ {}^{\sigma} f \circ \cdots \circ^{\sigma^{n-1}} f = \mathrm{id}_M$, we can define a ΛG-module structure on M as follows:

$$\lambda \sigma^r m := f \circ {}^{\sigma} f \circ \cdots \circ^{\sigma^{r-1}} f(\lambda m).$$

This establishes the equivalence between the category of $\Lambda G-$modules and the category of pairs (M, f). $\square$

Remark 3.7 *Consider the n-crown quiver, denoted by* $\mathcal{Q}_n$*, as described in* Fig. 1*. As discussed in Section* 3.1 *the group* $\langle \sigma_d \rangle$ *acts on the path algebra* $k_d \mathcal{Q}_n$ *via the automorphism* $\gamma_{d,n}$*. This allows us to construct the skew group algebra* $(k_d \mathcal{Q}_n)\langle \sigma_d \rangle$*, which has a basis consisting of elements of the form* $\lambda \gamma_{d,n}^r$*, where* λ *represents a path in* $k_d \mathcal{Q}_n$*. The multiplication in this algebra is defined as follows:*

$$\lambda \gamma_{d,n}^r \cdot \mu \gamma_{d,n}^s := \lambda \gamma_{d,n}^r(\mu) \gamma_{d,n}^{r+s}.$$

Recalling that $k_d\mathcal{Q}_n \cong k_d \otimes \Lambda_n$ *(Theorem 3.2) and since* $\langle\sigma_d\rangle$ *acts under this isomorphism, we have* $(k_d\mathcal{Q}_n)\langle\sigma_d\rangle \cong (k_d \otimes \Lambda_n)\langle\sigma_d\rangle$*. By Proposition 3.5 it follows that* $(k_d\mathcal{Q}_n)\langle\sigma_d\rangle \cong \mathrm{Mat}_{d\times d}(\Lambda_n)$*, as* k_n*-algebras.*

In particular, the categories of finitely generated modules $(k_d\mathcal{Q}_n)\langle\sigma_d\rangle$-mod *and* Λ_n-mod *are equivalents.*

On the other hand, consider $k_d \otimes X$ *as a* $k_d \otimes \Lambda_n$*-module, for every* $X \in \Lambda_n$-mod. *The action of* $\langle\sigma_d\rangle$ *on* $k_d \otimes X$ *is given by* ${}^{\sigma_d}(k_d \otimes X) := {}^{\sigma_d}k_d \otimes X$*. Moreover,* $\sigma_d^{-1} \otimes \mathrm{id}_X : k_d \otimes X \to {}^{\sigma_d}(k_d \otimes X)$ *is an isomorphism of* $k_d \otimes \Lambda_n$*-modules. This implies that* $k_d \otimes X$ *is an isomorphically invariant module. Furthermore, the isomorphism satisfies the following conditions:*

$$(\sigma_d^{-1} \otimes \mathrm{id}_X) \circ {}^{\sigma_d}(\sigma_d^{-1} \otimes \mathrm{id}_X) \circ \cdots \circ {}^{\sigma_d^{n-1}}(\sigma_d^{-1} \otimes \mathrm{id}_X) = \mathrm{id}_{k_d\otimes X}\,.$$

If $X \in \Lambda_n$*-mod is a regular simple module, then* $\mathrm{End}_{\Lambda_n}(X) \cong k_m$ *for some* $m \in \mathbb{Z}_{>0}$*. Consequently, if* $d = \mathrm{lcm}(n, m)$ *then, by Proposition 3.1, we have that*

$$\mathrm{End}_{k_d\otimes\Lambda_n}(k_d \otimes X) \cong k_d \otimes \mathrm{End}_{\Lambda_n}(X) \cong \underbrace{k_d \times \cdots \times k_d}_{m}. \tag{7}$$

Lemma 3.8 *Consider a regular simple* Λ_n*-mod* X *such that* $\mathrm{End}_{\Lambda_n}(X) \cong k_m$ *and* $d = \mathrm{lcm}(n, m)$*. Then,* $k_d \otimes X$ *is isomorphic to a direct sum of regular simple* $k_d\mathcal{Q}_n$*-modules* $N_1, N_2, \ldots, N_{m-1}$*, where each* N_i *satisfies* $\mathrm{Hom}_{k_d\mathcal{Q}_n}(N_i,\ N_j) \cong \delta_{i,j}k_d$*, with* $\delta_{i,j}$ *representing the Kronocker delta. Furthermore, for* $1 \leq i \leq m-2$*, the action induces the isomorphisms* ${}^{\gamma_{d,n}}N_i \cong N_{i+1}$*, while* ${}^{\gamma_{d,n}}N_{m-1} \cong N_1$*.*

Proof Let X be a regular simple Λ_n-module, with $\mathrm{End}_{\Lambda_n}(X) \cong k_m$. Then $\mathrm{End}_{k_d\otimes\Lambda_n}(k_d \otimes X) \cong \underbrace{k_d \times \cdots \times k_d}_{m}$. Additionally, $\tau_{k_d\mathcal{Q}_n}(k_d \otimes X) \cong k_d \otimes \tau_{\Lambda_n}(X) \cong k_d \otimes X$. Hence, $k_d \otimes X \cong N_1 \oplus N_2 \oplus \cdots \oplus N_{m-1}$, for regular simple $k_d\mathcal{Q}_n$-modules, where each N_i is a regular simple $k_d\mathcal{Q}_n$-module, with $\mathrm{Hom}_{k_d\mathcal{Q}_n}(N_i,\ N_j) \cong \delta_{i,j}k_d$, for $\delta_{i,j}$ the Kronocker delta.

From the description in Remark 3.7, we establish that $k_d \otimes X \cong {}^{\sigma_d}(k_d \otimes X)$. This allows us to deduce the isomorphism, $\bigoplus_{i=1}^{m-1} N_i \cong {}^{\gamma_{d,n}}(\bigoplus_{i=1}^{m-1} N_i)$. Since the functor $F(\gamma_{d,n})$ is additive, we can assume that ${}^{\gamma_{d,n}}N_i \cong N_{i+1}$, for $1 \leq i \leq m-2$, and ${}^{\gamma_{d,n}}N_{m-1} \cong N_1$, thus completing the proof. □

Theorem 3.9 *Consider an integer* $n > 0$*. There exists a bijective correspondence between the isomorphism classes of regular simple* Λ_n*-modules with endomorphism ring isomorphic to* k_m*, where* $d = \mathrm{lcm}(n, m)$*, and the orbits under* $\langle\sigma_d\rangle$ *of regular simple* $k_d\mathcal{Q}_n$*-modules* N *satisfying the following conditions:*

a) $\mathrm{End}_{k_d\mathcal{Q}_n}(N) \cong k_d$ *and* $\mathrm{Hom}_{k_d\mathcal{Q}_n}(N,\ {}^{\sigma_d^j}N) \cong 0$ *with* $j = 1, 2, \cdots, m-1$*,*
b) ${}^{\sigma_d^m}N \cong N$ *and* ${}^{\sigma_d^j}N \ncong N$ *with* $j = 1, 2, \cdots, m-1$*.*

Proof Let $X \in \Lambda_n$-mod be regular simple. By Lemma 3.8, there exists a $k_d\mathcal{Q}_n$-module N that satisfies both of the given conditions. Conversely, if N is a module that satisfies both conditions, we can consider the module $\bigoplus_{j=0}^{m-1} {}^{\sigma^j}N$ as a $(k_d\mathcal{Q}_n)\langle\sigma_d\rangle$-module. Since $(k_d\mathcal{Q}_n)\langle\sigma_d\rangle$-modules are equivalent to Λ_n-modules, there exists a regular simple Λ_n−module associated with $\bigoplus_{j=0}^{m-1} {}^{\sigma_d^j}N$. □

3.4 Homogenous and Non Homogenous Representations

As established in Theorem 3.9, our primary focus is on identifying representatives of regular simple representations of the n-crown $\mathcal{Q}_n$. To achieve this, we begin by noting that $\gamma_{d,n}$ commutes with τ, the Auslander-Reiten translate (see [11, Lemma 4.1]). Consequently, $\gamma_{d,n}$ acts on both the preprojective and preinjective components of the Auslander-Reiten quiver.

The indecomposable regular representations are precisely those with defect 0, and τ operates with finite period on each tube of the Auslander-Reiten quiver. Each regular indecomposable representation exhibits regular serial behavior and is uniquely determined, up to isomorphism, by its regular top and regular length. Therefore, the actions of both $\gamma_{d,n}$ and τ are determined by their actions on the regular simples.

Now, following 2.3, we consider the regular simple representations M_a, for $n \in \mathbb{Z}_{>0}$ and d be a multiple of n:

$$\begin{tikzcd}
k_d \arrow[d, "a"] \arrow[dr, "1"] & k_d \arrow[d, "1"] & \cdots \arrow[dr, "1"] & k_d \arrow[d, "1"] \arrow[dlll, "1"] \\
k_d & k_d & \cdots & k_d
\end{tikzcd}$$

with $a \neq 0$. Also, it is easy to see that $\mathrm{End}_{k_d\mathcal{Q}}(M_a) \cong k_d$. Moreover, for $a, b \in k_d$, we have $\mathrm{Hom}_{k_d\mathcal{Q}}(M_a, M_b) = 0$ if $a \neq b$. Considering M_a and M_b as representations as described above, they are isomorphic if and only $a = b$. As we observed in Section 3.2, $\gamma_{d,n}$ acts on M_a as follows:

$$\begin{tikzcd}
k_d \arrow[d, "1"] \arrow[dr, "1"] & k_d \arrow[d, "1"] & \cdots \arrow[dr, "1"] & k_d \arrow[d, "\sigma_d(a)"] \arrow[dlll, "1"] \\
k_d & k_d & \cdots & k_d
\end{tikzcd}$$

We denote this representation by ${}^{\gamma_{d,n}}M_a$. Since $a \neq 0$, it follows that $\sigma_d(a) \neq 0$ and therefore the morphism $f_{(\sigma_d(a))^{-1}} : k_d \to k_d$ is an isomorphism. Consequently, we have that ${}^{\gamma_{d,n}}M_a \cong M_{\sigma_d(a)}$, and further, ${}^{\gamma_{d,n}^j}M_a \cong M_{\sigma_d^j(a)}$, for all $1 \leq j \leq n-1$. Thus, we are interested in elements $a \in k_n$ that are *generic*, meaning that $\sigma_d^j(a) \neq a$, for $1 \leq j \leq n-1$. Thus, for $a \in k_m$, a generic element, we have the representation:

$$M := \bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j}M_a$$

which is an $ii-$indecomposable representation. These representations are of particular interest to us. As will be shown later, they provide the pairs described in 3.6. As demonstrated in Propositions 3.13 and 3.14, a crucial aspect in determining these pairs is the fact that the $\frac{n-m}{\gcd(n,m)}$-th power of these generic elements has a $\frac{d}{m}$-root in k_d. To establish this, we prove the following Lemma, which is slightly more general:

Lemma 3.10 *Consider the field of complex Laurent series $\mathbb{C}((x))$ and let n be a positive integer. For any element $a \in \mathbb{C}((x))$, there exists an element $b \in \mathbb{C}((x))$ such that $a = b^n x^m$, for some $0 \leq m \leq n-1$ and* $\mathrm{codeg}(a) \equiv m \pmod{n}$.

Proof Consider the element $b \in \mathbb{C}((x))$, let say $b = \sum_{j=0}^{\infty} b_j x^j$. Suppose that $b^n = \sum_{j=0}^{\infty} c_j x^j$, for some $n > 0$. We will demonstrate that this leads to a recurrence relation for the coefficients c_j in terms of the coefficients b_i. To achieve this, a common practice is to view b and b^n as "functions" and then obtain the derivative of b^n with respect to x. This will allow us to obtain the recurrence relations. More precisely, by definition of derivates, $(b^n)' = nb^{n-1}b'$, then $(b^n)'b = nb^n b'$. Since $(b^n)' = \sum_{j=1}^{\infty} jc_j x^{j-1} = \sum_{j=0}^{\infty}(j+1)c_{j+1}x^j$ and $b' = \sum_{j=0}^{\infty}(j+1)b_{j+1}x^j$, we obtain that

$$\sum_{j=0}^{\infty}(j+1)c_{j+1}x^j \sum_{j=0}^{\infty} b_j x^j = (b^n)'b = nb^n b' = n\sum_{j=0}^{\infty} c_j x^j \sum_{j=0}^{\infty}(j+1)b_{j+1}x^j,$$

equating coefficients, we obtain a recurrence relation for c_j.

In a similar way, if $a = \sum_{j=0}^{\infty} a_j x^j$, with $a_0 \neq 0$, we can define b_j recursively until we obtain $b^n = a$, for $b = \sum_{j=0}^{\infty} b_j x^j \in \mathbb{C}((x))$. Furthermore, if $a \in \mathbb{C}((x))$ is such that $\operatorname{codeg}(a)$ is a multiple of n, then $a = cx^{\operatorname{codeg}(a)}$, for some $c \in \mathbb{C}((x))$, and $a = cx^{\operatorname{codeg}(a)} = b^n x^{\operatorname{codeg}(a)} = (bx^{\operatorname{codeg}(a)/n})^n$.

Hence, for $a \in \mathbb{C}((x))$, with $\operatorname{codeg}(a)$ is a multiple of n, there exists $b \in \mathbb{C}((x))$ such that $a = b^n$. More generally, if $a \in \mathbb{C}((x))$ is such that $\operatorname{codeg}(a) \equiv m \pmod{n}$, with $1 \leq m \leq n-1$, then there exists $c \in \mathbb{C}((x))$, with $\operatorname{codeg}(c)$ a multiple of n such that $a = cx^m = b^n x^m$. $\square$

Remark 3.11 An interesting and useful remark is the following

$$\Pi_{i=0}^{n-1}\sigma_n^i(\varepsilon^{\frac{j}{n}}) = \Pi_{i=0}^{n-1}\zeta_n^{ij}(\varepsilon^{\frac{j}{n}}) = (\zeta_n^{\left(\sum_{i=0}^{n-1} i\right)j})\varepsilon^j = \varepsilon^j. \tag{8}$$

Now, our main objective is to prove that for $a \in k_m$, a generic element, there exists an isomorphism $f : \bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j} M_a \to {}^{\gamma_{d,n}}\left(\bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j} M_a\right)$, such that the induced automorphism ${}^{\gamma_{d,n}^{l-1}} f \circ \cdots \circ {}^{\gamma_{d,n}} f \circ f$ of M is the identity for some positive integer l. To this end, we will consider the following three cases: $m = n$, $m < n$ and $m > n$.

Proposition 3.12 *Let $M = \bigoplus_{j=0}^{n-1} {}^{\gamma_{d,n}^j} M_a$ be an ii-indecomposable representation. Then there exists an isomorphism $f : M \to {}^{\gamma_{d,n}} M$ such that the automorphism ${}^{\gamma_{d,n}^{n-1}} f \circ \cdots \circ {}^{\gamma_{d,n}} f \circ f$ of M is the identity.*

Proof Consider the $ii-$indecomposable representation M as the direct sum $\bigoplus_{j=0}^{n-1} {}^{\gamma_{d,n}^j} M_a$, where M_a is a regular simple representation defined earlier for $a \in k_n$, a generic element. We can assume that M has the form

$$\begin{tikzcd}
k_d^n \arrow[d, "A_1"] \arrow[dr, "\mathrm{id}_n"] & k_d^n \arrow[d, "A_n"] & \cdots \arrow[dr, "\mathrm{id}_n"] & k_d^n \arrow[d, "A_2"] \arrow[dlll, "\mathrm{id}_n"] \\
k_d^n & k_d^n & \cdots & k_d^n
\end{tikzcd}$$

with $A_j = \mathrm{diag}(1, \cdots, \sigma_n^{j-1}(a) \cdots, 1)$, that is, the diagonal matrix with 1 in all entries except the $\gamma_n^{j-1}(1)$-th entry. Thus, the induced representation ${}^{\gamma_{d,n}}M$ is given by:

$$\begin{tikzcd}
k_d^n \arrow[d, "\sigma_n(A_n)"] \arrow[dr, "\mathrm{id}_n"] & k_d^n \arrow[d, "\sigma_n(A_{n-1})"] & \cdots \arrow[dr, "\mathrm{id}_n"] & k_d^n \arrow[d, "\sigma_n(A_1)"] \arrow[dlll, "\mathrm{id}_n"] \\
k_d^n & k_d^n & \cdots & k_d^n
\end{tikzcd}$$

Now, we consider the linear morphism $k_d^n \longrightarrow k_d^n$ such that its associated matrix is given by

$$(E_n)_{ij} = \begin{cases} \delta_{i,\,j-1} \text{ if } 1 < j \leq n, \\ \delta_{i,\,1} \quad \text{if } \quad j = n, \end{cases} \tag{9}$$

where $\delta_{i,j}$ is the Kronocker delta, i.e.,

$$E_n = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Thus, we can define the morphism $f = ((f_i, f_{i'}))_{1 \leq i \leq n}$ from M to ${}^{\gamma_{d,n}}M$, where $f_i = f_{i'} = E_n$ as above. It is easy to see that

$$\begin{tikzcd}
k_d^n \arrow[r, "A_j"] \arrow[d, "E_n"] & k_d^n \arrow[d, "E_n"] \\
k_d^n \arrow[r, "\sigma_n(A_{j-1})"] & k_d^n
\end{tikzcd}$$

meanwhile, for $j = 1$,

$$\begin{tikzcd}
k_d^n \arrow[r, "A_1"] \arrow[d, "E_n"] & k_d^n \arrow[d, "E_n"] \\
k_d^n \arrow[r, "\sigma_n(A_n)"] & k_d^n
\end{tikzcd}$$

Consequently, ${}^{\gamma_{d,n}}f = ((g_i, g_{i'}))_{1 \leq i \leq n}$, where $g_i = f_{i+1},\ g_{i'} = f_{(i+1)'}$ *for* $1 \leq i \leq n-1$ *and* $g_n = f_1,\ g_{n'} = f_{1'}$. In conclusion, for the morphism f, we have that ${}^{\gamma_{d,n}}f = f$. Thus

$${}^{\gamma_{d,n}^{n-1}}f \circ \cdots \circ {}^{\gamma_{d,n}}f \circ f = f \circ f \circ \cdots \circ f = ((E_n^n, E_n^n))_{1 \leq i \leq n}.$$

Nevertheless, $E_n^n = \mathrm{id}_n$, is the identity linear map. Hence, we can conclude that $f = ((f_i, f_{i'}))_{1 \le i \le n}$ is the morphism satisfying the desired properties. $\square$

Proposition 3.13 *Let m be a positive integer, with $m < n$, and $M = \bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j} M_a$ be an ii-indecomposable representation. Then there exists an isomorphism $f : M \to {}^{\gamma_{d,n}} M$ such that the induced automorphism ${}^{\gamma_{d,n}^{d-1}} f \circ \cdots \circ {}^{\gamma_{d,n}} f \circ f$ of M is the identity.*

Proof Consider the $ii-$indecomposable representation M as the direct sum $\bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j} M_a$, where M_a is a regular simple representation defined earlier for a generic element $a \in k_m$. We assume that M has the form depicted in Fig. 4.

In this case, $A_j = \mathrm{diag}(1, \cdots, \sigma_d^{j-1}(a) \cdots, 1)$ represents the diagonal matrix with 1 in all entries except the $\gamma_m^{j-1}(1)$-th entry, for $1 \le j \le m$, and $A_j = \mathrm{id}_m$, for $m + 1 \le j \le n$. By Lemma 3.10, there exists $b \in k_d$ such that $b^{d/m} = a^{\frac{n-m}{\gcd(n,m)}}$. We analyze the following cases:

- If $\mathrm{lcm}(m, n) = n$, it suffices to take $f = ((E_m, E_m))_{1 \le i \le n}$.
- If $\mathrm{lcm}(m, n) \ne n$, we can consider f_i, $f_{i'} \in \mathrm{Mat}_{m \times m}(k_d)$, for $1 \le i \le n$, the pair of matrizes defined as follows:

$$f_i = \mathrm{diag}(1, x_i, \cdots, 1) E_m \text{ and } f_{i'} = \begin{cases} f_{i-1} & \text{if } 2 \le i \le n \\ f_n & \text{if } i = 1, \end{cases}$$

  where E_m is the matrix described in Eq. 9, $x_i = ab^{-1}$, if $1 \le i \le n - m$, and $x_i = b^{-1}$, otherwise.

Now, as established in the proof of Proposition 3.12, we obtain that $\sigma_d(A_{j-1}) E_m = E_m A_j$, for $1 \le j < m$, and $\sigma_d(A_m) E_m = E_m A_1$. Therefore:

$$\begin{aligned} \sigma_d(A_{j-1}) \,\mathrm{diag}(1, b^{-1}, \cdots, 1) E_m &= \mathrm{diag}(1, b^{-1}, \cdots, 1) \sigma_d(A_{j-1}) \\ E_m &= \mathrm{diag}(1, b^{-1}, \cdots, 1) E_m A_j, \end{aligned}$$

$$\begin{aligned} \sigma_d(A_m) \,\mathrm{diag}(1, b^{-1}, \cdots, 1) E_m &= \mathrm{diag}(1, b^{-1}, \cdots, 1) \sigma_d(A_m) \\ E_m &= \mathrm{diag}(1, b^{-1}, \cdots, 1) E_m A_1, \end{aligned}$$

and, for $m + 1 \le j \le n - 1$, it follows that

$$\begin{aligned} \sigma_d(A_j) f_{n+1-j} = \mathrm{id}_m \,\mathrm{diag}(1, ab^{-1}, \cdots, 1) E_m &= \mathrm{diag}(1, ab^{-1}, \cdots, 1) E_m \,\mathrm{id}_m \\ &= f_{n-j} A_{j+1}. \end{aligned}$$

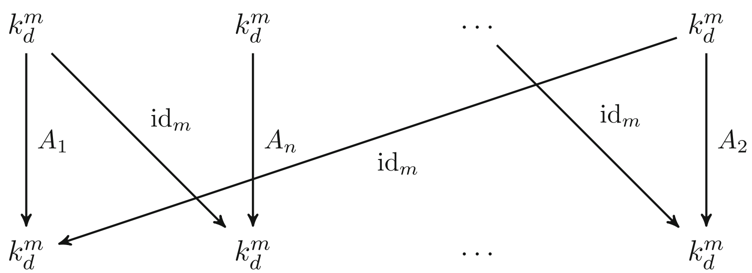

Fig. 4 A representation of $\bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j} M_a$

On the other hand, we have that

$$\begin{aligned}\sigma(A_n)f_1 &= \mathrm{id}_m \operatorname{diag}(1, ab^{-1}, \cdots, 1)E_m\\ &= \operatorname{diag}(1, b^{-1}, \cdots, 1)\operatorname{diag}(1, a, \cdots, 1)E_m\\ &= \operatorname{diag}(1, b^{-1}, \cdots, 1)E_m \operatorname{diag}(a, 1, \cdots, 1)\\ &= \operatorname{diag}(1, b^{-1}, \cdots, 1)E_m A_1.\end{aligned}$$

Thus, we prove that the pair $f = ((f_i, f_{i'}))_{1\le i\le n}$ defines a morphism from M to ${}^{\gamma_{d,n}}M$. Consequently,

$$\begin{aligned}{}^{\gamma_{d,n}}f &= ((\sigma_d(f_{\gamma_n^{-1}(i)}), \sigma_d(f_{\gamma_n^{-1}(i')})))_{1\le i\le n}, \quad \text{and}\\ {}^{\gamma_{d,n}^j}f &= ((\sigma_d^j(f_{\gamma_n^{-j}(i)}), \sigma_d^j(f_{\gamma_n^{-j}(i')})))_{1\le i\le n}.\end{aligned}$$

Now, we start grouping the first m-morphisms to obtain

$$\begin{aligned}{}^{\gamma_{d,n}^{m-1}}f \circ \cdots \circ {}^{\gamma_{d,n}}f \circ f &= ((\sigma_d^{m-1}(f_{\gamma_n^{-j}(i)})\cdots\sigma_d(f_{\gamma_n^{-1}(i)})f_i, \sigma_d^{m-1}(f_{\gamma_n^{-j}(i')})\cdots\sigma_d(f_{\gamma_n^{-1}(i')})f_{i'})))_{1\le i\le n}\\ &= ((\operatorname{diag}(x_i, \cdots, \sigma_d^{m-1}(x_{\gamma_n^{1-m}(i)})), g_{i',1}))_{1\le i\le n},\end{aligned}$$

where $g_{i',1} := g_{\gamma_n(i),1}$ and $g_{i,1} = \operatorname{diag}(x_i, \sigma_d(x_{\gamma_n^{-1}(i)}), \cdots, \sigma_d^{m-1}(x_{\gamma_n^{1-m}(i)}))$. Then, in the more general case, for $1 \le l \le d/m$, it follows that

$${}^{\gamma_{d,n}^{lm-1}}f \circ \cdots \circ {}^{\gamma_{d,n}^{m(l-1)}}f = ((g_{i,l}, g_{i',l}))_{1\le i\le n},$$

where $g_{i',l} := g_{\gamma_n(i),l}$ and $g_{i,l} = \operatorname{diag}(\sigma_d^{m(l-1)}(x_{\gamma_n^{m-ml}(i)}), \cdots, \sigma_d^{ml-1}(x_{\gamma_n^{1-ml}(i)}))$.

From the foregoing computations, we derive that $g_{1,d/n}g_{1,(d/n)-1}\cdots g_{1,1} = \operatorname{diag}(y_1, \cdots, y_m)$, where

$$y_j = \sigma_d^{m(\frac{d}{m}-1)+j-1}(x_{\gamma_n^{m(1-\frac{d}{m})+1-j}(1)})\cdots\sigma_d^{m+j-1}(x_{\gamma_n^{1-(m+j)}(1)})\sigma_d^{j-1}(x_{\gamma_n^{1-j}(1)}).$$

Since either $x_i = ab^{-1}$ or $x_i = b^{-1}$, a well-known combinatorial problem determines that the number of factors of y_j for which some $x_{\gamma_n^{m(1-l)+1-j}(1)}$ is equal ab^{-1}, is given by $\frac{n-m}{\gcd(n,m)}$. Now, using the fact of $a \in k_m$ and Eq. 8 we conclude that

$$y_j = \sigma_d^{j-1}(a^{\frac{n-m}{\gcd(n,m)}})\Pi_{i=0}^{\frac{d}{m}-1}\sigma_d^{mi+j-1}(b^{-1}) = \sigma_d^{j-1}(a^{\frac{n-m}{\gcd(n,m)}})\sigma_d^{j-1}\left(\Pi_{i=0}^{\frac{d}{m}-1}\sigma_d^{mi}(b^{-1})\right) = \sigma_d^{j-1}(a^{\frac{n-m}{\gcd(n,m)}}b^{-\frac{d}{m}}).$$

Consequently, $g_{1,d/n}g_{1,(d/n)-1}\cdots g_{1,1} = \mathrm{id}_m$. Moreover, it can be similarly proven that $g_{i,d/n}g_{i,(d/n)-1}\cdots g_{i,1} = \mathrm{id}_m$, for $1 \le i \le n$. In conclusion, ${}^{\gamma_{d,n}^{d-1}}f \circ \cdots \circ {}^{\gamma_{d,n}}f \circ f$ is the identity, which completes the proof of the proposition. □

Proposition 3.14 *Let m be an positive integer, with $m > n$, and $M = \bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j}M_a$ be an ii-indecomposable representation. Then there exists an isomorphism $f : M \to {}^{\gamma_{d,n}}M$ such that the induced automorphism ${}^{\gamma_{d,n}^{d-1}}f \circ \cdots \circ {}^{\gamma_{d,n}}f \circ f$ of M is the identity.*

Proof As in the proof of Proposition 3.13 we consider the $ii-$indecomposable representation M as the direct sum $\bigoplus_{j=0}^{m-1} {}^{\gamma_{d,n}^j}M_a$, where M_a is a regular simple representation and $a \in k_m$, a generic element. We can assume that M has the form of Fig. 4. In this case, $A_i = \operatorname{diag}(x_{i,1}, \cdots, x_{i,m}) \in \mathrm{Mat}_{m\times m}(k_d)$, where

$$x_{i,j} = \begin{cases}\sigma_d^{m+1-j}(a) & \text{if } j \equiv \gamma_n^{i-1}(1)\\ 1 & \text{otherwise.}\end{cases}$$

Let l be a positive integer such that $l \equiv m(\mathrm{mod}\, n)$ and we consider $b \in k_d$, such that $b^{d/m} = \sigma_d(a)^{\frac{n-l}{\gcd(n,l)}}$. If $\mathrm{lcm}(m,n) = m$, it suffices to take $f = ((E_m, E_m))_{1\leq i\leq n}$. Otherwise, that is $\mathrm{lcm}(m,n) \neq m$, we can consider $f_i,\ f_{i'} \in \mathrm{Mat}_{m\times m}(k_d)$, for $1 \leq i \leq n$, the pair of matrizes defined as follows:

$$f_i = \mathrm{diag}(x_i, 1, \cdots, 1)E_m \text{ and } f_{i'} = \begin{cases} f_{i-1} & \text{if } 2 \leq i \leq n \\ f_n & \text{if } i = 1 \end{cases}$$

where E_m is the matrix described in Eq. 9, $x_i = \sigma_d(a)b^{-1}$, if $l \leq i \leq n-1$, and $x_i = b^{-1}$, otherwise. Demonstrating that the pair $f = ((f_i, f_{i'}))_{1\leq i\leq n}$ is the morphism with the desired properties follows a similar line of reasoning to the proof of Proposition 3.13. □

To summarize the three cases above, we can state that:

Proposition 3.15 *Let $M = \bigoplus_{j=0}^{n-1} {}^{\gamma^j_{d,n}} M_a$ be an ii-indecomposable representation. Then there exists an isomorphism $f : M \to {}^{\gamma_{d,n}}M$ such that the induced automorphism ${}^{\gamma^{l-1}_{d,n}} f \circ \cdots \circ {}^{\gamma_{d,n}} f \circ f$ of M is the identity, for some positive integer l. Moreover, if $n = m$ then $l = n$ and if $n \neq m$ then $l = d$.*

In contrast, entire families of ii-indecomposable representations exist, specifically the $M_{(j)}$ and $N_{(j)}$ representations (illustrated graphically in Fig. 5). More precisely, the $M_{(j)}$ representantions are defined by:

- (M_j1) For $1 \leq i,\ i' \leq j$, the vector space associated with the vertex i and i' is k_n. For the remaining vertices, the vector space is 0.
- (M_j2) For $1 \leq i \leq j$ and $2 \leq i' \leq j$ the k_n-morphism associated to the arrows α_i and $\alpha_{i'}$ is id_{k_n}. For the remaining arrows, the morphisms are the null morphism.

And $N_{(j)}$ representations are defined by:

- (N_j1) For $1 \leq i \leq j$ and $2 \leq i' \leq j+1$ the vector space associated to the vertex i and i' is k_n. For the remaining vertices, the vector space is 0
- (N_j2) For $2 \leq i \leq j$ and $2 \leq i' \leq j+1$ the k_n-morphism associated to the arrows α_i and $\alpha_{i'}$ is id_{k_n}. For the remaining arrows, the morphisms are the null morphism.

The following remark establishes a crucial connection between the previously mentioned representations and the string representations of the crown quiver.

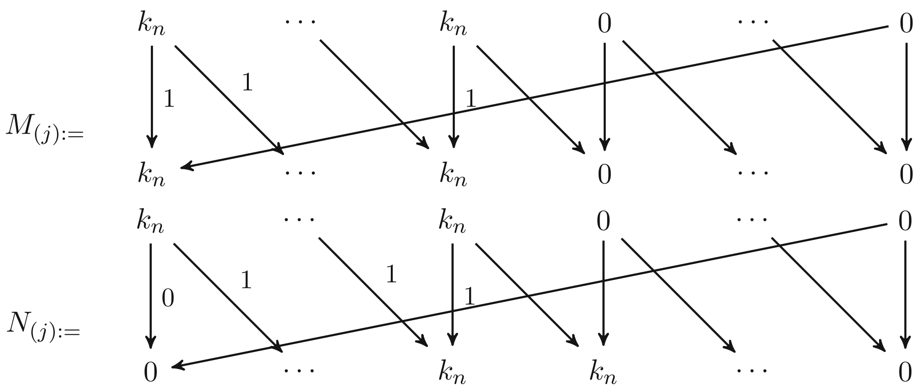

Fig. 5 For $1 \leq j \leq n$, the family of representations $M_{(j)}$ and $N_{(j)}$, respectively

Remark 3.16 *For $n \in \mathbb{Z}_{>0}$ and $1 \le j \le n$, we have $M_{(j)} = M(s_{1',2j-1})$ and $N_{(j)} = M(s_{1,2j-3})$. Furthermore, for $1 \le i \le n-1$, we have ${}^{\gamma^i_{d,n}}M_{(j)} = M(s_{(n+1-i)', 2j-1})$ and ${}^{\gamma^i_{d,n}}N_{(j)} = M(s_{n+1-i, 2j-1})$. These equalities follow directly from the definitions of string modules and the action of γ_n, as described in Sections 2.3 and 2.4. Consequently, for $j \ge 2$, the homomorphism rings $\mathrm{Hom}(M_{(j)}, {}^{\gamma_{n,n}}M_{(j)})$ and $\mathrm{Hom}(N_{(j)}, {}^{\gamma_{n,n}}N_{(j)})$ are both isomorphic to the field k_n, as is easily verified.Thus, by Theorem 3.9, we focus our attention on the representations $M_{(1)}$ and $N_{(1)}$.*

In the following lemma, we prove that these representations admit morphisms as described in Proposition 3.15.

Lemma 3.17 *There exists a natural automorphism $f : M \to {}^{\gamma_{n,n}}M$, such that*

$$ {}^{\gamma^{n-1}_{n,n}}f \circ \cdots \circ {}^{\gamma_{n,n}}f \circ f = \mathrm{id}, $$

where $M = \bigoplus_{i=0}^{n-1} {}^{\gamma^i_{n,n}}M_{(1)}$ or $M = \bigoplus_{i=0}^{n-1} {}^{\gamma^i_{n,n}}N_{(1)}$.

Proof It suffices to define $f = ((f_i, f_{i'}))_{1 \le i \le n}$ by setting $f_{i'} = f_i = \mathrm{id}_{k_n}$. □

Theorem 3.18 *Let N be a regular simple $k_d Q_n$-module. Then N satisfies the conditions of Theorem 3.9 if and only if N is isomorphic to either M_a, for a generic element $a \in k_m$, or is isomorphic to $M_{(1)}$ or $N_{(1)}$.*

Proof If N is isomorphic to either M_a, for a generic element $a \in k_m$, or is isomorphic to $M_{(1)}$ or $N_{(1)}$, then N satisfies the conditions of Theorem 3.9, due to Proposition 3.15 and Lemmas 3.17.

Conversely, suppose N is a regular simple $k_d Q_n$-module that satisfies the conditions of Theorem 3.9. Invoking Theorem 2.1, Lemmas 2.2 and 2.3 and Remark 3.16, we conclude that the indecomposable regular representations are isomorphic to $M_{(1)}$ or $N_{(1)}$ or $M(S', \varphi)$. If N is not isomorphic to $M_{(1)}$ or $N_{(1)}$, then $N \cong M(S', \varphi)$. Since N is also simple, this implies that $(V, \varphi) = (k_d, a)$, for some $a \in k_d$. The only elements that satisfy the conditions of Theorem 3.9 are the generic elements. Therefore, $N \cong M_a$ for some generic $a \in k_d$. □

A regular simple representation X with $\mathrm{End}_{\Lambda_n}(X) \cong k_m$ is said to be *homogeneous* if $k_d \otimes X \cong \bigoplus_{j=0}^{n-1} {}^{\gamma^j_{d,n}}M_a$, where $a \in k_n$ is a generic element. If $k_d \otimes X \cong \bigoplus_{j=0}^{n-1} {}^{\gamma^j_{n,n}}M_{(1)}$ or $k_d \otimes X \cong \bigoplus_{j=0}^{n-1} {}^{\gamma^j_{n,n}}N_{(1)}$, the representation X is said to be *non-homogenous*.

Remark 3.19 *Theorem 3.18 establishes that if M is homogeneous, there exist at most mn distinct generic elements in k_m that give rise to the "same" simple regular representation. Furthermore, for Λ_n, the homogeneous simple regular representations are parametrized by* $\mathrm{Spec}(k_n[x])$.

We now seek to establish a suitable representative for these regular simple representations. When X is non-homogenous we can construct a satisfactory representative, as demonstrated in Theorem 3.22.

For homogenous X, we only have a partial classification when $k_d \otimes X \cong \bigoplus_{j=0}^{n-1} {}^{\gamma^j_{d,n}}M_{x^n}$, where x^n is a generic element for $x \in k_n$ (see Theorem 3.21).

To accomplish this, for a positive integer m, we begin by defining the matrix $\alpha_m \in \mathrm{Mat}_{m \times m}(k)$ as follows:

$$ (\alpha_m)_{ij} = \begin{cases} \delta_{i+1,\,j} & \text{si } i < m, \\ \delta_{1,\,j}\varepsilon & \text{si } i = m, \end{cases} $$

i.e, in a more explicit form

$$\alpha_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \varepsilon & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For the sake of completeness, we define $\alpha_1 = \varepsilon$. The minimal polynomial of α_m, can be verified without difficulty, is $y^m - \varepsilon \in k[y]$.

Now, from Lemma 2.1 in [7], we can state that:

Proposition 3.20 *Consider x a generic element in k_m. Then, $\bigoplus_{j=0}^{m-1} {}^{\gamma^j_{d,n}} M_x$ is isomorphic to*

$$\begin{tikzcd}
k_d^m \arrow[d, "\alpha(x)"] \arrow[dr, "\mathrm{id}_m"] & \cdots \arrow[dr, "\mathrm{id}_m"] & k_d^m \arrow[dll, "\mathrm{id}_m"'] \arrow[d, "\mathrm{id}_m"] \\
k_d^m & \cdots & k_d^m .
\end{tikzcd}$$

Proof Suppose $x \in k_m$ is a generic element. Since ${}^{\gamma^j_{d,n}} M_x \cong M_{\sigma_d^j(x)}$, we can consider $\bigoplus_{j=0}^{n-1} {}^{\gamma^j_{d,n}} M_x$ as follows:

$$\begin{tikzcd}
k_d^m \arrow[d, "D_x"] \arrow[dr, "\mathrm{id}_m"] & \cdots \arrow[dr, "\mathrm{id}_m"] & k_d^m \arrow[dll, "\mathrm{id}_m"'] \arrow[d, "\mathrm{id}_m"] \\
k_d^m & \cdots & k_d^m
\end{tikzcd}$$

where $D_x = \mathrm{diag}(x, \sigma_m(x), \cdots, \sigma_m^{m-1}(x))$. By Lemma 2.1 in [7], it follows that there exist $A \in \mathrm{GL}_m(k_m)$ such that $\alpha(x)A = AD_x$. $\square$

Theorem 3.21 *Consider $x \in k_m$ such that x^n is a generic element. Then, the representation $\varphi_x : k_n^m \oplus^{\sigma_n} k_n^m \longrightarrow k_n$, whose matrix is $[\alpha(x)\ \mathrm{id}_m]$, is a regular simple representation.*

Proof We know that $k_d \otimes \varphi_x$ can be identified with the representation

$$\begin{tikzcd}
k_d^m \arrow[d, "\alpha(x)"] \arrow[dr, "\sigma_n^{n-1}(\mathrm{id}_m)"] & k_d^m \arrow[d, "\sigma_n^{n-1}(\alpha(x))"] & \cdots \arrow[dr, "\sigma_n(\mathrm{id}_m)"] & k_d^m \arrow[d, "\sigma_n(\alpha(x))"] \arrow[dlll, "\mathrm{id}_m"] \\
k_d^m & k_d^m & \cdots & k_d^m
\end{tikzcd}$$

where $d = \mathrm{lcm}(n, m)$. Since $\sigma_n^j(\mathrm{id}_m) = \mathrm{id}_m$ and $\sigma_n^j(\alpha(x)) = \alpha(x)$, we have that the corresponding representation is

$$\begin{tikzcd}
k_d^m \arrow[d, "\alpha(x)"] \arrow[dr, "\mathrm{id}_m"] & k_d^m \arrow[d, "\alpha(x)"] & \cdots \arrow[dr, "\mathrm{id}_m"] & k_d^m \arrow[d, "\alpha(x)"] \arrow[dlll, "\mathrm{id}_m"] \\
k_d^m & k_d^m & \cdots & k_d^m
\end{tikzcd}$$

Nevertheless, this representation is isomorphic to

$$\begin{tikzcd}
k_d^m \arrow[d, "\alpha(x)^n"] \arrow[dr, "\mathrm{id}_m"] & k_d^m \arrow[d, "\mathrm{id}_m"] & \cdots \arrow[dr, "\mathrm{id}_m"] & k_d^m \arrow[d, "\mathrm{id}_m"] \arrow[dlll, "\mathrm{id}_m"] \\
k_d^m & k_d^m & \cdots & k_d^m
\end{tikzcd}$$

where the isomorphism is given by $f = ((f_i, f_{i'}))_{1 \le i \le n-1}$, where $f_i = \alpha(x)^{n-i}$ and $f_{i'} = f_j$, $i \equiv j - 1 (\mathrm{mod}\, n)$. From the equation $\alpha(x^n) = \alpha(x)^n$, and due to Proposition 3.20 and Theorem 3.9, we can conclude that φ_x is regular simple representation. $\square$

Theorem 3.22 *The representation $\varphi : k_n \oplus {}^{\sigma_n} k_n \longrightarrow k_n$, whose matrix is $[0 \ \mathrm{id}_{k_n}]$ or $[\mathrm{id}_{k_n} \ 0]$, is a regular simple representation.*

Proof Consider the representation $\varphi : k_n \oplus {}^{\sigma_n} k_n \longrightarrow k_n$, defined by the matrix $[0 \ \mathrm{id}_{k_n}]$ or $[\mathrm{id}_{k_n} \ 0]$. We will analyze the first case, $[0 \ \mathrm{id}_{k_n}]$, as the second case is entirely analogous. As established in Section 3.2, $k_n \otimes \varphi$ has the following description:

$$\begin{tikzcd}
k_n \arrow[d, "0"] \arrow[dr, "\sigma_n^{n-1}(\mathrm{id}_{k_n})"] & \cdots \arrow[dr, "\sigma_n(\mathrm{id}_{k_n})"] & k_n \arrow[d, "\sigma_n(0)"] \arrow[dll, "\mathrm{id}_{k_n}"] \\
k_n & \cdots & k_n
\end{tikzcd}$$

where $\sigma_n^i(0) = 0$ and $\sigma_n^i(\mathrm{id}_{k_n}) = \mathrm{id}_{k_n}$, for any $1 \leq i \leq n-1$. However, we have that it is the representation $\bigoplus_{i=0}^{n-1} \gamma^i_{n,n} N_{(1)}$ and, by Theorem 3.9, we conclude that $k_n \otimes \varphi$ is a regular simple representation. $\square$

3.5 The Classification on Λ_1

In this case, we have that $\Lambda_1 = \begin{bmatrix} k & k^2 \\ 0 & k \end{bmatrix}$, which is isomorphic to kQ_1 the path algebra of 2-Kronecker quiver:

$$Q_1 = \ 1 \leftleftarrows 2 \ .$$

The classification of all irreducible representations of the Kronecker quiver over a field is a well-established problem, as evidenced in [13] for $n-$Kronecker quiver and [9] for the specific case 2-Kronecker quiver. A well-known fact is that the regular simple representations of k_nQ_1, whose dimension vector is $(1, 1)$, are classified up to isomorphism by:

$$M_a := \ k_n \underset{1}{\overset{a}{\leftleftarrows}} k_n \qquad M_{(1)} := \ k_n \underset{0}{\overset{1}{\leftleftarrows}} k_n \ ,$$

with $a \in k_n$.

According to [9], the regular simple modules of kQ_1 with dimension vector (n, n), where $n \geq 2$, are in bijective correspondence with the irreducible monic polynomials over $k[x]$. Thus, for any irreducible monic polynomial $p(x) \in k[x]$ with $\deg(p) = n$, we can construct the following representation:

$$k_n^n \underset{\mathrm{id}_n}{\overset{C(p)}{\leftleftarrows}} k_n^n \ ,$$

where $C(p)$ is the companion matrix of $p(x)$. If $a \in k_n$ is a root of $p(x)$, then a is generic, i.e., $\sigma_n^j(a) \neq a$ for $1 \leq j \leq n-1$. Therefore, $C(p) = V^{-1} \operatorname{diag}(a, \sigma_n(a), \cdots, \sigma_n^{n-1}(a))V$, where V is the Vandermonde matrix:

$$V = \begin{bmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ 1 & \sigma_n(a) & (\sigma_n(a))^2 & \cdots & (\sigma_n(a))^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \sigma_n^{n-1}(a) & (\sigma_n^{n-1}(a))^2 & \cdots & (\sigma_n^{n-1}(a))^{n-1} \end{bmatrix}.$$

Consequently, the representations of $p(x)$ are equivalents to

$$k_n^n \underset{\mathrm{id}_n}{\overset{D_a}{\leftleftarrows}} k_n^n \ ,$$

where $D_a = \mathrm{diag}(a, \sigma_n(a), \cdots, \sigma_n^{n-1}(a))$.

Alternatively, a representation that is not part of this family is

$$k \underset{0}{\overset{1}{\leftleftarrows}} k \;\oplus\; k \underset{1}{\overset{0}{\leftleftarrows}} k \;.$$

Proposition 3.23 *Let n a positive integer and M a Λ_1−module such that either $M = \bigoplus_{j=0}^{n-1} {}^{\gamma_n^j} M_a$ or $M = M_{(1)} \oplus {}^{\gamma_2} M_{(1)}$. Then, there exists a morphism $f : M \longrightarrow {}^{\gamma_n} M$ such that the induced automorphism ${}^{\gamma_n^{n-1}} f \circ \cdots \circ {}^{\gamma_n} f \circ f$ of M is the identity.*

Proof It suffices to take $f = (E_n, E_n)$.

It is straightforward to adapt Lemma 3.8 to the case of $\mathcal{Q}_1$. Therefore, we can conclude the following proposition. □

Proposition 3.24 *Each regular simple representation of Λ_1 is isomorphic to either $\bigoplus_{j=0}^{n-1} {}^{\gamma_n^j} M_a$, for some generic element $a \in k_n$, or $M_{(1)} \oplus {}^{\gamma_2} M_{(1)}$.*

Finally, for Λ_1, we can effectively describe the structure of a regular simple representation.

Theorem 3.25 *Let $a \in k_n$ be a generic element. Then, the representation $\varphi_a : k^n \oplus k^n \to k^n$, whose matrix is given by $[\alpha(a)\ \mathrm{id}_n]$, is a regular simple representation.*

Proof Since a is generic it suffices to show that $k_n \otimes_k \varphi_a$ is isomorphic to

$$k_n^n \underset{\mathrm{id}_n}{\overset{D_a}{\leftleftarrows}} k_n^n \;,$$

where $D_a = \mathrm{diag}(a, \sigma_n(a), \cdots, \sigma_n^{n-1}(a))$. Recall that $k_n \otimes_k \varphi_a$ can be identified with the following representation:

$$k_n^n \underset{\mathrm{id}_n}{\overset{\alpha(a)}{\leftleftarrows}} k_n^n \;.$$

Thus, the conclusion is a consequence of Lemma 2.1 in [7]. □

Acknowledgements The authors thank the anonymous referee for their valuable suggestions and corrections, which improved the quality of this article. They are especially grateful for the referee's identification of significant errors in an earlier version of Theorem 3.22. The second author gratefully acknowledges the support provided by CODI (Universidad de Antioquia, UdeA) through its postdoc program. The second author is especially grateful to Christof Geiss (Instituto de Matemáticas, UNAM, México) for the several enriching discussions about this work.

Author Contributions All authors whose names appear on the submission: 1. made substantial contributions to the conception or design of the work; 2. drafted the work or revised it critically for important intellectual content; 3. approved the version to be published; and 4. agree to be accountable for all aspects of the work in ensuring that questions related to the accuracy or integrity of any part of the work are appropriately investigated and resolved.

Funding Open Access funding provided by Colombia Consortium The first and third authors were partially supported by CODI (Universidad de Antioquia, UdeA) by project numbers 2022-52654 and 2023-62291, respectively. The second author was supported by CODI (Universidad de Antioquia, UdeA) through its Postdoc program by project number 2022-47830.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

Ethical Approval Not applicable.

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